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UNIT ROOT TESTING FOR FINANCIAL BUBBLES**

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Specification Sensitivities in Right-Tailed Unit Root Testing for Financial Bubbles*

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Abstract

Right-tailed unit root tests have proved promising for detecting exuberance in economic and financial activities. Like left-tailed tests, the limit theory and test performance are sensitive to the null hypothesis and the model specification used in parameter estimation. This paper aims to provide some empirical guidelines for the practical implementation of right-tailed unit root tests, focusing on the sup ADF test of Phillips, Wu and Yu (2011), which implements a right-tailed ADF test repeatedly on a sequence of forward sample recursions. We analyze and compare the limit theory of the sup ADF test under different hypotheses and model specifications. The size and power properties of the test under various scenarios are examined in simulations and some recommendations for empirical practice are given. An empirical application to Nasdaq data reveals the practical importance of model specification on test outcomes.

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1. Introduction

In distinguishing between two hypotheses, such as a unit root null and a stationary alternative, results are often sensitive to model formulation. In effect, the maintained hypothesis or *technical lens* through which the properties of the data are explored can influence outcomes in a major way. Formulating a suitable maintained hypothesis is particularly difficult in the presence of nonstationarity because of the different roles that parameters can play under the null hypothesis of a unit root and the alternative of stationarity. Many of these issues of formulation have already been extensively studied in unit root testing.

Suppose, for example, that the null hypothesis is that the data is difference stationary and the alternative is that the data is stationary. If we run the ADF regression

$$R_1 : \Delta y_t = \beta y_{t-1} + \sum_{i=1}^k \phi_i \Delta y_{t-i} + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} (0, \sigma^2), \quad (1)$$

and test the null $\beta = 0$ against the alternative $\beta < 0$, we also (implicitly) assume that the mean of y_t is zero under the alternative. Under this lens any evidence of a non-zero mean in the sample is likely to be interpreted as evidence in favor of the null and the test procedure tends to have poor power. A more suitable lens allows for a non zero mean in y_t under the alternative through the regression

$$R_2 : \Delta y_t = \alpha + \beta y_{t-1} + \sum_{i=1}^k \phi_i \Delta y_{t-i} + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} (0, \sigma^2), \quad (2)$$

even though α is zero under the null. Similarly, if the null is difference stationarity and the alternative trend stationarity, then the regression model (2) will be inappropriate because an empirical trend may be misinterpreted as evidence of a unit root, leading to the augmented formulation

$$R_3 : \Delta y_t = \alpha_0 + \alpha_1 t + \beta y_{t-1} + \sum_{i=1}^k \phi_i \Delta y_{t-i} + \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} (0, \sigma^2), \quad (3)$$

where we can test the null $\beta = 0$ against the alternative $\beta < 0$, even if $\alpha_1 = 0$ under the null. Use of the maintained hypothesis R_3 allows for both a unit root with drift ($\alpha_0 \neq 0$ and $\alpha_1 = 0$) under the null and trend stationarity ($\alpha_0 \neq 0$ and $\alpha_1 \neq 0$) under the alternative. Similar issues, of course, arise with more complex maintained hypotheses that allow for trend breaks and other deterministic components. The

regression model of a left-tailed unit root test (against stationary or trend stationary alternatives) needs to nest the alternative hypothesis.¹

Right-tailed unit root tests are also of empirical interest, particularly in detecting exuberance in financial markets or mildly explosive alternatives (Diba and Grossman, 1988; Hall, Psaradakis and Sola, 1999; Phillips, Wu and Yu, 2011, PWY hereafter). With these right-tailed tests, there are related issues of model formulation. The present work examines appropriate ways of formulating empirical regressions when the null hypothesis is difference stationarity and the alternative is a mildly explosive process (Phillips and Magdalinos, 2007) of the type

$$H_A : y_t = \delta_T y_{t-1} + \varepsilon_t \text{ with } \delta_T = 1 + cT^{-\theta} \text{ and } \varepsilon_t \sim N(0, \sigma^2), \quad (4)$$

where $c > 0$, $\theta \in (0,1)$ and T is the sample size. H_A is formulated with a zero intercept since a non-zero intercept produces a dominating deterministic component that has an empirically unrealistic explosive form (Phillips and Yu, 2009). Similar characteristics apply in the case of inclusion of a deterministic trend term in H_A . Since these forms are unreasonable for most economic and financial time series, the model (4) is formulated without an intercept or a deterministic trend.

Suppose we run R_1 to investigate evidence for mildly explosive behavior as in (4). Analogous to the effects in a left-tailed unit root test, in a regression of the form R_1 any evidence of non-zero mean in Δy_t may be misjudged as evidence in favor of the alternative - in this case, mildly explosive behavior. To elaborate, consider the following cases where under the null the mean of Δy_t is not necessarily zero:

$$H_{01} : y_t = y_{t-1} + \varepsilon_t, \varepsilon_t \sim N(0, \sigma^2), \quad (5)$$

$$H_{02} : y_t = dT^{-\eta} + y_{t-1} + \varepsilon_t \text{ with } d > 0, \eta > 1/2 \text{ and } \varepsilon_t \sim N(0, \sigma^2), \quad (6)$$

$$H_{03} : y_t = \tilde{\alpha} + y_{t-1} + \varepsilon_t \text{ with } \tilde{\alpha} \neq 0 \text{ and } \varepsilon_t \sim N(0, \sigma^2). \quad (7)$$

In all three specifications, y_t is (asymptotically) difference stationary. The mean of Δy_t in H_{01} is zero. In H_{02} , Δy_t has a local-to-zero mean (i.e. $dT^{-\eta}$) which is of order of $O(T^{-\eta})$, while in H_{03} the time series Δy_t has a non-zero constant mean and y_t is a stochastic trend with deterministic linear drift. Now suppose that the true null model under a right-tailed unit root test is H_{02} or H_{03} and the regression

¹ Similar arguments can be found in Dickey, Bell and Miller (1986) and Davidson and MacKinnon (2004).

model is R_1 . Due to the fact that the regression model does not allow for deterministic-trend-like behavior in y_t under the null, the presence of a non-zero mean in Δy_t (i.e. $dT^{-\eta}$ in H_{02} and $\tilde{\alpha}$ in H_{03}) may likely be misinterpreted as evidence that supports an explosive alternative.

Table 1 summarizes five scenarios that are considered in this paper. Since y_t does not have deterministic trend behavior under the null model H_{01} , Cases 1 and 2 are expected to be less empirically reasonable formulations given the mildly explosive alternative. Further, although R_3 has a constant as well as a deterministic trend and both of these may generate deterministic-trend-like behavior under the null, the presence of either of these two terms is empirically unrealistic when $\beta > 0$. Thus, Case 5 also seems inappropriate. By contrast, Cases 3 and 4 are both empirically more realistic. Diba and Grossman (1988) implemented a unit root test based on Case 5, while the test given in PWY is based on Case 2.

This paper illustrates the practical importance of the null hypothesis and regression model specification in right-tailed unit root testing in the context of the sequential procedures of the type proposed by PWY to detect bubbles in economic and financial data. This test implements a right-tailed unit root test repeatedly on a sequence of forward expanding samples. We discuss the asymptotic distributions of the test statistic and examine the size and the power properties of the test under different scenarios. Based on the simulation findings, we provide guidelines for the selection of an appropriate null hypothesis and a suitable regression model formulation with associated test critical values.

The rest of the paper is organized as follows. Section 2 reviews the respective limit distributions of the ADF statistic under Cases 1 - 5 and examines the finite sample performance of the right tailed unit root tests under Cases 3 and 4. Section 3 introduces several different types of exuberant behavior for the alternative hypothesis: the periodically collapsing explosive behavior of Evans (1991); the locally explosive behavior introduced by Phillips and Yu (2009, PY hereafter); and a modification of the PY model which produces a more realistic generating process for locally explosive behavior. The sup ADF test (i.e., the sequential right-tailed ADF test), along with the behavior of the sup ADF statistic (including its limiting and finite sample distributions), are explored in Section 4. Section 5 reports size and power properties for the sup ADF test under Cases 3 and 4. We apply the sup ADF test using model formulations corresponding to Cases 3 and 4 to NASDAQ market data in Section 6. Section 7 concludes. Proofs of propositions are collected in a separate technical note which is available for download from https://sites.google.com/site/shupingshi/Hypothesis_Appendix.pdf?attredirects=0.

2. Right-Tailed Unit Root Tests

Right-tailed unit root tests, like their left-tailed counterparts, have asymptotic distributions which depend on the null hypothesis and the regression model.

Proposition 2.1 *Under Case 2 and Case 3 (with $\eta > 1/2$), the asymptotic distribution of the ADF statistic is*

$$ADF \xrightarrow{L} \frac{\frac{1}{2}[W^2(1)-1] - W(1)\int_0^1 W(s)ds}{\left\{ \int_0^1 W^2(s)ds - \left[\int_0^1 W(s)ds \right]^2 \right\}^{1/2}} := F_{23}(W), \quad (8)$$

where W is a standard Wiener process and \xrightarrow{L} denotes the convergence in distribution. Under Case 1, the asymptotic distribution of the ADF statistic is

$$ADF \xrightarrow{L} \frac{1}{2}[W(1)^2 - 1] \left[\int_0^1 W^2(s)ds \right]^{-1/2} := F_1(W); \quad (9)$$

Under Case 4, the asymptotic ADF distribution is

$$ADF \xrightarrow{L} \left[\int_0^1 s dW(s) - \int_0^1 W(s)ds \right] \left(\int_0^1 s^2 ds \right)^{-1/2} := F_4(W), \quad (10)$$

which is identical to the standard normal; Under Case 5, the asymptotic distributions of the ADF statistic is

$$ADF \xrightarrow{L} \frac{E - 12CF - 6AF + 6CD - 4AD}{(12AC - 12C^2 + B - 4A^2)^{1/2}} := F_5(W) \quad (11)$$

with $A = \int_0^1 W(s)ds$, $B = \int_0^1 W^2(s)ds$, $C = \int_0^1 W(s)sds$, $D = W(1)$, $E = \frac{1}{2}[W(1)^2 - 1]$ and

$$F = W(1) - \int_0^1 W(s)ds.$$

Remark 2.1 *The asymptotic ADF distribution in Case 3 is identical to that of Case 2 despite the inclusion of an intercept in the null hypothesis model. The reason that the inclusion of an intercept does not affect the limit distribution is that the intercept effect is of a smaller order of magnitude than the stochastic trend.*

Remark 2.2 *If $\eta = 1/2$ in Case 3, then the asymptotic ADF distribution is*

$$ADF \xrightarrow{L} (D_\sigma - A_\sigma C_\sigma) (B_\sigma - A_\sigma^2)^{-1/2} := F_{30}(W, \sigma), \quad (12)$$

with $A_\sigma = \frac{1}{2} + \sigma \int_0^1 W(s) ds$, $B_\sigma = \frac{1}{3} + \sigma^2 \int_0^1 W(s)^2 ds + 2\sigma \int_0^1 W(s) s ds$, $C_\sigma = W(1)$ and

$D_\sigma = \left[W(1) - \int_0^1 W(s) ds \right] + \frac{1}{2} \sigma [W(1)^2 - 1]$. *Importantly, the limit theory depends on the nuisance parameter σ and hence it is not invariant unless we include a trend in the regression or adjust for the trend in some other way (for example, Schmidt and Phillips, 1992 and Phillips and Lee, 1996).*

Remark 2.3 *Suppose $\eta < 1/2$ in Case 3, then the asymptotic ADF distribution is equivalent to that of Case 4, (10). This result arises because the intercept is of higher order of magnitude and behaves like a linear deterministic trend.*

Remark 2.4 *The asymptotic ADF distributions under Case 1, Case 2 and Case 5 are well documented in the unit root literature (as is the fact that the asymptotic ADF distribution under Case 4 is standard normal); see Phillips (1987) and Phillips and Perron (1988). We provide an alternative expression for the asymptotic ADF distribution under Case 4, (10), because this aids the derivation below.*

Remark 2.5 *As discussed above, Case 3 and Case 4 are empirically more reasonable than the other cases. This observation is in contrast to the left-tailed unit root test where Case 2 and Case 5 are found to be empirically more reasonable. In Case 3 we compare a unit root model with an asymptotically negligible intercept with a mildly explosive model. In Case 4, we compare a unit root model with a intercept with a mildly explosive model. In finite samples, the null hypothesis in both case may exhibit a linear trend but the alternative hypothesis has a nonlinear trend behavior.*

2.1 The Finite Sample Distributions of the Unit Root Test

The finite sample distributions of the ADF test under Case 1, Case 2 and Case 5 are well documented; see, for example, Fuller (1995) and Hamilton (1994). In this Section we only compare the finite sample distribution of the ADF statistic with the corresponding asymptotic distribution under Case 3 and Case 4. The finite sample distributions are obtained from 2,000 Monte Carlo simulations. The lag order is

determined by the significance test proposed by Campbell and Perron (1991) with the maximum lag length 12. The asymptotic distribution is obtained by numerical simulation with 2,000 iterations. The Wiener process is approximated by partial sums of $N(0,1)$ with 5,000 steps.

Figure 1 plots the finite sample distributions of the ADF statistic under Case 3 when $\tilde{\alpha} = T^{-1}$ (i.e. $d = 1$ and $\eta = 1$) and Case 4 when $\tilde{\alpha} = 1$ (i.e. $d = 1$ and $\eta = 0$ in Case 2). The dotted lines in the figure correspond to the finite sample distributions of the ADF statistic with sample size $T = \{40, 80, 200, 400\}$ and the solid lines are the asymptotic distributions. As we can see, the finite sample distribution of the ADF statistic converges to the asymptotic distribution $F_{23}(W)$ under Case 3 in Figure 1a and $F_4(W)$ under Case 4 in Figure 1b as the sample size increases.

Figure 2 displays the finite sample distributions of the ADF statistic when the regression model is R_2 , $T = 400$ and $\tilde{\alpha} = T^{-\eta}$ (i.e. $d = 1$) with $\eta = \{1, 0.9, \dots, 0.1, 0\}$. We can observe the following phenomena. First, when $\eta > 0.5$ (Case 3) the finite sample distribution moves towards the asymptotic distribution $F_{23}(W)$ as η increases. Nevertheless, the discrepancy among the finite sample distributions with $\eta = \{0.6, 0.7, 0.8, 0.9, 1\}$ is negligible. Second, the finite sample distribution of the ADF statistic with $\eta = 0.5$ is significantly different from those with $\eta > 0.5$. Third, when $\eta < 0.5$ (Case 4) the discrepancy among the finite sample distributions with $\eta = \{0.4, 0.3, 0.2, 0.1, 0\}$ is quite visible. However, we observe a tendency of convergence towards the asymptotic distribution $F_4(W)$ as η decreases (or the drift value $\tilde{\alpha}$ in H_{03} increases).

3. Exuberant Behavior

Exuberance may manifest in various forms. In this Section, we focus on the periodically collapsing explosive process of Evans (1991) and a new locally explosive process as possible alternatives.

3.1 Periodically Collapsing Explosive Process

The DGP proposed by Evans (1991) consists of a market fundamental component P_t^f , which follows a random walk process

$$P_t^f = \tilde{u} + P_{t-1}^f + \sigma_f \varepsilon_t, \varepsilon_t \stackrel{iid}{\sim} N(0,1) \quad (13)$$

and a periodically collapsing explosive bubble component such that

$$B_{t+1} = \rho^{-1} B_t \varepsilon_{B,t+1}, \quad \text{if } B_t < b \quad (14)$$

$$B_{t+1} = [\zeta + (\pi\rho)^{-1} \theta_{t+1} (B_t - \rho\zeta)] \varepsilon_{B,t+1}, \quad \text{if } B_t \geq b, \quad (15)$$

where $\rho^{-1} > 1$ and $\varepsilon_{B,t} = \exp(y_t - \tau^2/2)$ with $y_t \sim N(0, \tau^2)$. θ_t follows a Bernoulli process which takes the value 1 with probability π and 0 with probability $1 - \pi$. ζ is the remaining size after the bubble collapse. The bubble component has the property that $E_t(B_{t+1}) = \rho^{-1} B_t$. By construction, the bubbles collapse completely in a single period when triggered by the Bernoulli process realization.

The market fundamental equation, (13), is equivalent to the combination of a random walk dividend process and the Lucas asset pricing equation

$$D_t = \mu + D_{t-1} + \varepsilon_{Dt}, \quad \varepsilon_{Dt} \stackrel{iid}{\sim} N(0, \sigma_D^2) \quad (16)$$

$$P_t^f = \frac{\mu\rho}{(1-\rho)^2} + \frac{\rho}{1-\rho} D_t, \quad (17)$$

where μ is the drift of the dividend process, σ_D^2 is the variance of the dividend. The drift of the market fundamental process \tilde{u} equals $\mu\rho(1-\rho)^{-1}$ and the standard deviation $\sigma_f = \sigma_D\rho(1-\rho)^{-1}$. In Evans (1991), the parameter values for μ and σ_D^2 were matched to the sample mean and sample variance of the first differences of real S&P500 dividends from 1871 to 1980. The value for the discount factor ρ is equivalent to a 5% annual interest rate. In other words, the parameter settings in Evans (1991) correspond to a yearly frequency. In accordance with our empirical application, we consider a set of the parameters calibrated to monthly data. Parameters μ and σ_D^2 are set to be the sample mean and the sample variance of the monthly first differences of real NASDAQ dividends as described in the application section (normalized to unity at the beginning of the sample period). These are $\mu = 0.0020$ and $\sigma_D^2 = 0.0034$ respectively. The discount factor equals 0.985. We can then calculate the values of \tilde{u} , σ_f , P_0^f based on those of μ , σ_D^2 , D_0 .

The setting of parameters in the bubble component, (14) - (15), are the same as those in Evans (1991). The asset price P_t is equal to the sum of the market fundamental component and the bubble component,

namely $P_t = P_t^f + \kappa B_t$, where κ controls the relative magnitudes of these two components. These two settings are provided in Table 2 and are labeled *yearly* and *monthly* respectively.

Figure 3a illustrates a realization of this DGP with the yearly parameter settings (sample size $T = 100$) and Figure 3b displays a realization of this DGP with the monthly parameter settings ($T = 200$).

3.2 Locally Explosive Process

Locally explosive behavior can be expressed in terms of an AR process with time-varying coefficients such that

$$y_t = u_t + \rho_t y_{t-1} + \sigma_t \varepsilon_t, \quad \varepsilon_t \stackrel{iid}{\sim} N(0,1), \quad (18)$$

where u_t is the intercept, ρ_t is the autoregressive coefficient and σ_t is the disturbance standard deviation.

In PY, it is assumed that $u_t = 0$ and $\sigma_t = \sigma$ for all $t = 1, \dots, T$. The autoregressive coefficient ρ_t is greater than 1, namely $\rho_t = 1 + cT^{-\alpha}$ with $c > 0$ and $\alpha \in (0,1)$, for the bubble expansion period, but otherwise equals unity, viz., $\rho_t = 1$. More specifically,

$$y_t = y_{t-1} \mathbf{1}(t < T_e) + \rho_T y_{t-1} \mathbf{1}(T_e \leq t \leq T_f) + \left(\sum_{k=T_f+1}^t \varepsilon_k + y_{T_f}^* \right) \mathbf{1}(t > T_f) + \varepsilon_t \mathbf{1}(t \leq T_f) \quad (19)$$

where $\rho_T = 1 + cT^{-\alpha}$, $y_{T_f}^* = y_{T_e} + y^*$ with $y^* = O_p(1)$, $\mathbf{1}(\cdot)$ is an indicator function, T_e is the origination date of the bubble and T_f is the termination date.

Notice that y_t is re-initialized to y_{T_e} (with a small perturbation) upon the bubble collapse. Although bubbles frequently collapse rapidly, in many cases it is unrealistic to require complete collapse within one period. For instance, according to PWY, the dot-com bubble began to collapse in March 2000 and the termination date was between September 2000 and March 2001. Therefore, instead of a sudden collapse as in equation (19), we assume that y_t switches to a (mildly) stationary regime when the bubble starts to burst. The new DGP can be specified as

$$y_t = \begin{cases} u_t + y_{t-1} + \sigma_1 \varepsilon_t, & t \in [1, T_e) \cup (T_c, T] \\ \phi_T y_{t-1} + \sigma_2 \varepsilon_t, & t \in [T_e, T_f] \\ \gamma_T y_{t-1} + \sigma_3 \varepsilon_t, & t \in (T_f, T_c] \end{cases}, \quad (20)$$

where T_c marks the conclusion of the bubble collapse, $\phi_T = 1 + c_1 T^{-\alpha}$ and $\gamma_T = 1 - c_2 T^{-\beta}$ with $c_1, c_2 > 0$ and $\alpha, \beta \in [0, 1)$. The formulation of the AR coefficients ϕ_T and γ_T both involve mild deviations from unity in the sense of Phillips and Magdalinos (2007), one in the explosive direction for the bubble expansion, the other in the stationary direction for the bubble collapse. Equation (20) corresponds with (18) if we set

$$\begin{aligned} u_t &= s_{nt} u_1, \\ \rho_t &= s_{nt} + s_{bt} \phi_T + s_{ct} \gamma_T, \\ \sigma_t &= s_{nt} \sigma_1 + s_{bt} \sigma_2 + s_{ct} \sigma_3, \end{aligned}$$

where $s_{nt} = \mathbf{1}(t \in [0, T_e) \cup (T_c, T])$, $s_{bt} = \mathbf{1}(t \in [T_e, T_f])$, $s_{ct} = \mathbf{1}(t \in (T_f, T_c])$, which are the regime indicators for the market fundamental, the bubble expansion and the bubble collapse respectively.

We illustrate the process (20) by setting the market fundamental regime as in Table 2 (monthly): $y_0 = 94.122$, $u_0 = 0.131$, $\sigma_1 = 3.829$. We set other parameters relating to the bubble expansion and collapsing regime to be: $c_1 = c_2 = 1$, $\alpha = 0.6$, $\beta = 0.5$, $\sigma_2 = \sigma_1$, $\sigma_3 = 2\sigma_1$, $T_e = [0.6T]$, $T_f = [0.70T]$, $T_c = [0.75T]$ (we explore different settings for parameters $\alpha, \beta, T_e, T_f, T_c$ in the size and power comparison Section). The sample size T is equal to 200. The implied autoregressive coefficients $\phi_{200} = 1.042$ and $\gamma_{200} = 0.929$. Figure 4 illustrates one realization of the DGP. Compared with the PY and Evans DGPs, a distinguishing feature of this DGP is that the bubble collapsing process is a gradual one and hence it is more realistic.

4. The Sup ADF Test

The sup ADF (SADF) test of PWY was suggested to test the existence of exuberant behavior in economic and financial time series. The alternative hypothesis of the test therefore includes both periodically collapsing explosive behavior and locally explosive behavior. The null hypotheses are exactly the same as those for the right-tailed unit root test in equation (5) - (7). In the sup ADF test, the right-tailed unit root

test is implemented repeatedly on a forward expanding sample sequence and inference is based on the sup value of the corresponding ADF sequence.

Suppose r is the window size of the regression (proportional to the full sample size) for the right-tailed unit root test. In the sup ADF test, the window size r expands from r_0 to 1 through the recursive calculations. The smallest window size r_0 is selected to ensure that there are sufficient observations to achieve estimation efficiency. The number of observations in the regression is $T_r = [Tr]$, where $[\cdot]$ signifies the integer part of its argument and T is the total number of observations.

The regression models for the sup ADF test are:

$$R_1^s : \Delta y_t = \beta_r y_{t-1} + \sum_{i=1}^k \phi_r^i \Delta y_{t-i} + \varepsilon_t; \quad (21)$$

$$R_2^s : \Delta y_t = \alpha_r + \beta_r y_{t-1} + \sum_{i=1}^k \phi_r^i \Delta y_{t-i} + \varepsilon_t; \quad (22)$$

$$R_3^s : \Delta y_t = \alpha_r + \beta_r y_{t-1} + \gamma_r t + \sum_{i=1}^k \phi_r^i \Delta y_{t-i} + \varepsilon_t, \quad (23)$$

where $t = 1, \dots, T_r$ and k is the lag order, which is determined by a significance test (Campbell and Perron, 1991). The corresponding ADF t -statistic is denoted by ADF_r . To test for the existence of bubbles, inferences are made based on the sup ADF statistic, which is defined as $\sup_{r \in [r_0, 1]} ADF_r$ and denoted by $SADF(r_0)$. It is important to highlight the dependence of $SADF$ on r_0 although little attention has been paid to this dependency in the literature.

Like the ADF test, there are five cases for the $SADF$ test, summarized in Table 2, by replacing regression models (1)-(3) with (21)-(23).

4.1 The Limiting Distribution of Sup ADF

Proposition 4.1 *Under Case 2 and Case 3 (with $\eta > 1/2$), the asymptotic distribution of the sup ADF statistic is*

$$SADF(r_0) \xrightarrow{L} \sup_{r \in [r_0, 1]} \left\{ \frac{\frac{1}{2} r [W(r)^2 - r] - \int_0^r W(s) ds W(r)}{r^{1/2} \left\{ r \int_0^r W(s)^2 ds - \left[\int_0^r W(s) ds \right]^2 \right\}^{1/2}} \right\} := F_{78}(W, r_0); \quad (24)$$

Under Case 1, the sup ADF statistic converges to

$$SADF(r_0) \xrightarrow{L} \sup_{r \in [r_0, 1]} \left\{ \frac{1}{2} [W(r)^2 - r] \left[\int_0^r W(s)^2 ds \right]^{-1/2} \right\} := F_6(W, r_0); \quad (25)$$

Under Case 4, the sup ADF statistic converges to

$$SADF(r_0) \xrightarrow{L} \sup_{r \in [r_0, 1]} \left\{ \left[\int_0^r s dW(s) - \int_0^r W(s) ds \right] \left(\int_0^r s^2 ds \right)^{-1/2} \right\} := F_9(W, r_0); \quad (26)$$

Under Case 5, the sup ADF statistic converges to

$$SADF(r_0) \xrightarrow{L} \sup_{r \in [r_0, 1]} \left\{ \frac{E_r r^3 - 6F_r(2C_r - A_r r) + 2D_r r(3C_r - 2A_r r)}{r^{3/2} (12A_r C_r r - 12C_r^2 + B_r r^3 - 4A_r^2 r^2)^{1/2}} \right\} := F_{10}(W, r_0), \quad (27)$$

with $A_r = \int_0^r W(s) ds$, $B_r = \int_0^r W(s)^2 ds$, $C_r = \int_0^r W(s) s ds$, $D_r = W(r)$, $E_r = \frac{1}{2} [W(r)^2 - r]$ and

$$F_r = rW(r) - \int_0^r W(s) ds.$$

Remark 4.1 The asymptotic SADF distributions are obtained by applying the sup function to the asymptotic ADF_r distributional space (based on the continuous mapping theorem). It implies that the limsup and the suplim operations are equivalent, namely

$$\lim_{T \rightarrow \infty} \sup_{r \in [r_0, 1]} \{ADF_r\} = \sup_{r \in [r_0, 1]} \{ \lim_{T \rightarrow \infty} ADF_r \}, \quad (28)$$

for all cases.

Remark 4.2 If $\eta = 1/2$ in Case 3, then the asymptotic distribution of the SADF statistic is

$$SADF(r_0) \xrightarrow{L} \sup_{r \in [r_0, 1]} \left[r^{-1/2} \left(rD_{r, \sigma_r} - A_{r, \sigma_r} C_{r, \sigma_r} \right) \left(rB_{r, \sigma_r} - A_{r, \sigma_r}^2 \right)^{-1/2} \right] := F_{62}(W, r_0),$$

with $A_{r, \sigma_r} = \frac{1}{2}r + \sigma_r \int_0^r W(s) ds$, $B_{r, \sigma_r} = \frac{1}{3}r^3 + \sigma_r^2 \int_0^r W(s)^2 ds + 2\sigma_r \int_0^r W(s) s ds$, $C_{r, \sigma_r} = W(r)$ and $D_{r, \sigma_r} = \left[rW(r) - \int_0^r W(s) ds \right] + \frac{1}{2} \sigma_r [W(r)^2 - r]$. Similar to the ADF statistic, the limit theory depends on the nuisance parameters σ_r for all $r \in [r_0, 1]$.

Remark 4.3 The asymptotic ADF_r distribution under Case 4 is

$$ADF_r \xrightarrow{L} \left[\int_0^r s dW(s) - \int_0^r W(s) ds \right] \left(\int_0^r s^2 ds \right)^{-1/2}, \quad (29)$$

which is identically distributed as standard normal. Suppose $r_A, r_B \in [r_0, 1]$ and $r_A \neq r_B$, the asymptotic ADF_{r_A} distribution and the asymptotic ADF_{r_B} distribution are correlated due to the fact that both of them are functions of a standard Wiener process.

Remark 4.4 The asymptotic SADF distribution in Case 2, (24), is identical to that in PWY. The asymptotic SADF distributions under the other four cases have not been discussed in the literature. However, as pointed out in Remark 2.5, we believe that only Case 3 and Case 4 are empirically reasonable for economic and financial time series.

In Figures 5 we examine the sensitivity of the asymptotic distributions of $SADF$ with respect to r_0 . In both cases, the asymptotic distributions are obtained by numerical simulation based on 2,000 iterations. The Wiener process is approximated by partial sums of $N(0,1)$ with 5,000 steps. The smallest window size r_0 is set to be $\{0.2, 0.15, 0.10, 0.05\}$.

Figure 5a displays the asymptotic distributions under Case 3 while Figure 5b is for Case 4. Under both cases, the asymptotic distributions of the SADF statistic move sequentially to the right as r_0 decreases.²

² Intuitively, when r_0 is smaller, the feasible range of r (i.e. $[r_0, 1]$) becomes wider and hence the distributional space of $\lim_{r \rightarrow \infty} ADF_r$ expands. The asymptotic SADF distribution, which applies the sup function to the aforementioned distributional space, should move sequentially towards the right as r_0 decreases.

In addition, like the left-tailed unit root test, the asymptotic distribution under Case 4 has larger values for the 90%, 95% and 99% quantiles. For example, the 95% asymptotic critical values for Case 3 with $r_0 = \{0.2, 0.15, 0.10, 0.05\}$ are respectively 1.39, 1.44, 1.54, 1.58 and those for Case 4 are respectively 2.79, 2.86, 2.91, 2.96. Obviously, the critical values are sensitive to r_0 .

4.2 The Finite Sample Distribution of Sup ADF

The finite sample distribution of the SADF statistic depends on the sample size T , the value of the drift in the null hypothesis (d, η in Case 3 and $\tilde{\alpha}$ in Case 4) and the smallest window size r_0 . Figure 6 displays the finite sample distributions of the SADF statistic when $r_0 = 0.1$ and the sample sizes are 400, 600, 800, 1000. The parameters d and η in Case 3 and $\tilde{\alpha}$ in Case 4 are set to unity. As we can see, the finite sample distribution of SADF moves towards the asymptotic distribution $F_{78}(W, 0.1)$ under Case 3 and moves towards $F_9(W, 0.1)$ under Case 4 as the sample size T increases.

The convergence illustrates the validity of interposing the \limsup and \suplim operations in equation (28) under Case 3 and Case 4. The left-hand side variable in this equation can be approximated by the finite sample SADF distribution with a reasonably large sample size (i.e. $T \geq 1000$) while $F_{78}(W, 0.1)$ and $F_9(W, 0.1)$ are the right-hand side variables for Case 3 and Case 4.

Figure 7 describes the finite sample distributions of the SADF statistic when the regression model is R_2^s , $T = 400$, $r_0 = 0.1$ and the drift value $\tilde{\alpha}_T = T^{-\eta}$ (i.e. $d = 1$) with $\eta = \{1, 0.9, \dots, 0.1, 0\}$. The solid line on the left is the $F_{78}(W, 0.1)$ distribution, and that on the right hand side is $F_9(W, 0.1)$. The dotted lines in between are the finite sample distributions. We observe a similar pattern as in Figure 2. For a given T and r_0 , the finite sample distribution moves towards $F_{78}(W, 0.1)$ as η increases and shifts towards $F_9(W, 0.1)$ as η decreases. An obvious separation occurs when $\eta = 0.5$. The discrepancy among the finite sample distributions is negligible with $\eta = \{0.6, 0.7, 0.8, 0.9, 1\}$, but becomes considerably large with $\eta = \{0.4, 0.3, 0.2, 0.1, 0\}$.

Like the finite sample ADF distribution described in Figure 2, the finite sample SADF distribution is invariant to η under Case 3 (when $\eta > 0.5$) while it varies significantly with η when it is less than 0.5 (which is equivalent to Case 4). Combining with the fact that the true value of η is usually unknown in practice, we may not be able to obtain an accurate finite sample distribution under Case 4 and hence an

exact implementation of the test (using the finite sample critical values) under this case may not be feasible.

5. Size and Power Comparison

The 90%, 95% and 99% quantiles of the asymptotic and finite sample distributions of the SADF statistic under Cases 3 and Case 4 are presented in Table 4. The asymptotic critical values are obtained by numerical simulations with 2,000 iterations. The Wiener process is approximated by partial sums of $N(0,1)$ with 5,000 steps. The finite sample critical values are obtained from the 2,000 Monte Carlo simulations. The parameters d and η in Case 3 and $\tilde{\alpha}$ in Case 4 are equal to unity.

Table 5 gives sizes for the SADF test based on nominal asymptotic critical values for Cases 3 and 4 and with sample sizes $T = 100, 200$ and 400 . The nominal size is 5%. The DGP is specified according to the respective null hypotheses ($d = \eta = 1$ in Case 3 and $\tilde{\alpha} = 1$ in Case 4). The number of iterations for size calculations is 2,000. The smallest window size has 40 observations. Table 5 shows that there are significant size distortions under both cases when using the asymptotic critical values.³ The size distortions (when the sample size is 400) can also be observed from the discrepancy between the finite sample distributions and their corresponding asymptotic distributions in Figure 7.

5.1 Periodically Collapsing Explosive Behavior

To calculate the power of the tests, we need to specify the alternative hypothesis. First, we assume the DGP is Evans (1991) periodically collapsing explosive process, with both yearly and monthly parameters settings (see Table 2). For the yearly parameters setting, we calculate powers of the sup ADF test under Cases 3 and 4 with sample sizes 100 and 200. The sample size is set to 100, 200 and 400 for the DGP with the monthly parameter setting. The power calculations are based on the 95% quantiles of the finite sample distributions.

From Table 6 power of the test evidently increases with sample size. Under the yearly parameter setting and $T = 200$, power under Cases 3 and 4 is 21% and 17% higher than when $T = 100$.

Furthermore, Case 3 always outperforms Case 4 in terms of power. From the left panel of Table 6 (yearly parameters setting), the power of the SADF test under Case 3 is 12% and 16% higher than Case 4 when

³ Suppose one keeps the smallest fractional window size r_0 unchanged for all sample sizes. The size of the SADF test will decrease as the sample size increases. For example, if $r_0 = 0.4$, the size of SADF test under Case 3 is 0.077 when sample size is 200 and it is 0.062 when sample size is 400. However, when T is large, there is some advantage to using a small value for r_0 so that the sup ADF test does not miss any opportunity to capture an explosive phase.

$T = 100$ and 200 . With the monthly parameters setting (right panel), when $T = 100, 200$ and 400 , the power of the SADF test under Case 3 is 21%, 20% and 14% higher than Case 4.

5.2 Locally Explosive Behavior

Second, we let the DGP be the locally explosive model defined by equation (20). The parameter settings are the same as in Section 3.2. As we mentioned, this DGP is more realistic than both PY and Evans in the sense that the explosive behavior does not collapse completely within one period. Instead, the bubble collapsing process is assumed to be a (mildly) stationary process. The parameter β controls the contraction rate of the bubble, the duration of which is $T_c - T_f$. To explore the sensitivity of the SADF test to these two coefficients, we calculate powers of the test by setting β equal to 0.4, 0.5 and 0.6 (see Table 7) and $T_c - T_f$ equal to $[0.05T]$, $[0.10T]$ and $[0.15T]$ (Table 8). In general, we find that the power of the SADF test is invariant to the contraction rate and the contraction duration of the bubble.

The explosive rate of the bubble is determined by parameter α and the duration of the bubble expansion $T_f - T_e$. In simulations, we allow α to be 0.6, 0.55 and 0.5 (Table 7) and $T_f - T_e$ to be $[0.10T]$, $[0.15T]$ and $[0.20T]$ (see Table 8). From Table 7, we can see that, *ceteris paribus*, the power of the SADF test increases as α decreases. That is, the frequency of successfully detecting the existence of exuberant behavior is higher when the expansion rate is faster. For example, under Case 3, when $T = 200$, $\beta = 0.5$ and α takes the values 0.6, 0.55 and 0.5, the power is 62%, 67% and 72% respectively. Moreover, we can see from Table 8 that the power of the SADF test is higher when the duration of the bubble expansion is longer. For instance, when $T = 200$ and $T_c - T_f = [0.10T]$, the power under Case 3 with $T_f - T_e = [0.10T]$, $[0.15T]$, $[0.20T]$ is 61%, 79% and 88% respectively.

The location of the bubble episode is indicated by T_e . Table 9 illustrates the power of the SADF test with $T_e = [0.2T]$, $[0.4T]$, $[0.6T]$. We observe that given an identical expansion rate and expansion duration of the bubble, if the bubble episode occurs at an earlier stage of the sample period, the frequency of successfully detecting a bubble episode is higher. For instance, when $T = 200$, $\alpha = 0.6$ and $T_f - T_e = [0.15T]$, the power under Case 3 is 89%, 82% and 79% for $T_e = [0.2T]$, $[0.4T]$, $[0.6T]$ respectively.

Table 10 illustrates the power of the SADF with different sample sizes under Cases 3 and Case 4. First, it is clear that the power of the test is higher when the sample size is larger. The power under Case 3 are

57%, 62% and 73% for $T = 100, 200, 400$. Second, Case 3 is always superior to Case 4 in terms of power. For example, when the sample size T equals 200, the power under Case 3 is 20% greater than that under Case 4. Most importantly, the last observation apply to Table 7, Table 8, Table 9 and Table 10.

6. Application to the NASDAQ

We apply the sup ADF test with different hypotheses and model specifications to the NASDAQ stock market over the period from February 1973 to July 2009 (constituting 438 observations). The NASDAQ composite index and the NASDAQ dividend yield are obtained from DataStream International. The consumer price index, which is used to convert stock prices and dividends into real series, is downloaded from the Federal Reserve Bank of St. Louis.

Figure 8 illustrates the dynamics of the real NASDAQ index and the real NASDAQ dividend (normalized to 100 at the beginning of the data series) during the sample period. The real NASDAQ index grows steadily, manifesting an upward drift, until the early 90s. This is followed by a rapid increase to a peak that is 944.4 times bigger than the starting point of the series. The NASDAQ index, then dropped quickly to a level of less than 248 times of the starting point at April 2003. It recovers gradually until October 2008, however, followed by another sudden crash. Relative to the NASDAQ index, the dividend process changes are of a much smaller magnitude (although it is volatile throughout the sample period).

Table 11 displays the SADF statistics for the logarithmic real NASDAQ index and the logarithmic real NASDAQ dividend, along with respective finite sample critical values, under Case 3 and Case 4. The critical values are obtained from 2,000 Monte Carlo simulations with sample size 348. The parameters d and η in Case 3 and $\tilde{\alpha}$ in Case 4 are set to 1. The smallest window is set to have 40 observations. For the logarithmic real NASDAQ index, we reject the unit root null hypothesis against the explosive alternative at the 10% significance level under Case 3 whereas we fail to reject the null hypothesis at the 10% significance level under Case 4. Furthermore, we cannot reject the null hypothesis of unit root at the 10% significance level for the logarithmic real NASDAQ dividend under both cases.

In other words, with the specification of Case 3, we find evidence of exuberance in the NASDAQ stock market using the sup ADF test. However, if the null hypothesis and the regression model are specified as in Case 4, the sup ADF suggests no evidence of bubble existence in the NASDAQ stock market during the sample period. These results reveal that the empirical evidence of exuberance in the NASDAQ is sensitive to model specification.

7. Conclusion

This paper has investigated various formulations of the null and alternative hypotheses and the effect of the chosen regression model on the detection of exuberance in economic and financial time series. In particular, we identify two empirically reasonable setups and neither setup includes a linear deterministic trend in the regression. In both cases, we estimate the autoregressive coefficient from the following model:

$$\Delta y_t = \alpha + \beta y_{t-1} + \sum_{i=1}^k \phi_i \Delta y_{t-i} + \varepsilon_t.$$

In one case the null hypothesis has an asymptotically negligible intercept while in the other case the intercept is a constant. The limiting distributions of the ADF statistic and the SADF statistic are derived in both cases. The asymptotic critical values are obtained via simulations.

The size and power properties have been examined and compared. When asymptotic critical values are used, the SADF test shows significant size distortions under both cases. Therefore, when the sample size is small (i.e. $T \leq 400$), we suggest using finite sample critical values, instead of the asymptotic critical values, for the SADF test.

For the power calculation, we consider two DGPs: Evans (1991) periodically collapsing explosive process (with both yearly and monthly parameter settings) and the locally explosive process proposed in this paper (with monthly parameters setting). The conclusion drawn from these two DGPs is consistent. Our findings indicate that the preferred procedure for practical implementation is to estimate the regression model of equation (2) and specify the null hypothesis to be an asymptotically negligible intercept in the right-tailed unit root test. The empirical application of these methods to the NASDAQ stock market demonstrates the importance of hypothesis and model specification in the right-tailed unit root test, revealing some sensitivity in the outcomes of the test to these modeling decisions.

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Table 1. Different Model Formulations for Right-Tailed Unit Root Tests

	Case 1	Case 2	Case 3	Case 4	Case 5
Null Model	H_{01}	H_{01}	H_{02}	H_{03}	$H_{01}/H_{02}/H_{03}$
Regression Model	R_1	R_2	R_2	R_2	R_3

Table 2. Parameter Settings

	\tilde{u}	σ_f	P_0^f	ρ	b	B_0	π	ζ	τ	κ
<i>Yearly</i>	0.740	7.869	41.195	0.952	1	0.50	0.85	0.50	0.05	20
<i>Monthly</i>	0.131	3.829	94.122	0.985	1	0.50	0.85	0.50	0.05	150

Table 3. Different Cases for the Sequential Right-Tailed Unit Root Test

	Case 1	Case 2	Case 3	Case 4	Case 5
Null Hypothesis	H_{01}	H_{01}	H_{02}	H_{03}	$H_{01}/H_{02}/H_{03}$
Regression Model	R_1^s	R_2^s	R_2^s	R_2^s	R_3^s

Table 4. Critical Values of the SADF Statistic (Against Explosive Alternative)

	Case 3			Case 4		
	90%	95%	99%	90%	95%	99%
The asymptotic critical values of the SADF statistic						
$r_0 = 0.4$	0.88	1.20	1.87	2.27	2.62	3.20
$r_0 = 0.2$	1.10	1.39	1.95	2.48	2.79	3.39
$r_0 = 0.1$	1.23	1.54	2.04	2.58	2.92	3.42
The finite sample critical values of the SADF statistic						
$T = 100$ and $r_0 = 0.4$	1.32	1.77	2.87	2.80	3.25	4.05
$T = 200$ and $r_0 = 0.2$	1.49	1.93	2.83	2.95	3.36	4.17
$T = 400$ and $r_0 = 0.1$	1.63	2.01	2.85	3.04	3.44	4.25

Note: the asymptotic critical values are obtained by numerical simulations with 2,000 iterations. The Wiener process is approximated by partial sums of $N(0,1)$ with 5,000 steps. The finite sample critical values are obtained from the 2,000 Monte Carlo simulations. The parameters d and η in Case 3 and $\tilde{\alpha}$ in Case 4 are set to unity.

Table 5. Sizes of the SADF Test (Using Asymptotic Critical Values). The Data Generating Process is Specified According to the Respective Null Hypothesis. Parameters d, η in Case 3 and $\tilde{\alpha}$ in Case 4 are Set to Unity. The Nominal Size is 5%.

	Case 3	Case 4
$T = 100$ and $r_0 = 0.4$	0.117	0.127
$T = 200$ and $r_0 = 0.2$	0.126	0.118
$T = 400$ and $r_0 = 0.1$	0.118	0.122

Note: the number of iterations for size calculation equals 2,000.

Table 6. Powers of the SADF Test Under Evans (1991) Periodically Collapsing Explosive Behavior

	Yearly		Monthly	
	Case 3	Case 4	Case 3	Case 4
$T = 100$ and $r_0 = 0.4$	0.39	0.27	0.54	0.33
$T = 200$ and $r_0 = 0.2$	0.60	0.44	0.73	0.53
$T = 400$ and $r_0 = 0.1$	-	-	0.86	0.72

Note: the number of iterations for power calculation equals 2,000.

Table 7. Powers of the SADF Test for the Locally Explosive Behavior (The Rates of Bubble Expansion and Contraction). Parameters are Set as:

$$y_0 = 94.122, u_0 = 0.131, c_1 = c_2 = 1, \sigma_1 = \sigma_2 = 3.829, \sigma_3 = 2\sigma_1, T_e = [0.6T], T_f = [0.7T], \\ T_c = [0.75T], T = 200, r_0 = 0.2$$

β	0.4		0.50		0.60	
	Case 3	Case 4	Case 3	Case 4	Case 3	Case 4
$\alpha = 0.60, \phi_T = 1.04$	0.63	0.42	0.62	0.42	0.62	0.42
$\alpha = 0.55, \phi_T = 1.05$	0.67	0.47	0.67	0.47	0.67	0.46
$\alpha = 0.50, \phi_T = 1.07$	0.72	0.49	0.72	0.48	0.71	0.47

Note: the number of iterations for power calculation equals 2,000.

Table 8. Powers of the SADF Test for the Locally Explosive Behavior (The Duration of Bubble Expansion and Contraction). Parameters are Set as:

$$y_0 = 94.122, u_0 = 0.131, c_1 = c_2 = 1, \sigma_1 = \sigma_2 = 3.829, \sigma_3 = 2\sigma_1, \alpha = 0.6, \beta = 0.5, T = 200, r_0 = 0.2, T_e = [0.6T]$$

$T_c - T_f$	[0.05T]		[0.10T]		[0.15T]	
	Case 3	Case 4	Case 3	Case 4	Case 3	Case 4
$T_f - T_e = [0.10T]$	0.63	0.43	0.61	0.42	0.62	0.42
$T_f - T_e = [0.15T]$	0.79	0.61	0.79	0.63	0.79	0.62
$T_f - T_e = [0.20T]$	0.87	0.71	0.88	0.72	0.88	0.72

Note: the number of iterations for power calculation equals 2,000.

Table 9. Powers of the SADF Test for the Locally Explosive Behavior (The Location of the Bubble Episode). Parameters are Set as:

$$y_0 = 41.195, u_0 = 0.740, c_1 = c_2 = 1, \sigma_1 = \sigma_2 = 7.869, \sigma_3 = 2\sigma_1, \alpha = 0.6, \beta = 0.5, T = 200, r_0 = 0.2, T_c - T_f = [0.05T]$$

T_e	[0.2T]		[0.4T]		[0.6T]	
	Case 3	Case 4	Case 3	Case 4	Case 3	Case 4
$T_f - T_e = [0.10T]$	0.78	0.61	0.68	0.51	0.63	0.42
$T_f - T_e = [0.15T]$	0.89	0.79	0.82	0.67	0.79	0.62
$T_f - T_e = [0.20T]$	0.94	0.86	0.90	0.78	0.88	0.72

Note: the number of iterations for power calculation equals 2,000.

Table 10. Powers of the SADF Test for the Locally Explosive Behavior (The Sample Size). Parameters are Set as:

$$y_0 = 41.195, u_0 = 0.740, c_1 = c_2 = 1, \sigma_1 = \sigma_2 = 7.869, \sigma_3 = 2\sigma_1, \alpha = 0.6, \beta = 0.5, T_e = [0.6T], \\ T_f - T_e = [0.10T], T_c - T_f = [0.05T]$$

	Case 3	Case 4
$T = 100$ and $r_0 = 0.4$	0.57	0.36
$T = 200$ and $r_0 = 0.2$	0.62	0.42
$T = 400$ and $r_0 = 0.1$	0.73	0.52

Note: the number of iterations for power calculation equals 2,000.

Table 11. The Sup ADF Test of the NASDAQ Stock Market

	Case 3	Case 4
Log Real NASDAQ Index	1.90	1.90
Log Real NASDAQ Dividend	-1.07	-1.07
Finite sample critical values		
90%	1.60	3.02
95%	1.97	3.41
99%	2.89	4.18

Note: Critical values of the sup ADF test are obtained from 2,000 Monte Carlo simulations with sample size 438. The parameters d and η in Case 2 and $\tilde{\alpha}$ in Case 3 are set to 1. The smallest window is set to have 40 observations.

Figure 1. The Finite Sample Distribution of the ADF Statistic under Case 3 and Case 4 with $T = 40, 80, 200, 400$.

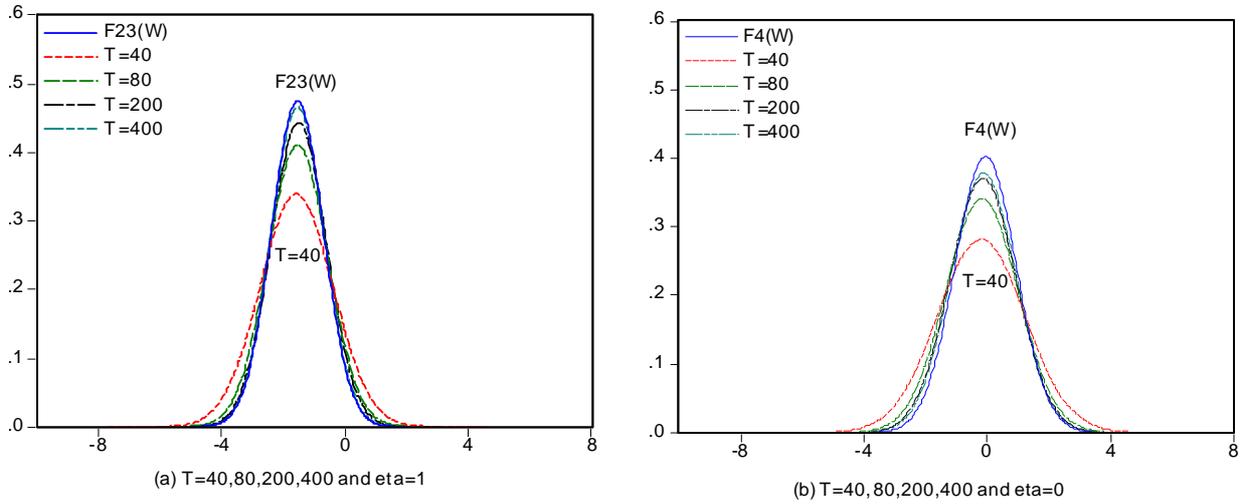


Figure 2. The Finite Sample Distribution of the ADF Statistic when the Regression Model is $R_2, T = 400$ and $\tilde{\alpha} = T^{-\eta}$ (i.e. $d = 1$) with $\eta = 1, 0.9, \dots, 0.1, 0$

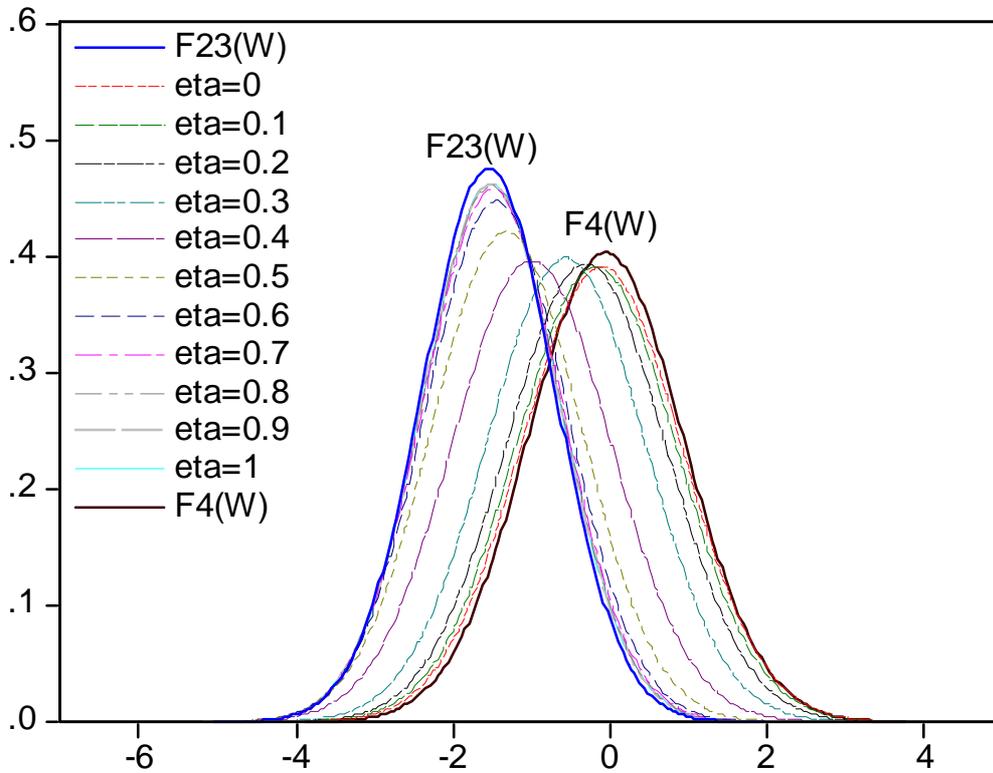


Figure 3. The Simulated Time Series based on Evans' DGP

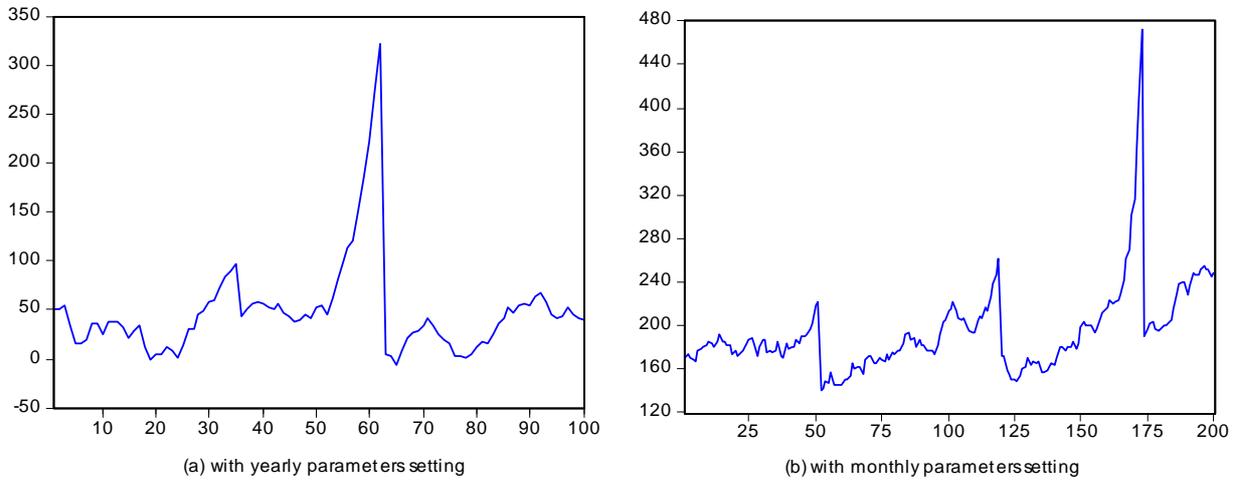


Figure 4. The Simulated Time Series based on Equation (20)

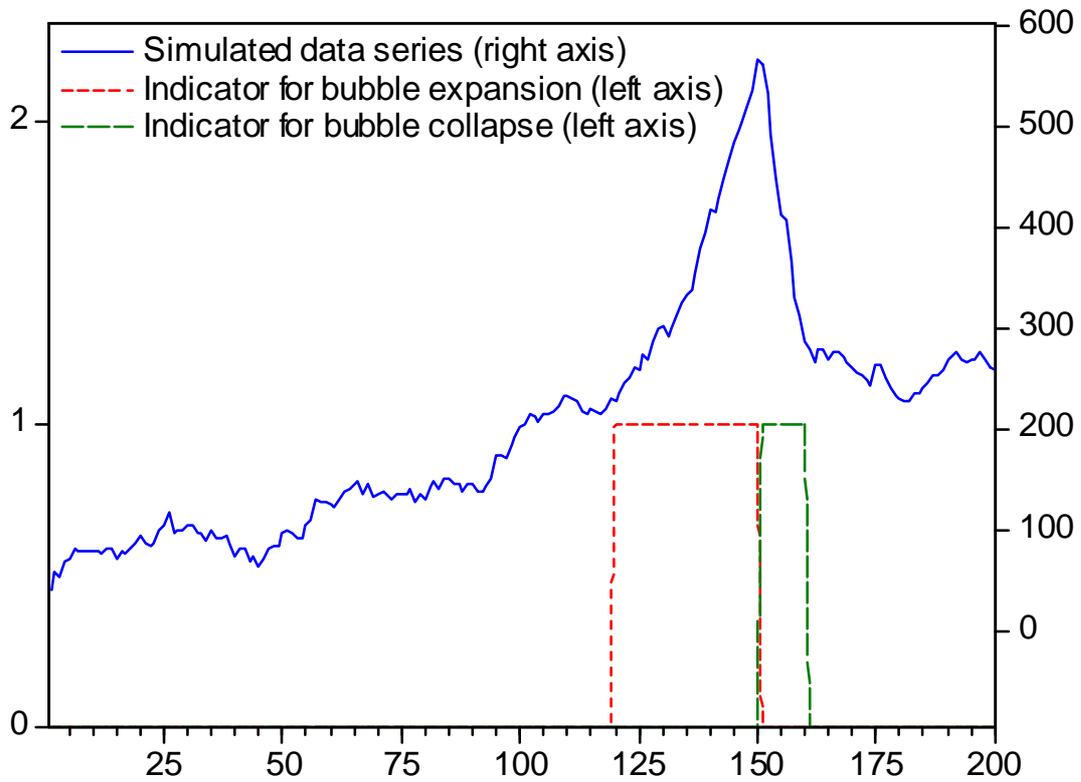


Figure 5. The Asymptotic Distributions of the SADF Statistic with $r_0 = 0.20, 0.15, 0.10, 0.05$

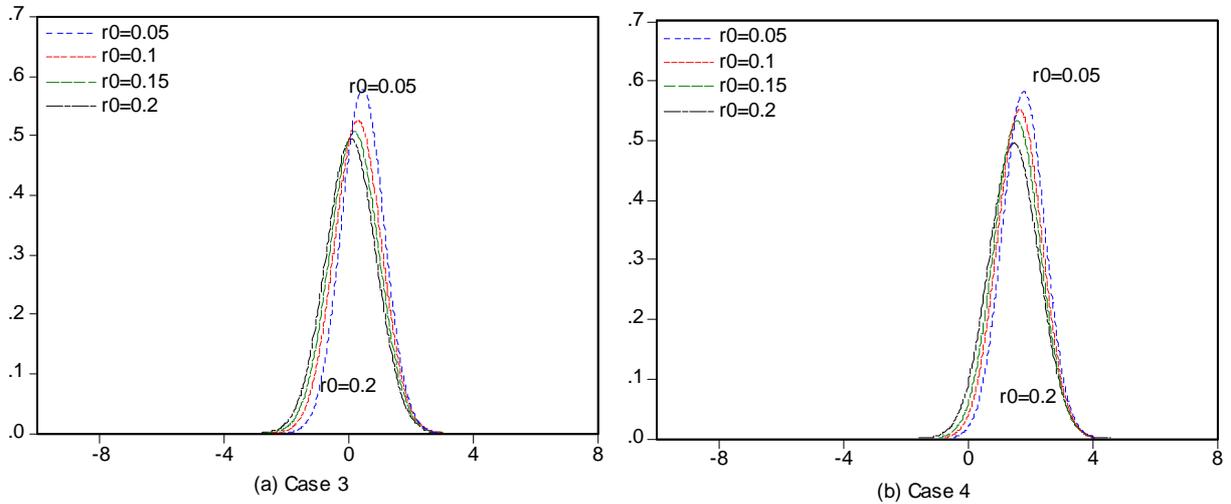


Figure 6. The Finite Sample Distributions of the SADF Statistic when $r_0 = 0.1$ and the Sample Sizes are 400, 600, 800, 1000. The Parameters d and η in Case 3 and $\tilde{\alpha}$ in Case 4 are Set to 1

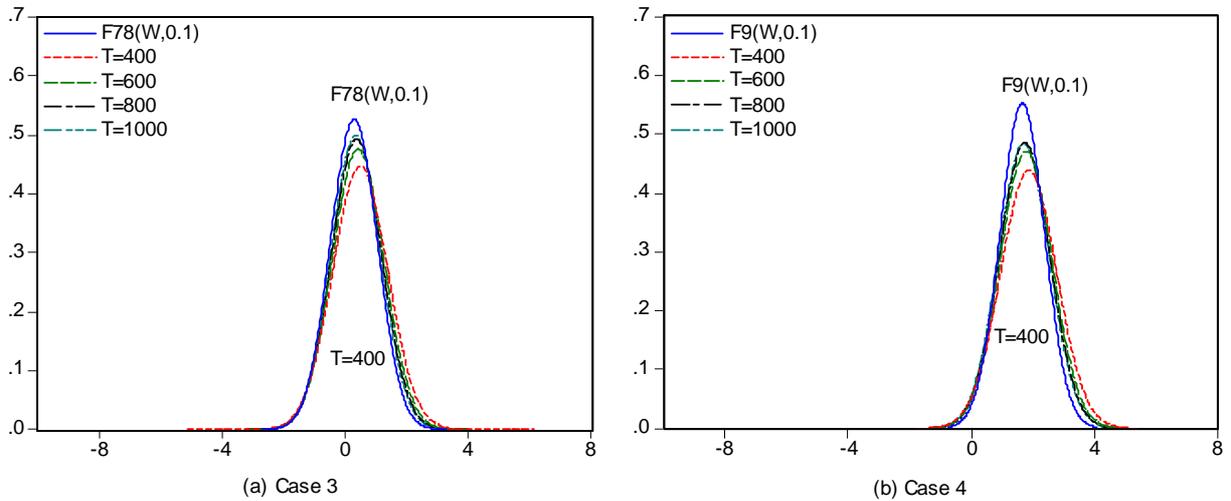


Figure 7. The Finite Sample Distributions of the SADF Statistic when the Regression Model is R_2^S , $T = 400, r_0 = 0.1$ and $\tilde{\alpha} = T^{-\eta}$ (i.e. $d = 1$) with $\eta = \{1, 0.9, \dots, 0.1, 0\}$

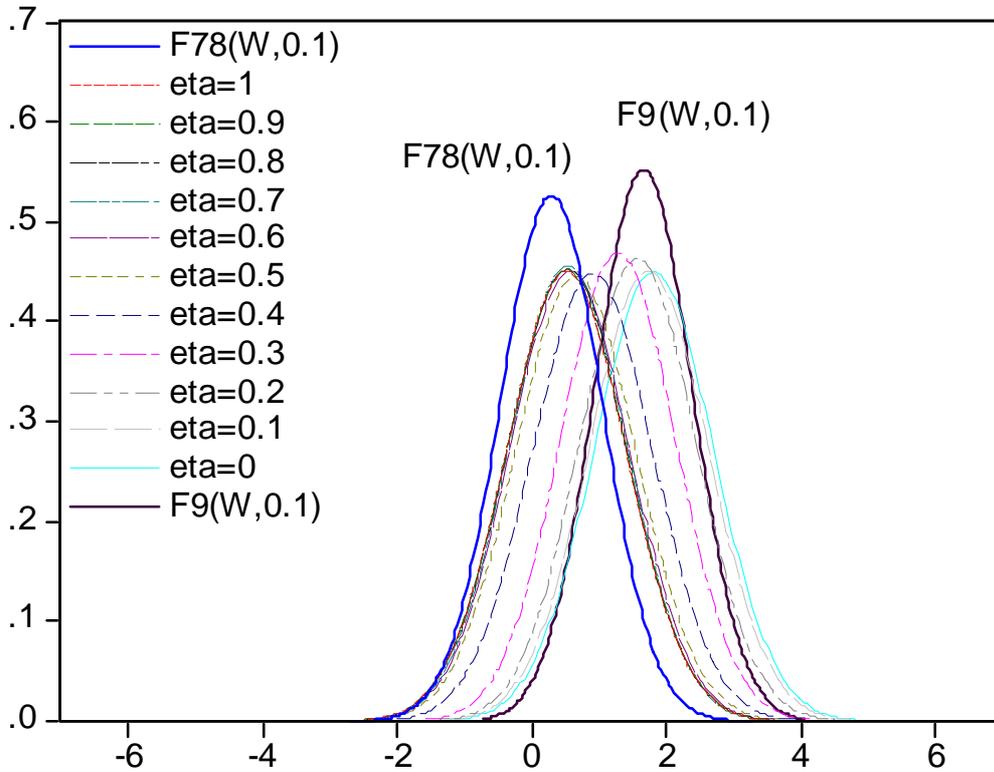


Figure 8. NASDAQ Stock Market Sampled from February 1973 to September 2009 (Normalized to 100 at the Beginning of Data Series)

