HONG KONG INSTITUTE FOR MONETARY RESEARCH

THE FISCAL THEORY OF THE PRICE LEVEL IN A WORLD OF LOW INTEREST RATES

Marco Bassetto and Wei Cui

HKIMR Working Paper No.02/2018

January 2018

Hong Kong Institute for Monetary Research
香港金融研究中心
(a company incorporated with limited liability)

The Fiscal Theory of the Price Level in a World of Low Interest Rates

Marco Bassetto [*]

Federal Reserve Bank of Chicago, University College London, and IFS

Wei Cui

University College London and Centre for Macroeconomics

January 2018

Abstract

A central equation for the fiscal theory of the price level (FTPL) is the government budget constraint (or "government valuation equation"), which equates the real value of government debt to the present value of fiscal surpluses. In the past decade, the governments of most developed economies have paid very low interest rates, and there are many other periods in the past in which this has been the case. In this paper, we revisit the implications of the FTPL in a world where the rate of return on government debt may be below the growth rate of the economy, considering different sources for the low returns: dynamic inefficiency, the liquidity premium of government debt, or its favorable risk profile.

Keywords: the fiscal theory of price level, risks, dynamic inefficiency, and liquidity

JEL classification: E31, E62

[*] Email addresses: Bassetto: bassetto@nber.org and Cui: w.cui@ucl.ac.uk

We are indebted to Gadi Barlevy, John Cochrane, Mariacristina De Nardi, Pedro Gomis-Porqueras, Franc¸ois Gourio, John M. Roberts, Martin Uribe, Christopher J. Waller, Stephen D. Williamson, and Randall Wright for useful comments and to Matthew Easton for valuable research assistance. Financial support from the ADEMU (H2020, No. 649396) project and from the ESRC Grant #ES/L500343/1 through the Centre for Macroeconomics is gratefully acknowledged. Wei Cui is also grateful for the financial support from the Hong Kong Institute for Monetary Research.

1. Introduction

Models of monetary economies are plagued by the presence of multiple equilibria, which weakens the ability to make tight predictions.[1] To select among them, it has become common to appeal to what Leeper [25] defined as "active monetary policies." However, these rules imply *local* determinacy, but not global uniqueness, and are therefore not universally accepted as an equilibrium selection criterion.[2]

An alternative approach to price-level determinacy follows what Leeper [25] defined "active fiscal policies," in which the requirement that government debt follows a stable trajectory is used to select an equilibrium. In particular, since Sims [36] and Woodford [39], this approach is known as the "fiscal theory of the price level" (FTPL from now on). According to the FTPL, price-level determinacy follows when the present value of government (primary) surpluses does not react to the price level in a way that ensures government budget balance; rather, government debt is a promise to deliver "dollars" (either purely a unit of account, or the underlying currency which by assumption can be freely printed by the government), and the value of a dollar adjusts in equilibrium so that the present value of surpluses and the real value of debt match.

Whether the FTPL is a valid selection criterion has been controversial.[3] Bassetto [2] analyzes the issue in a game-theoretic environment, where the price level arises explicitly from the actions of the agents in the economy and their interaction; in this more complete description of the economic environment, the FTPL remains valid when surpluses are always positive, but it requires adoption of more sophisticated strategies when the desired equilibrium path involves periods of net borrowing. This distinction is particularly important in the context of our analysis, because we find that, across a variety of models, low interest rates are compatible with a stable and positive path for debt only when

[1] This issue is addressed in any graduate textbook in macroeconomics; see e.g. Sargent [33], Woodford [40], or Ljungqvist and Sargent [26]. Woodford [39] contains an exhaustive description of the nature of equilibrium multiplicity for a cash-in-advance economy under money-supply and interest-rate rules.

[2] See Cochrane [13] for a particularly stinging critique of active rules as a device to achieve uniqueness.

[3] Examples of criticisms appear in Buiter [7], Kocherlakota and Phelan [23], and Niepelt [31].

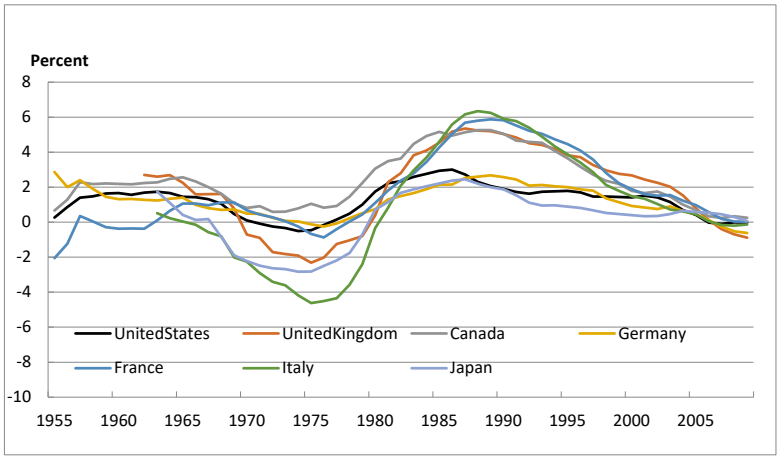

Figure 1: Real Interest Rates in the G7 countries (11-year centered moving averages of annual real interest rates). Reproduced from Yi and Zhang [41].

the government runs primary deficits, at least on average, which is precisely the environment in which the theory is on weakest ground.

In this paper, we sidestep the controversy and assume that the government can indeed commit to a sequence of real taxes, independent of the realization of the price level, but we reassess whether the uniqueness result of the FTPL continues to hold in economies in which the interest rate on government debt is persistently below the growth rate of the economy. This question is motivated by the long decline in real interest rates on government debt from the high values of the 1980s and early 1990s. As an example, Figure 1 (reproduced from Yi and Zhang [41]) plots the experience of the G7 countries and shows that real interest rates on government debt below the growth rate of the economy might well be the norm rather than the exception for those countries. When interest rates fall short of the growth rate of the economy, the present value of primary surpluses may not be well defined, posing a challenge for the FTPL.

The possibility that low interest rates may affect the interaction between fiscal and monetary policy was discussed by Darby [16] in the context of Sargent and Wallace's [35] unpleasant monetarist arithmetic. However, as emphasized by Miller and Sargent [30], the way in which real interest rates

respond to government debt is important to reach valid conclusions: it is thus important to analyze economies where interest rates are endogenous. We study three main reasons why interest rates may be low, and we show that the validity of the FTPL is sensitive to the specific reason.

Our first experiment is the most favorable to the FTPL. It studies a stochastic economy and analyzes what happens when real rates of return are low because of high risk premia, similarly to Bohn [6]. In this case, an equilibrium still requires the present value of government surpluses to be (positive and) well defined and the FTPL remains valid, although without the risk adjustment expected surpluses may well be negative.

Second, we entertain the possibility that the economy might be dynamically inefficient. In this context, we show that the FTPL is no longer able to select a unique equilibrium, and multiple price levels are consistent with an equilibrium even when taxes are set in real terms and do not adjust. Even then, the FTPL can select a *range* of equilibrium prices even when monetary policy is run as in Sargent and Wallace [34] and no prediction would be possible otherwise.

The hypothesis of dynamic inefficiency has been revived by Geerolf [20], who rebuts the negative evidence in Abel et al. [1]. Nonetheless, other authors have emphasized that the rate of return on capital has not declined in line with that of government debt.[4]

Our final explanation for the prevalence of low rates of return on government debt is that it provides liquidity services, so that it has itself some of the characteristics usually associated with money. This new environment is described by equations that are very similar to the previous one, since debt played the role of money there too, but with the important difference that negative holdings are now ruled out. Restricting our attention to deterministic paths, we can prove that the FTPL holds if primary surpluses are positive (at least asymptotically), but in this case we can also show that the interest rate will necessarily exceed the growth rate of the economy (normalized to zero in our case). When instead primary surpluses are zero or negative, many equilibria where debt has positive value exist, and our results mimic what happens in a dynamically inefficient economy. This economy is similar to that

[4]See e.g. Yi and Zhang [41], Marx, Mojon, and Velde [27], and Del Negro et al. [18].

analyzed by Berentsen and Waller [5] and by Domínguez and Gomis-Porqueras [19]. Berentsen and Waller find results similar to ours, but choose to focus on cases in which the fiscal theory holds, while Domínguez and Gomis-Porqueras [19] revisit Leeper's [25] analysis of active vs. passive monetary and fiscal rules and find that the link between Leeper's original classification and determinacy and uniqueness of an equilibrium breaks down when government debt plays a liquidity role.[5]

Concerning the conduct of fiscal policy, these results suggest three main conclusions:

- The ability of the FTPL to select a unique equilibrium when interest rates are low is not robust across specifications; appealing to fiscal policy to achieve a price level target is thus fraught with even greater difficulties than those that arise in economies where returns are such that computing present values is always possible.

- It is difficult to blame an excessively conservative fiscal policy for the recent experience of low inflation, because, over extended periods of time, low interest rates and stable (or increasing) government debt levels coexist only when fiscal policy entails primary deficits on average. Moreover, the presence of multiple equilibria makes it problematic to use comparative statics to study the effects of fiscal expansions.

- While unsuccessful at uniquely pinning down the price level, the FTPL still provides a lower bound on prices across all the environments that we consider here, which implies that it is not completely devoid of content.

In Section 2, we start by purely analyzing the government budget constraint and show how sustained primary deficits are needed to keep debt positive when interest rates fall short of the growth rate of the economy. The following three sections derive further implications by considering specific

[5] In a model where the liquidity premium is tied to financing constraints, Cui [14] analyzes the local dynamics around a steady state with positive primary surplus and finds that Leeper's [25]'s classification still holds; this matches with our observation that the FTPL holds when the government always delivers surpluses. Canzoneri and Diba [10] analyze a different transaction technology, which imposes a lower bound on the equilibrium real value of government-issued liabilities, no matter how high their opportunity cost is. Such a setup is more favorable to the FTPL.

reasons for low interest rates: risk premia, dynamic inefficiency, or liquidity. Section 6 summarizes the lessons we draw and suggests future directions for analysis.

2. Preliminaries

Before we embark on analyzing the validity of the FTPL in specific models that feature low interest rates, it is useful to probe the implications of low interest rates just from analyzing the government budget constraint, an equation that holds across all of the models we will consider. For simplicity, we will concentrate our analysis on one-period debt. Let B_t be promised nominal debt repayments by the government due at the beginning of period t, R_t be the nominal interest rate, P_t be the price level, and τ_t be real taxes. The government budget constraint is

$$\frac{B_{t+1}}{1+R_t} = B_t - P_t\tau_t, \tag{1}$$

where we abstract from government spending, since its presence does not affect any of our subsequent results.[6]

Since the government tax base is related to the size of the economy, it is convenient to rescale debt and taxes by real output y_t. Defining $x_t := \tau_t/y_t$, (1) can be rewritten as

$$\frac{B_{t+1}}{P_{t+1}y_{t+1}} = \frac{(1+R_t)P_ty_t}{P_{t+1}y_{t+1}}\left(\frac{B_t}{P_ty_t} - x_t\right). \tag{2}$$

In this paper, we are interested in environments in which the real return on government debt is below the growth rate of the economy. Consider first a deterministic economy; then, the condition becomes simply

$$\frac{(1+R_t)P_ty_t}{P_{t+1}y_{t+1}} < \alpha < 1. \tag{3}$$

Let the (gross) growth rate be $1+g_{t+1} = y_{t+1}/y_t$ and let the (gross) real interest rate be $1+r_{t+1} = (1+R_t)P_t/P_{t+1}$. Iterating (2) from period 0 forward and assuming that the economy starts with positive

[6] When government spending is present, our results about the signs of taxes should be reinterpreted as applying to the sign of primary surpluses instead.

initial debt, this yields

$$\begin{aligned}\frac{B_t}{P_t y_t} &= \frac{B_0}{P_0 y_0}\prod_{s=1}^{t}\left(\frac{1+r_s}{1+g_s}\right) - \sum_{s=0}^{t-1} x_s \prod_{v=s+1}^{t}\left(\frac{1+r_v}{1+g_v}\right) \\ &< \alpha^t \frac{B_0}{P_0 y_0} - \sum_{s=0}^{t-1} x_s \prod_{v=s+1}^{t}\left(\frac{1+r_v}{1+g_v}\right).\end{aligned} \tag{4}$$

If taxes converge asymptotically to some value $\bar{x}$ and government debt is to remain positive (and bounded away from zero), equation (4) implies that $\bar{x} < 0$. This is intuitive: when the interest rate is below the growth rate of the economy, the debt/GDP ratio shrinks to zero on its own, and continuing primary deficits are required to prevent debt from vanishing in the limit (or even becoming negative). If taxes do not converge to a steady state, equation (4) still implies that a distributed sum must remain negative in order for government debt not to disappear or become negative.

In stochastic environments, the real rate of return on government debt may or may not exceed the growth rate of the economy, and we instead study what happens when the *expected* return is low. More precisely, the corresponding version of condition (3) that we study is

$$E_t\left[\frac{(1+R_t)P_t y_t}{P_{t+1} y_{t+1}}\right] < \alpha < 1. \tag{5}$$

We then obtain

$$\begin{aligned}E_0\frac{B_t}{P_t y_t} &= E_0\left\{\frac{B_0}{P_0 y_0}\prod_{s=1}^{t}\left(\frac{1+r_s}{1+g_s}\right) - \sum_{s=0}^{t-1} x_s \prod_{v=s+1}^{t}\left(\frac{1+r_v}{1+g_v}\right)\right\} \\ &< \alpha^t \frac{B_0}{P_0 y_0} - E_0\left[\sum_{s=0}^{t-1} x_s \prod_{v=s+1}^{t}\left(\frac{1+r_v}{1+g_v}\right)\right].\end{aligned} \tag{6}$$

In order for the expected debt/GDP ratio to remain bounded away from zero in the limit, we need

$$\lim_{t\to\infty} E_0\left[\sum_{s=0}^{t-1} x_s \prod_{v=s+1}^{t}\left(\frac{1+r_v}{1+g_v}\right)\right] < 0. \tag{7}$$

Equation (6) is slightly more involved than (4), since the expectation about the future sum of taxes also involves covariance terms with realized rates of return; even in this case, under a suitable change in measure that takes into account the role of risk premia, the sum of future taxes needs to be negative to sustain a positive debt/GDP ratio.

Without taking a specific stance on the nature of the economy at hand, pure accounting implies that, ineconomies with low interest rates, the government will necessarily run recurrent primary deficits.

As discussed in Bassetto [2], this by itself has significant implications for the FTPL, since the fiscal strategies that support a unique equilibrium are more involved when the equilibrium features primary deficits.

Following Cochrane's [12] analogy, when the government always runs primary surpluses, government debt looks like a corporate stock paying a stream of positive dividends, and the price level can be viewed as the inverse of the value of this stock. Just as any mispricing of the stock would not require an adjustment in the dividends paid by the corporation, any deviation in the general level of prices would not require an adjustment in the surpluses that the government raises to reabsorb the money created when nominal debt is repaid. However, when primary deficits are part of the picture, the proper analogy is with a corporation that may have trouble raising fresh funds from investors (such as AIG in 2008): in this case, mispricing of the stock could force the corporation to alter its investment/divestment plans and thus its future dividends and likewise would be true in the case of the government and its debt.[7]

In Sections 4 and 5, we highlight even greater challenges that low interest rates may pose for the FTPL, but first we consider a more benign case.

3. An Economy with High Risk Premia

We study an economy in which the presence of risk implies that households are happy saving in the form of government debt for precautionary reasons, even though its expected return falls short of the expected growth of the economy.

We consider a pure-exchange economy with a continuum of identical infinitely lived agents and a government, similar to that analyzed by Cochrane [12].[8]

[7] The question whether the AIG crisis in 2008 was the result of a coordination failure or whether it reflected the true weak fundamentals of the company is beyond the scope of our work. The analogy entertains the hypothesis that it was at least in part the former.

[8] As discussed by Cochrane, the presence of cash is not essential for the results, and we continue to abstract from it.

Private agents can save by buying one-period nominal government debt B_t; they are subject to a lower bound on debt holdings $\frac{B_t}{P_t y_t} \geq -\underline{B}$, which we assume never to be binding (other than in the limit, through the transversality condition). The government sets a fixed and constant nominal interest rate R at which debt is issued. In this and the following sections, we abstract from any role for money, so that R measures government promises in an abstract unit of account. However, introducing a role for money does not change any of our results, as long as money is elastically supplied at the interest rate R; we illustrate this in Appendix B for the economy of Section 5.

The representative consumer discounts utility at $\beta \in (0,1)$, pays (real) lump-sum taxes τ_t, and chooses a sequence $\{c_t, B_t\}_{t=0}^{\infty}$ to solve

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$

subject to

$$P_t c_t + \frac{B_{t+1}}{(1+R)} + P_t \tau_t = P_t y_t + B_t,$$

taking as given $\{y_t, \tau_t, R_t, p_t\}_{t=0}^{\infty}$ and the initial bond holdings B_0.

Letting $\pi_{t+1} = P_{t+1}/P_t$ be gross inflation from t to $t+1$ and

$$z_t := \beta^t c_t^{-\gamma}$$

be the real stochastic discount factor, the first-order condition for the consumer reduces to

$$\mathbb{E}_t \left[\frac{1+R}{\pi_{t+1}} \cdot \frac{z_{t+1}}{z_t} \right] = 1, \tag{8}$$

along with the transversality condition,[9]

$$\lim_{s \to \infty} \mathbb{E}_t \left[\frac{B_{t+s} z_{t+s}}{P_{t+s}(1+R)^s} \right] = 0.$$

The government uses direct lump-sum taxes and new debt to repay its existing obligations every period, subject to the budget constraint (1). In equilibrium, the transversality condition is necessary to ensure that consumers are willing to hold debt, which leads to an intertemporal budget constraint (the core equation of FTPL)

$$\frac{B_t}{P_t} = \mathbb{E}_t \sum_{s=0}^{\infty} \left[\frac{z_{t+s}}{z_t} \tau_{t+s} \right]. \tag{9}$$

[9] For a discussion of the necessity of transversality conditions in stochastic environments, see Kamihigashi [22].

This economy is a textbook version of the models used to illustrate the FTPL. The present value of future primary surpluses must be well defined in a competitive equilibrium, since the transversality condition is necessary for household optimization. If taxes are set exogenously in real terms, the present value pins down the price level both in period 0 and in all subsequent periods, as long as the present value itself is positive as of time 0 (otherwise no equilibrium would exist with $B_0 > 0$).

The only difference with the standard treatment is the observation that equation (9) does not rule out the possibility that (5) and (7) hold as well.[10] In particular, while the present value of taxes has to be positive, it is quite possible that $\mathbb{E}(\tau_{t+s})$ is negative in all periods. We illustrate this possibility in Appendix A by considering a specific endowment process and fiscal policy for which government debt is risk free in real terms; equation (5) holds, and expected taxes are always negative.

4. Dynamic Inefficiency

While the present value of government surpluses was well defined in the previous section, we now turn to environments where this is no longer the case. In this section, we illustrate the implications of the FTPL in dynamically inefficient economies. We work with an overlapping-generations (OLG) economy where people live for two periods, based on Sargent [33], chapter 7. This setup is useful to obtain analytical results that readily generalize to more complex environments in which other frictions imply dynamic inefficiency.

We consider a pure exchange economy populated by overlapping generations of constant size, normalized to 1. Each generation lives for two periods. To keep the notation simple, we assume that the endowment is constant over time, and we abstract from any uncertainty. The income of each household is w^y when young and w^o when old. Households have preferences given by

$$U(c_t^y, c_{t+1}^o), \tag{10}$$

where c_t^y (c_t^o) is consumption by the young (old) in period t. The only asset in the economy is one-

[10] On this point, see Bohn [6] as well.

period government debt, as introduced in Section 2. In period 0, there is an initial stock of nominal debt B_0. As in the previous section, the government sets a constant nominal interest rate R. Taxes are raised in a real amount τ_t from the old;[11] this is a transfer if $\tau_t < 0$.

The budget constraint for the generation born in period t is given by

$$P_t c_t^y + \frac{B_{t+1}}{1+R} \leq P_t w^y, \tag{11}$$

$$P_{t+1} c_{t+1}^o \leq P_{t+1}(w^o - \tau_{t+1}) + B_{t+1}. \tag{12}$$

The old cohort in period 0 simply consumes all of its after-tax endowment and its savings:

$$P_0 c_0^o = P_0(w^o - \tau_0) + B_0. \tag{13}$$

With a constant nominal rate R, the real interest rate in the economy between periods t and $t+1$ is

$$1 + r_{t+1} := (1+R)\frac{P_t}{P_{t+1}}, \tag{14}$$

and the problem of maximizing (10) subject to (11) and (12) yields a (real) saving function $f(1 + r_{t+1})$. We assume that U is strictly increasing in both arguments, strictly quasiconcave, continuously differentiable, that consumption when young and old are gross substitutes, and that Inada conditions apply. Under these assumptions, f is strictly increasing in the real interest rate, which allows us to invert the function and obtain the equilibrium real interest rate as a function of savings by the young: $r_{t+1} = r(s_{t+1})$, where $s_{t+1} = w^y - c_t^y$. Since the growth rate of this economy is normalized to zero, we are interested in the case in which $f(1) > 0$, or, equivalently, $r(0) < 0$:[12] young households have a sufficient need to save for their old age that they are willing to do so even at a zero interest rate, which allows for dynamically inefficient equilibria with positive debt to a rise. The government budget constraint in period t is given by (1).

[11] Our results do not depend on the way taxes are allocated across generations, as long as the allocation is fixed over time. Of course, the range of parameters for which dynamic inefficiency arises, with its implications for the FTPL, does depend on this allocation.

[12] If preferences satisfy Inada conditions, this will necessarily happen provided the endowment when young is sufficiently larger than the endowment when old.

A competitive equilibrium of this economy is given by a sequence $\{c_t^y,\, c_t^o,\, P_t,\, r_t,\, B_{t+1}\}_{t=0}^{\infty}$, such that households maximize their utility subject to their budget constraints, the government budget constraint holds in each period t, the definition of real rate (14) applies, and markets clear, i.e.,

$$P_t f(1+r_{t+1}) = B_{t+1}/(1+R). \tag{15}$$

We compute competitive equilibria following Sargent [33]. Define $\pi_{t+1} := P_{t+1}/P_t$ to be the gross inflation rate between t and $t+1$. Combining equations (1), (14), and (15), an equilibrium must satisfy the following difference equation:

$$f(1+r_{t+1}) = (1+r_t)f(1+r_t) - \tau_t, \quad t \geq 1 \tag{16}$$

with an initial condition

$$f(1+r_1) = \frac{B_0}{P_0} - \tau_0. \tag{17}$$

It is easier to analyze this equation in terms of the savings by the young, which gives

$$s_{t+1} = (1+r(s_t))s_t - \tau_t \quad t \geq 1 \tag{18}$$

and

$$s_1 = \frac{B_0}{P_0} - \tau_0. \tag{19}$$

We concentrate our attention on the case of constant taxes, $\tau_t = \tau$, $t \geq 0$, in which the difference equation is time invariant and clearer results can be established analytically.

Proposition 1.

- *If $\tau > 0$, the difference equation (18) admits exactly two steady states, one with positive and one with negative savings.*
- *If $\tau = 0$, the difference equation (18) admits exactly two steady states, one with zero and one with positive savings.*
- *If $\tau < 0$, there generically exists an even number of steady states, all of which feature positive saving. If τ is sufficiently close to zero, the number of steady states is two; and if it is sufficiently negative, it is zero (no steady states exist).*

Proof. A steady state $\bar{s}$ requires $r(\bar{s})\bar{s} = \tau$. The Inada conditions imply $\lim_{s\to w^y} r(s) = \infty$ and $\lim_{s\to-\infty} r(s) = -1$, which, in turn, means that $\lim_{s\to w^y} r(s)s = \lim_{s\to-\infty} r(s)s = \infty$. Furthermore,

$$\frac{d[r(s)s]}{ds} = sr'(s) + r(s).$$

Thus, $sr(s)$ is monotonically increasing when both s and $r(s)$ are positive, and monotonically decreasing when they are both negative. $sr(s) = 0$ when either $s = 0$ or $r(s) = 0$. Since $r(0) < 0$ and r is increasing, $r(s) = 0$ occurs exactly at one point, at a value $\hat{s} > 0$, which proves the statement for the case in which $\tau = 0$. For $\tau > 0$, the derivative and limits above imply that there exists exactly one steady state in $(-\infty, 0)$ and one in $(\hat{s}, \infty)$, which again proves the relevant statement. Finally, for $\tau < 0$, we know $sr(s) > \tau$ for all values of $s \in (-\infty, 0] \cup [\hat{s}, w^y)$. By continuity, the number of steady states in the interval $(0, \hat{s})$ must be generically even.[13] We can also remark that, again by continuity, the number of steady states for $\tau < 0$ but sufficiently close to zero must be exactly two (as in the case of $\tau = 0$), and no steady state will exist if τ is sufficiently negative, since $sr(s)$ attains an interior minimum, which completes the proof. □

To complete the characterization of the equilibria, the following proposition analyzes the dynamics of the system away from the steady state.

Proposition 2.

- *If no steady state of the difference equation (18) exists, then s_{t+1} is monotonically increasing and it would eventually exceed w^y, where the difference equation ceases to be defined: hence, no competitive equilibrium exists.*
- *Let there be $2N$ steady states, ordered $(\bar{s}^1, \ldots, \bar{s}^{2N})$. If the initial saving rate is $s_1 < \bar{s}^2$, then s_t converges monotonically to $\bar{s}^1$. If $N > 1$ and $s_1 \in (\bar{s}^{2k}, \bar{s}^{2k+2}), \quad k = 1, ...N-1$, s_t converges monotonically to $\bar{s}^{2k+1}$. Finally, if $s_1 > \bar{s}^{2N}$, then s_t is monotonically increasing and eventually exceeds w^y, which implies that no equilibrium exists for such an initial condition.*

[13] For a measure-zero set of values of τ, $sr(s)$ will have a tangency point to τ, in which case an odd number of steady states can occur. Proposition 2 applies to this case if one interprets the tangency point as two coincident steady states.

Proof. First, $\lim_{s\to w^y} r(s)s = \infty$ (or $\lim_{s\to -\infty} r(s)s = \infty$), along with continuity, implies that $s_{t+1} > s_t$ always if no steady state exists. In this case, equilibrium would require a monotonically increasing sequence $\{s_t\}_{t=1}^{\infty}$, which cannot converge, because any convergence point would have to be a steady state. Since $\{s_t\}_{t=1}^{\infty}$ is bounded by the endowment w^y, a contradiction ensues, and no equilibrium can exist.

When $2N$ steady states exist, the same limits imply $s_{t+1} > s_t$ when $s_t < \bar{s}^1$, $s_t > \bar{s}^{2N}$, or $s_t \in (\bar{s}^{2k}, \bar{s}^{2k+1})$, $k = 1, \ldots, N-1$. Conversely, $s_{t+1} < s_t$ when $s_t \in (\bar{s}^{2k-1}, \bar{s}^{2k})$, $k = 1, \ldots, N$.

Furthermore, given any steady state $\bar{s}^i$, we have

$$s_{t+1} = s_t + r(s_t)s_t - \tau = s_t + r(s_t)s_t - r(\bar{s}^i)\bar{s}^i.$$

For $s_t > \bar{s}^i$,

$$s_{t+1} = s_t + r(s_t)s_t - r(\bar{s}^i)\bar{s}^i > s_t + r(\bar{s}^i)(s_t - \bar{s}^i) > \bar{s}^i,$$

with the converse being true for $s_t < \bar{s}^i$. Hence given any initial condition s_1, the sequence of saving will never "jump over" a steady state, but rather it will monotonically converge to an odd-numbered steady state. The exception is the case in which $s_1 > \bar{s}^{2N}$, where the sequence increases monotonically up to the point at which no solution exists.

Propositions 1 and 2 describe the behavior of the economy given s_1; more specifically, provided τ is not too negative, they show that a continuum of values of $s_1 \in (-\infty, \bar{s}^{\text{MAX}}]$ is consistent with a competitive equilibrium, where $\bar{s}^{\text{MAX}}$ is the largest steady state of the difference equation.

While in the representative-agent economy it is impossible for debt to have positive value when $\tau \leq 0$ for sure, this is not the case in the OLG economy, where dynamic inefficiency implies that money (or unbacked debt) can have value: hence, provided τ is not too negative, the economy can still admit positive values of debt.

Figures 2–4 illustrate the evolution of the economy for the simplest case in which s_{t+1} is a convex function of s_t, so that there are exactly two steady states. When $\tau > 0$ (Figure 2), one steady state features positive saving, whereas the other one has negative saving. The steady state with positive

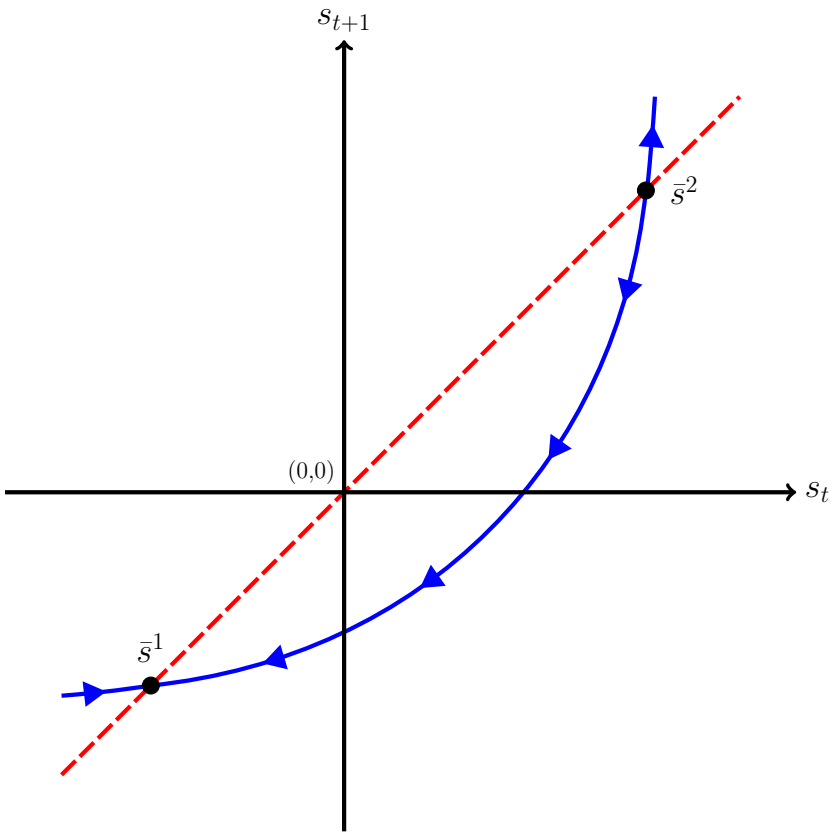

Figure 2: Evolution of s_t when $\tau > 0$.

saving ($\bar{s}^2$) corresponds to the traditional solution that would apply in a standard treatment of the FTPL: with constant endowment, a representative agent, and lump-sum taxes, the real interest rate would be independent of government debt, and the solid line would become straight and steeper than the 45-degree line. In this steady state, the interest rate is positive. When household saving is at $\bar{s}^2$, taxes are exactly sufficient to pay interest on government debt. For higher household saving (and hence higher government debt), taxes are insufficient to repay interest and debt explodes. Vice versa, for lower saving, taxes are more than enough and debt decreases. While this is true both in the representative-agent and in the OLG economy, in the latter the interest rate drops as s_t becomes lower, flattening the relationship between s_{t+1} and s_t. Eventually the solid line is flatter than the 45-degree line and a second, stable steady state emerges. In this steady state, young households are borrowing from the

government at a negative interest rate, so that their repayment in old age is smaller than what they receive from the government when young. Taxes are used by the government to replenish its assets and be able to lend the same amount to the next generation. Any value of $s_1 \in (-\infty, \bar{s}^2]$ leads to a stable evolution going forward: the economy always converges to $\bar{s}^1$, except if it starts (and stays) at $\bar{s}^2$.

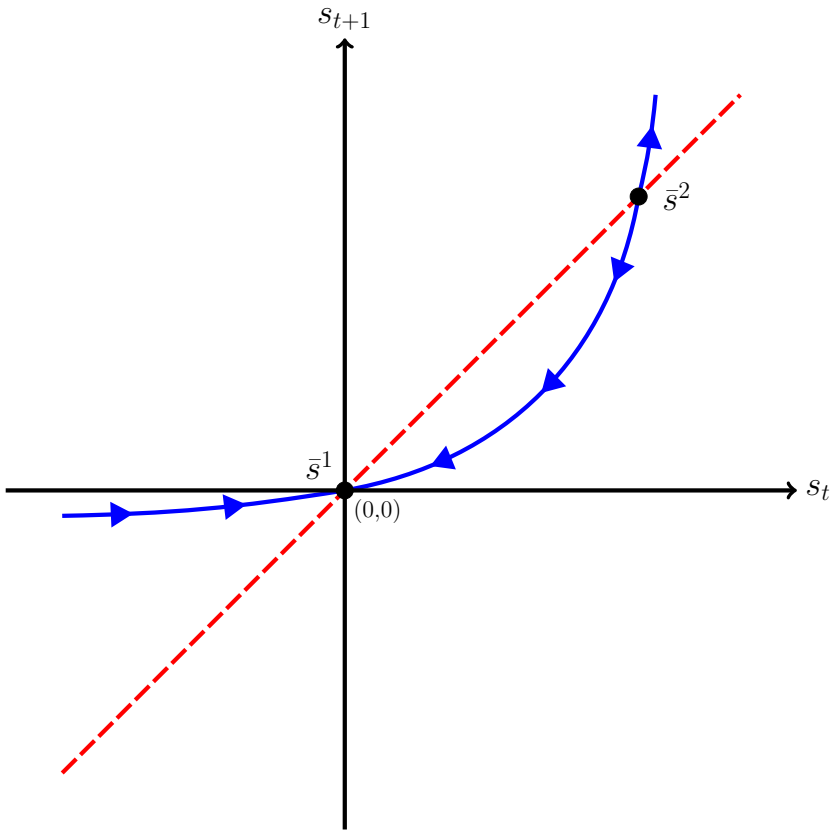

Figure 3: Evolution of s_t when $\tau = 0$.

When taxes are zero, the solid line shifts to the position indicated by Figure 3. Here, the interest rate in the higher steady state $\bar{s}^2$ is exactly zero, but government debt still has value as fiat money: it is passed from one generation to the next at a constant price. Any initial value above $\bar{s}^2$ would imply a positive interest rate, so that debt would explode in the absence of taxes, which is inconsistent with an equilibrium, since household saving is bounded above by w^y. Conversely, any lower value implies

a negative interest rate; in the absence of taxes or transfers, household saving shrinks to zero over time, where the lower steady state $\bar{s}^1$ is located.

Finally, if $\tau < 0$, but not too large, Figure 4 emerges: in this case, both steady states feature positive saving and a negative interest rate, so that the government repays to the households less than it borrowed, but the additional transfer payments make up for the negative interest. If τ is too negative, then the solid line is always above the 45-degree line, and no equilibrium can exist, since debt would always explode.

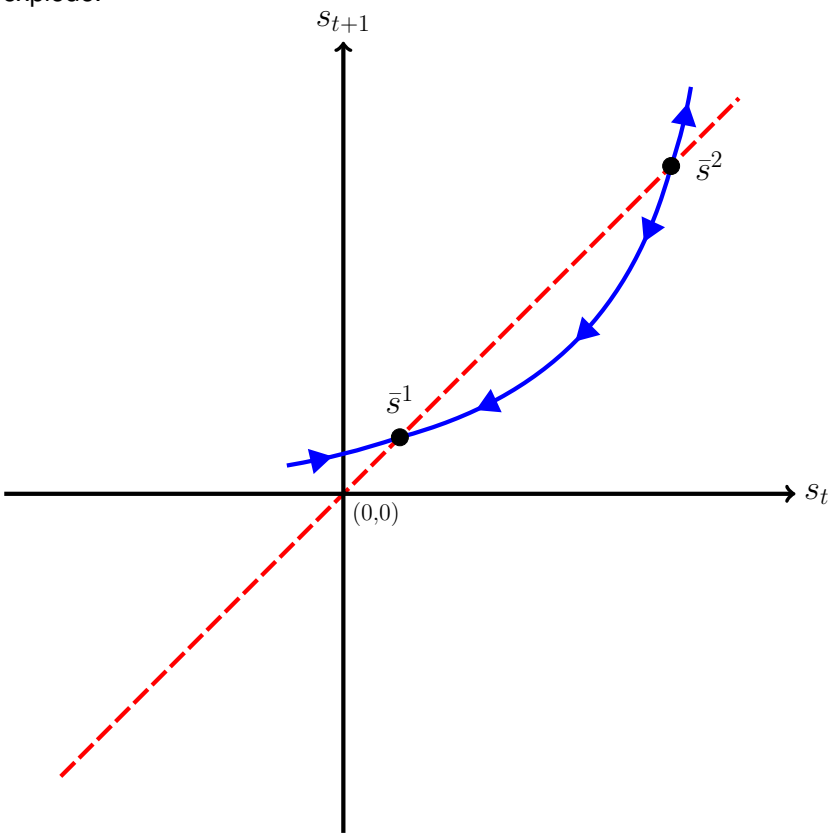

Figure 4: Evolution of s_t when $\tau < 0$.

In order to fully characterize the set of competitive equilibria of the economy, one last step is required: we need to relate the starting level of saving s_1 to the initial price level, P_0. We obtain

$$\frac{B_0}{P_0} = \tau + s_1.$$

Assuming that the economy starts with positive values of government debt and that $\bar{s}^{\text{MAX}} + \tau > 0$ ($\bar{s}^2 + \tau > 0$ in the figures), the requirement that $s_1 \in (-\infty, \bar{s}^{\text{MAX}}]$ implies that there exists a continuum of equilibria indexed by the initial price level, with[14]

$$P_0 \in [(\bar{s}^{\text{MAX}} + \tau)/B_0, \infty). \tag{20}$$

Equation (20) proves that the FTPL breaks down in our OLG economy: even with a fixed nominal interest rate and a fixed amount of real taxes, a continuum of possible price levels emerges. In Woodford [39], the FTPL explicitly appears as a way of selecting among equilibria in monetary economies, to remedy the fact that the monetary side alone typically does not achieve global uniqueness of an equilibrium. However, in OLG economies in which dynamic inefficiency is possible, government debt itself is akin to money, and hence the original multiplicity reemerges.

With a continuum of possible equilibrium price levels, we cannot rely on comparative statics for tight predictions on the inflationary consequences of lowering τ, providing a cautionary tale for using fiscal policy as a substitute for monetary policy to achieve a desired price target. While uniqueness fails in our environment, equation (20) still imposes a lower bound on prices, so that the FTPL is not completely devoid of content even in a dynamically inefficient economy.

Remark 1. *A possible criticism of the analysis above is that the general specification of the FTPL does not require that a sequence of real taxes should be independent of the initial price level, but rather that the* present value *of the sequence should be. With the possibility of dynamic inefficiency, computing present values becomes problematic, because they may not be well defined. However, we can easily construct an example in which even this version of the FTPL fails. Specifically, consider a sequence in which* $\tau_0 > 0$ *and* $\tau_t = 0, t > 0$. *For this sequence, the present value of taxes is always* τ_0, *independent*

[14] The condition $\bar{s}^{\text{MAX}} + \tau > 0$ is an implicit characterization, since $\bar{s}^{\text{MAX}}$ is itself a function of τ. However, from proposition 1, we know that $\bar{s}^{\text{MAX}} > 0$, and it is straightforward to prove that it is a decreasing function of τ. Hence, $\bar{s}^{\text{MAX}} + \tau > 0$ will always hold for $\tau \geq 0$, and also for $\tau < 0$, provided it is not too negative. Finally, note that there is a range of negative values of τ for which steady states exist but $\bar{s}^{\text{MAX}} + \tau \leq 0$: for these values, competitive equilibria would exist for negative initial amounts of debt, but none exists if $B_0 > 0$.

of the initial price level, and yet there exists a continuum of equilibria indexed by

$$P_0 \in [(\bar{s}^{MAX}_{\tau=0} + \tau_0)/B_0, \infty),$$

where the subscript to $\bar{s}^{MAX}$ makes it explicit that it refers to the (maximal) steady state of the economy with no taxes (since taxes will indeed be zero from period 1 onwards).

Remark 2. *The two-period OLG economy and our assumptions on f imply relatively that all deterministic equilibria feature monotone dynamics. By relaxing these assumptions, it would be possible to obtain cycles or even chaotic dynamics. An early survey of these possibilities appears in Woodford [38]. Furthermore, in a stochastic environment, sunspot equilibria would emerge. Our conclusion is robust to these more exotic environments: specifying an exogenous path for real taxes is insufficient to pin down the initial price level, and the FTPL breaks down.*

5 Debt as a Source of Liquidity

In the previous section, government debt bears a low rate of return because households have a large desire to save and there are no other assets that would allow them to earn a better rate of return.[15] Here, we study an economy which is dynamically efficient and in which private assets pay a rate of return that is higher than the growth rate of the economy (normalized to zero); government debt pays a lower rate of return instead because it plays a special liquidity role, allowing some transactions that cannot be completed by exchanging private assets.

In common with the previous section, government debt has itself the characteristics of money, and the only difference is the role money plays. This leads to many similarities, but also to some important differences. When government debt is an asset like any other, as in the OLG context, equilibria in which households borrow from the government are possible. However, when government debt provides a special liquidity service, this service is predicated on the government being the debtor and

[15] As is well known, similar results apply even if physical capital was present, as long as it is subject to a sufficiently decreasing rate of return.

the households being creditors. We prove below that this restores the validity of the FTPL for $\tau > 0$, albeit it also implies that interest rates below the growth rate of the economy are impossible in this case.

We develop our analysis in the context of the model developed by Lagos and Wright [24]; the analysis would be similar if we considered a cash-in-advance model, or alternative models where debt facilitates transactions and/or relaxes liquidity constraints.[16]

5.1 The Basic Environment

We consider an economy populated by a continuum of identical infinitely lived households and a government. Each period is divided into two subperiods.

In the first subperiod (the "morning"), households disperse in bilateral anonymous markets, where they have an opportunity to buy a good that they like with probability $\chi \in (0, 1)$, and they have an opportunity to produce the good that the other party likes with the same probability. Double-coincidence meetings are ruled out. In these meetings, private credit and privately issued assets cannot be recorded and/or recognized: only government debt can be used in exchange for the desired good.

We assume that buyers make a take-it-or-leave-it offer to the sellers, which, given the preferences

[16] One important assumption that is implied by the Lagos-Wright model is that the demand for government-issued means of transacting vanishes as their opportunity cost is sufficiently high. The importance of this assumption is well understood from the analysis of monetary models. In models in which the value of liquidity services may become unbounded, the indeterminacy we describe here may not arise. In the context of monetary models, this is what happens when preferences for cash goods are unbounded from below (see e.g. Woodford [39]). To mention an extreme example, if money velocity is unresponsive to the interest rate and output is exogenous, then real balances are fixed and the nominal quantity of money automatically pins down the price level. An example in which this channel is at work is Canzoneri and Diba [10], where velocity is variable but bounded above. Assuming that the utility from liquidity services is unbounded from below (or, equivalently, that velocity of the liquidity instruments is bounded above, no matter the cost) strikes us as implausible: if households faced unboundedly large costs from acquiring government-issued means of exchange, they would presumably find alternatives to money or government bonds, such as other currencies, gold, cigarettes, etc. In the context of the Lagos-Wright model, such an alternative is implicitly represented by having finite utility, even without consuming any of the goods supplied in the market for which money or government bonds are essential.

below, is equivalent to competitive pricing.[17]

In the second subperiod (the "evening") a centralized market opens, where a good is traded that all households value and can produce. In this market, a record-keeping technology is present, and households can trade the evening good, privately issued claims, and government debt. The government levies taxes according to an exogenous real sequence $\{\tau_t\}_{t=0}^{\infty}$, repays maturing debt, and supplies new nominally risk-free debt at a set interest rate R. In Appendix B, we add money paying no interest, so that R is the opportunity cost of holding money vs. government debt.

Preferences of each household are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u(q_t) - n_t + c_t - y_t \right],$$

where q_t is the consumption of the morning good, u is strictly increasing and strictly concave with $u(0) = 0$, c_t is the consumption of the evening good, and n_t and y_t are the production of the morning good and evening good, respectively. Production enters negatively in the utility function because it requires labor effort. We assume that there exists $q^* \in (0, +\infty)$ such that $u'(q^*) = 1$. We will concentrate on equilibria where aggregates are deterministic, hence the expected value is taken with respect to the idiosyncratic history of matches encountered by a household.

5.2 Characterizing the Economy

Let $W_t(b, a)$ be the value for an agent who enters the centralized market with b units of bonds and a units of private claims, and let $V_t(b, a)$ be the value of the same position upon entering the decentralized market. We have

$$W_t(b, a) = \max_{c, y, b', a'} \{c - y + \beta \mathbb{E}_t V_{t+1}(b', a')\} \text{ s.t.} \tag{21}$$

$$P_t c + \frac{a'}{1 + R_t^p} + \frac{b'}{1 + R} \leq P_t(y - \tau_t) + a + b, \tag{22}$$

$$a' \geq -\underline{a} P_t, \tag{23}$$

[17] Similar results would apply to different bargaining protocols.

and

$$b' \geq 0. \tag{24}$$

P_t is the price of goods in term of nominal claims in the centralized market, and R_t^p is the nominal interest rate on private claims between periods t and $t+1$. Equation (23) imposes a borrowing limit on households so that they cannot engage in Ponzi schemes; equation (24) bans naked short selling of government debt. While privately-issued claims and government bonds are perfect substitutes when entering into the centralized market, the two assets are different in the decentralized market, and equation (24) formalizes the constraint that only the government can issue the latter.

The solution to maximizing (21) subject to (22), (23), and (24) yields

$$W_t(b,a) = \hat{W}_t + \frac{a+b}{P_t}, \tag{25}$$

where $\hat{W}_t$ depends on subsequent choices, but is independent of the current state. When buyers and sellers meet in the decentralized market, in any meeting in which buyers exchange $\tilde{b}$ units of government bonds for $\tilde{q}$ units of goods, equation (25) implies that the sellers' participation constraint is

$$-\tilde{q} + \tilde{b}/P_t \geq 0. \tag{26}$$

Since buyers make a take-it-or-leave-it offer, equation (26) is equivalent to a "bonds-in-advance" constraint in which buyers can purchase the good at the linear price P_t (the same price that will prevail in the centralized market), up to their endowment of government bonds.

Using equation (26), and exploiting the fact that all the surplus from decentralized trade is appropriated by the buyers, the value function at the beginning of the period, before it is known who will be a seller, who will be a buyer, and who will not trade, is given by

$$V_t(b,a) = W_t(b,a) + \chi \max_q [u(q) - q] = \hat{W}_t + \frac{a+b}{P_t} + \chi \max_q [u(q) - q], \tag{27}$$

subject to

$$P_t q \leq b. \tag{28}$$

It is straightforward to verify that $\frac{\partial V_t(b,a)}{\partial b} = \frac{1}{P_t}$ for any $b > P_t q^*$ and $\frac{\partial V_t(b,a)}{\partial b} = \chi u'(b/P_t) + (1-\chi)(1/P_t)$ for any $b < P_t q^*$, and that $\frac{\partial V_t(b,a)}{\partial a} = \frac{1}{P_t}$ always. Taking this into account, the necessary and sufficient conditions for optimality in maximizing (21) subject to (22), (23), and (24) are given by

$$\frac{\beta(1+R_t^p)P_t}{P_{t+1}} = 1 \tag{29}$$

and

$$\frac{P_{t+1}}{\beta(1+R)P_t} - 1 = \chi\left[\max\left\{u'\left(\frac{B_{t+1}}{P_{t+1}}\right), 1\right\} - 1\right]. \tag{30}$$

Unless households are satiated with liquidity ($B_{t+1}/P_{t+1} \geq q^*$), the rate of return on government bonds will be lower than the one on private assets, and it could even be negative in real terms.

5.3 Does the FTPL Hold when Debt Provides Liquidity?

We can invert equation (30) to obtain the demand for real bonds as a correspondence of the real interest rate on government debt, which yields the same equation as (15) for the OLG economy, with f defined as

$$f(1+r_{t+1}) : \begin{cases} = \frac{1+R}{1+r_{t+1}}(u')^{-1}\left[\frac{\frac{1}{\beta(1+r_{t+1})}-(1-\chi)}{\chi}\right] & \text{for } r_{t+1} < 1/\beta - 1 \\ \in [\beta(1+R)q^*, \infty) & \text{for } r_{t+1} = 1/\beta - 1. \end{cases} \tag{31}$$

Since we are interested in equilibria where the real rate on government debt is below the growth rate of the economy (zero, in our case), a necessary condition is that $f(1) > 0$, as in the OLG economy, and we assume this to be the case: the demand for liquidity must be enough for households to be willing to hold government debt even at a zero real interest rate. As in the OLG economy, we assume f to be increasing in r.[18] Furthermore, no equilibrium is possible if $r_{t+1} > 1/\beta - 1$, since households would have an incentive to accumulate indefinite savings at that interest rate.

With this definition of f, an equilibrium is characterized by the same difference equation as in the previous section, which is (16) and (17) in terms of the interest rate and (18) and (19) in terms of real

[18] As an example, if u exhibits constant relative risk aversion γ, a sufficient condition is that $\gamma < 1$, which is already necessary to obtain $u(0) = 0$. See the discussion in footnote 16.

purchases of government bonds.

While the equilibria of the two economies are described by the same difference equation, so that several results will be similar, nonetheless some important differences must be noted:

- The domain of s_t is different: in an OLG economy, s_t can take values in $(-\infty, w^y)$, while in the liquidity economy its values must be contained in $[0, \infty)$. While the difference at the top has no effect on the results, the inability to borrow from the government has important implications for equilibrium selection. This is not surprising, since we encountered in the previous section an instance in which the stable steady state of the difference equation is negative.

- In the OLG economy, $r(s_t)$ approaches infinity as $s_t \to w^y$; in contrast, here $r(s_t)$ is constant at $1/\beta - 1$ when $s_t \geq \beta(1+R)q^*$.

- In the OLG economy, $r(s_t)$ is finite at $s_t = 0$. In contrast, $r(s_t)$ may approach -1 at $s_t = 0$ if u satisfies Inada conditions.

To characterize the set of equilibria, we now study the steady states and convergence properties of the difference equations, as we did in Propositions 1 and 2 for an OLG economy.

Proposition 3.

- *If $\tau > 0$, the difference equation (18) admits exactly one steady state, with positive savings.*

- *If $\tau = 0$, the difference equation (18) admits exactly two steady states, one with zero and one with positive savings.*

- *If $\tau < 0$, there generically exists an even number of steady states, all of which feature positive saving. If τ is sufficiently close to zero, the number of steady states is two; and if it is sufficiently negative, it is zero (no steady states exist).*

Proof. As in Proposition 1, a steady state $\bar{s}$ requires $r(\bar{s})\bar{s} = \tau$, and $sr(s)$ is monotonically increasing when both s and $r(s)$ are positive. We also know $\underline{r} := \lim_{s\to 0} r(s) \geq -1$, with equality if $\lim_{q\to 0} u'(q) = \infty$, and $r(s) = 1/\beta - 1$ for $s \geq \beta q^*(1+R)$, so that $\lim_{s\to\infty} sr(s) = \infty$. $sr(s) = 0$ when either $s = 0$ or

$r(s) = 0$. From $f(1) > 0$ we know $r(0) < 0$, so $r(s) = 0$ occurs exactly at one point, at a value $\hat{s} > 0$, which proves the statement for the case in which $\tau = 0$. For $\tau > 0$, the monotonicity properties of $sr(s)$ and its limits imply that there exists exactly one steady state, in $(\hat{s}, \infty)$, which again proves the relevant statement. Finally, for $\tau < 0$, we know $sr(s) > \tau$ for all values of $s \in [\hat{s}, \infty)$, and at $s = 0$. By continuity, the number of steady states in the interval $(0, \hat{s})$ must be generically even.[19] Continuity also implies the remaining properties of the last bullet.

Away from a steady state, the dynamics are described in the following proposition:

Proposition 4.

- *If no steady state of the difference equation (18) exists, then s_{t+1} is monotonically increasing and government debt eventually explodes exponentially, violating a household's transversality condition, which cannot happen in an equilibrium.*
- *If $\tau > 0$, such that a unique steady state $\bar{s}$ exists, then for $s_1 > \bar{s}$, $\{s_{t+1}\}_{t=0}^{\infty}$ is monotonically increasing and government debt eventually explodes exponentially, violating a household's transversality condition. For $s_1 < \bar{s}$, $\{s_{t+1}\}_{t=0}^{T}$ is monotonically decreasing until $s_{T+1}[1 + r(s_{T+1})] - \tau < 0$, at which point the difference equation no longer has a solution.*
- *Let there be $2N$ steady states, ordered $(\bar{s}^1, \ldots, \bar{s}^{2N})$. If the initial saving rate is $s_1 < \bar{s}^2$, then s_t converges monotonically to $\bar{s}^1$. If $N > 1$ and $s_1 \in (\bar{s}^{2k}, \bar{s}^{2k+2}), \quad k = 1, ...N - 1$, s_t converges monotonically to $\bar{s}^{2k+1}$. Finally, if $s_1 > \bar{s}^{2N}$, then government debt eventually explodes exponentially, violating a household's transversality condition.*

Proof. First, $\lim_{s\to\infty} r(s)s = \infty$, along with continuity, implies that $s_{t+1} > s_t$ always if no steady state exists. In this case, equilibrium would require a monotonically increasing sequence $\{s_t\}_{t=0}^{\infty}$, which cannot converge, since any convergence point would have to be a steady state. Once $s_t \geq \beta q^*(1+R)$, the difference equation becomes $s_{t+1} = s_t/\beta - \tau_{t+1}$, so that $\lim_{t\to\infty} s_{t+1}/s_t = 1/\beta$.

The household transversality condition requires

$$\lim_{t\to\infty} \beta^t u' \left(\max \left\{ \frac{B_t}{P_t}, q^* \right\} \right) \frac{B_t}{P_t} = 0.$$

[19] Footnote 13 applies here as well.

For $s_t \geq \beta q^*(1+R)$, $\frac{B_{t+1}}{P_{t+1}} \geq q^*$, hence exponential growth in s_t at a rate β would not be optimal from the households' perspective. The same reasoning applies if $\tau > 0$ and $s_1 > \bar{s}$, or if $\tau \leq 0$ and $s_1 \geq \bar{s}^{2N}$.

When $2N$ steady states exist, continuity and the boundary properties of $sr(s)$ at $s = 0$ and $s = \infty$ imply $s_{t+1} > s_t$ when $s_t < \bar{s}^1$, $s_t > \bar{s}^{2N}$, or $s_t \in (\bar{s}^{2k}, \bar{s}^{2k+1})$, $k = 1, \ldots, N-1$. Conversely, $s_{t+1} < s_t$ when $s_t \in (\bar{s}^{2k-1}, \bar{s}^{2k})$, $k = 1, \ldots, N$. The rest of the proof is identical to Proposition 2.

While in the OLG economy a continuum of initial values of s_1 is consistent with an equilibrium (provided τ was such that an equilibrium exists), for the economy with liquidity this is only true if $\tau \leq 0$. When $\tau \leq 0$, Figures 3 and 4 are still valid, provided we only concentrate on the upper-right quadrant. In this case, the FTPL breaks down in the same way it did in the OLG economy. A continuum of values of $s_1 \in [0, \bar{s}^{\text{MAX}}]$ is consistent with a competitive equilibrium. Unless the economy starts at the highest steady state $\bar{s}^{\text{MAX}}$, it converges to a lower steady state, which involves positive debt if $\tau < 0$ and no debt if $\tau = 0$. Correspondingly, if the economy starts with positive values of government debt and $\bar{s}^{\text{MAX}} + \tau > 0$, a continuum of initial levels of prices is consistent with an equilibrium, as described by equation (20); the same considerations about comparative statics and the presence of a lower bound for prices that we discussed in Section 4 apply here as well. All of these results are reminiscent of the properties of equilibria with money-supply rules in cash-in-advance economies, as in Matsuyama [28, 29] or Woodford [39]. This is not surprising, since debt plays the same role as fiat money for this environment.[20]

When $\tau > 0$, the dynamics of the OLG economy and the economy in which debt provides liquidity services differ. The restriction to $s_t > 0$ bites in this case, as shown by Figure 5: now all trajectories that do not start at the positive steady state $\bar{s}$ end up violating either the nonnegativity constraint or the transversality condition, hence $\bar{s}$ is the unique solution to the difference equation when debt provides liquidity services. Hence, when $\tau > 0$, equilibrium requires $s_1 = \bar{s}$ and we can recover the price level

[20] Remarks 1 and 2 apply to this section as well. In addition to the papers by Matsuyama and Woodford, the potential for complicated dynamics is analyzed by Rocheteau and Wright [32] in an environment closer to ours.

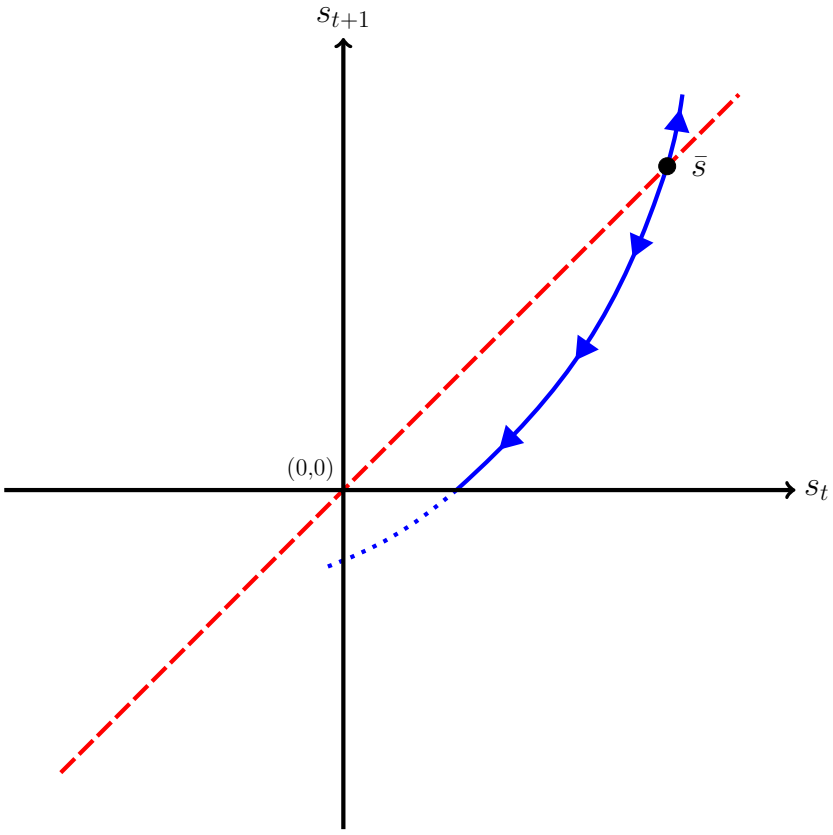

Figure 5: Evolution of s_t when $\tau > 0$.

uniquely from the condition

$$\frac{B_0}{P_0} = \tau + s_1. \tag{32}$$

However, when $\tau > 0$, the steady state is such that $\bar{s}r(\bar{s}) > 0$, which implies $r(\bar{s}) > 0$: the real interest rate on government debt is necessarily positive (above the zero growth rate of the economy). The FTPL holds in this case, but it negates the premise of our paper.

Our analysis has important implications when evaluating the effectiveness of fiscal policy to fight deflation. According to the standard FTPL, lowering τ will increase prices.[21] Hence, a commitment to smaller fiscal revenues will lead to an immediate jump to higher prices. This policy prediction ceases

[21] In our perfect foresight economy, lowering τ corresponds to a surprise revaluation of government debt, and is subject to Niepelt's [31] criticism. However, the same result applies in an economy in which τ is ex ante stochastic and we are comparing across different realizations. See Daniel [15].

to be true in an economy in which government debt offers liquidity services and the real interest rate is negative.[22] As discussed in Section 2, observing positive debt and a persistently negative interest rate in this economy is by itself evidence that households already expect *primary deficits*, at least in the long run, and that government debt retains positive value only because it also provides liquidity services.[23]

6. Conclusion

The FTPL is not a robust equilibrium selection criterion when the interest rate is persistently below the growth rate of the economy: whether the theory does or does not hold depends on the specific economic forces that lead to low rates. In this paper, we have shown three broad classes of models in which government bonds feature low returns (or low expected returns); the situation is further complicated by the possibility that these reasons interact with each other. As an example, government debt might have a specific liquidity role because of its favorable risk profile; see Caballero and Farhi [8]. In turn, the greater liquidity role of debt in recessions might limit the need for procyclical fiscal policy to support the debt's favorable risk profile.[24]

In this paper, we have concentrated on stationary environments. As Figure 1 showed, the real return on bonds has varied a great deal in the past decades, and it is possible that interest rates will exceed the growth rate again in the future. In such richer environments, the validity of the FTPL would depend on the frequency and duration of low-rate episodes, in ways that could be analyzed using a

[22] More precisely, what is relevant is whether the interest rate is negative asymptotically, since our proofs rely on limiting dynamics of the difference equation.

[23] A knife-edge situation arises for time-varying paths of taxes in which $\tau_t > 0$ (at least asymptotically) but taxes decay exponentially to zero. Adapting Proposition 1 in Tirole [37], one can then prove that there exists a unique equilibrium, even though the interest rate is asymptotically negative. The FTPL would hold in this knife-edge case. We are indebted to Gadi Barlevy for pointing this out.

[24] The role of term premia in explaining the recent experience of low interest rates is discussed in Campbell, Sunderam, and Viceira [9] and Gourio and Ngo [21], among others. Del Negro et al. [18] present evidence in favor of the hypothesis that the low return on government debt is due to a combination of liquidity services ("convenience yields") and a favorable risk profile.

References

[1] Andrew B. Abel, Gregory N. Mankiw, Lawrence H. Summers, and Richard J. Zeckhauser. Assessing dynamic efficiency: Theory and evidence. *The Review of Economic Studies*, 56(1):1–19, 1989.

[2] Marco Bassetto. A Game-Theoretic View of the Fiscal Theory of the Price Level. *Econometrica*, 70(6):2167–2195, 2002.

[3] Marco Bassetto. Equilibrium and Government Commitment. *Journal of Economic Theory*, 124(1):79–105, 2005.

[4] Marco Bassetto and Christopher Phelan. Speculative Runs on Interest Rate Pegs. *Journal of Monetary Economics*, 73:99–114, 2015.

[5] Aleksander Berentsen and Christopher Waller. Liquidity Premiums on Government Debt and the Fiscal Theory of the Price Level. Working Paper 2017-008A, Federal Reserve Bank of St. Louis, 2017.

[6] Henning Bohn. The Sustainability of Budget Deficits in a Stochastic Economy. *Journal of Money, Credit and Banking*, 27(1):257–271, 1995.

[7] Willem H. Buiter. The Fiscal Theory of the Price Level: A Critique. *Economic Journal*, 112(481):459–480, 2002.

[8] Ricardo J. Caballero and Emmanuel Farhi. The Safety Trap. *The Review of Economic Studies*, Forthcoming.

[9] John Y. Campbell, Adi Sunderam, and Luis M. Viceira. Inflation Bets or Deflation Hedges? The Changing Risks of Nominal Bonds. *Critical Finance Review*, 6(2):263–301, 2017.

[10] Matthew B. Canzoneri and Behzad T. Diba. Interest Rate Rules and Price Determinacy: The Role of Transactions Services of Bonds. *Journal of Monetary Economics*, 52:329–343, 2005.

[11] Hess Chung, Troy Davig, and Eric M. Leeper. Monetary and Fiscal Policy Switching. *Journal of Money, Credit and Banking*, 39(4):809–842, 2007.

[12] John H. Cochrane. Money as Stock. *Journal of Monetary Economics*, 52(3):501–528, 2005.

[13] John H. Cochrane. Determinacy and Identification with Taylor Rules. *Journal of Political Economy*, 119(3):565–615, 2011.

[14] Wei Cui. Monetary-Fiscal Interactions with Endogenous Liquidity Frictions. *European Economic Review*, 87:1–25, 2016.

[15] Betty C. Daniel. The Fiscal Theory of the Price Level and Initial Government Debt. *Review of Economic Dynamics*, 10(2):193–206, 2007.

[16] Michael R. Darby. Some Pleasant Monetarist Arithmetic. *Federal Reserve Bank of Minneapolis Quarterly Review*, 8(2):15–20, 1984.

[17] Troy Davig and Eric M. Leeper. Generalizing the Taylor Principle: Reply. *The American Economic Review*, 100(1):618–624, 2010.

[18] Marco Del Negro, Domenico Giannone, Marc P. Giannoni, and Andrea Tambalotti. Safety, Liquidity, and the Natural Rate of Interest. *Brookings Papers on Economic Activity*, 2017(1):235–294, 2017.

[19] Begoña Domínguez and Pedro Gomis-Porqueras. The Effects of Secondary Markets for Government Bonds on Inflation Dynamics. `https://mpra.ub.uni-muenchen.de/82448/`, 2016. Munich Personal RePEc Archive.

[20] François Geerolf. Reassessing Dynamic Efficiency. http://www.econ.ucla.edu/fgeerolf/research/Efficiency_Emp.pdf, 2013. Mimeo, University of California at Los Angeles.

[21] François Gourio and Phuong Ngo. Risk Premia at the ZLB: A Macroeconomic Interpretation. https://docs.google.com/viewer?a=v&pid=sites&srcid=ZGVmYXVsdGRvbWFpbnxmZ291cmlvfGd4OjI2MDhmMDIxMzY2ZDIxMTU, 2016. Mimeo, Federal Reserve Bank of Chicago and Cleveland State University.

[22] Takashi Kamihigashi. Necessity of Transversality Conditions for Stochastic Problems. *Journal of Economic Theory*, 109(1):140–149, 2003.

[23] Narayana R. Kocherlakota and Christopher Phelan. Explaining the Fiscal Theory of the Price Level. *Federal Reserve Bank of Minneapolis Quarterly Review*, 23(4):14–23, 1999.

[24] Ricardo Lagos and Randall Wright. A Unified Framework for Monetary Theory and Policy Analysis. *Journal of Political Economy*, 113(3):463–484, 2005.

[25] Eric Leeper. Equilibria under 'Active' and 'Passive' Monetary Policies. *Journal of Monetary Economics*, 27(1):129–147, 1991.

[26] Lars Ljungqvist and Thomas J. Sargent. *Recursive Macroeconomic Theory*. MIT Press, 3rd edition edition, 2012.

[27] Magali Marx, Benoît Mojon, and François R. Velde. Why Have Interest Rates Fallen Far Below the Return on Capital. Working Paper 630, Banque de France, 2017.

[28] Kiminori Matsuyama. Sunspot Equilibria (Rational Bubbles) in a Model of Money-in-the-Utility-Function. *Journal of Monetary Economics*, 25(1):137–144, 1990.

[29] Kiminori Matsuyama. Endogenous Price Fluctuations in an Optimizing Model of a Monetary Economy. *Econometrica*, 59(6):1617–1631, 1991.

[30] Preston J. Miller and Thomas J. Sargent. A Reply to Darby. *Federal Reserve Bank of Minneapolis Quarterly Review*, 8(2):21–26, 1984.

[31] Dirk Niepelt. The Fiscal Myth of the Price Level. *The Quarterly Journal of Economics*, 119(1):277–300, 2004.

[32] Guillaume Rocheteau and Randall Wright. Liquidity and Asset-Market Dynamics. *Journal of Monetary Economics*, 60(2):275–294, 2013.

[33] Thomas J. Sargent. *Dynamic Macroeconomic Theory*. Harvard University Press, 1987.

[34] Thomas J. Sargent and Neil Wallace. "Rational" Expectations, the Optimal Monetary Instrument, and the Optimal Money Supply Rule. *Journal of Political Economy*, 83(2):241–254, 1975.

[35] Thomas J. Sargent and Neil Wallace. Some Unpleasant Monetarist Arithmetic. *Federal Reserve Bank of Minneapolis Quarterly Review*, 5(1):1–17, 1981.

[36] Christopher A. Sims. A Simple Model for Study of the Determination of the Price Level and the Interaction of Monetary and Fiscal Policy. *Economic Theory*, 4(3):381–399, 1994.

[37] Jean Tirole. Asset Bubbles and Overlapping Generations. *Econometrica*, 53(5):1071–1100, 1985.

[38] Michael Woodford. Indeterminacy of Equilibrium in the Overlapping Generations Model: A Survey. `http://www.columbia.edu/~mw2230/Woodford84.pdf`, 1984. Mimeo, Columbia University.

[39] Michael Woodford. Monetary Policy and Price Level Determinacy in a Cash-in-Advance Economy. *Economic Theory*, 4(3):345–380, 1994.

[40] Michael Woodford. *Interest & Prices*. Princeton University Press, 2003.

[41] Kei-Mu Yi and Jing Zhang. Understanding Global Trends in Long-Run Interest Rates. *Economic Perspectives, Federal Reserve Bank of Chicago*, 41(2):1–20, 2017.

Appendix A A Positive Net Present Value with Negative Expected Taxes

To discuss a specific example of Section 3, we posit that the (log) endowment grows stochastically over time according to:

$$\ln y_t - \ln y_{t-1} = \ln \Delta + \varepsilon_t, \tag{33}$$

where ε_t is independent across time, with an exponential distribution with coefficient λ.

By using (33) and imposing market clearing ($y_t = c_t$), we rewrite the first-order condition (8) at time t as

$$1 = \beta(1 + r_{t+1})\Delta^{-\gamma}\mathbb{E}_t\left[\exp\{-\gamma\varepsilon_{t+1}\}\right].$$

It then follows that

$$1 + r_{t+1} = \frac{\Delta^{\gamma}(\gamma + \lambda)}{\beta\lambda}. \tag{34}$$

In this economy, the level of the real interest rate is independent of government policy, as long as an equilibrium exists. Given the real interest rate computed above, equation (5) is satisfied if and only if

$$\frac{\Delta^{\gamma-1}(\gamma + \lambda)}{\beta(\lambda + 1)} < 1. \tag{35}$$

In order for the household problem to be well defined, parameters must be such that the utility of consuming the endowment must be finite, which requires

$$\lambda\beta\Delta^{1-\gamma} < (\gamma + \lambda - 1). \tag{36}$$

Equations (35) and (36) are mutually compatible only if $\gamma > 1$, i.e., when agents are sufficiently risk averse; in this case, there is a range of values for Δ and λ such that the downside risk (the inverse of Δ) and volatility (λ) are sufficiently elevated that (35) hold, but not as large as would yield infinitely negative utility for the household.

Consider taxes next. We study a tax rule in which $x_t(= \tau_t/y_t)$ is a function $x(\varepsilon_t)$ of the contemporaneous shock ε_t only and government debt is risk free in real as well as in nominal terms. In order for

this to be the case, the price level P_{t+1} must be time-t measurable. From equation (9),

$$\begin{aligned} P_{t+1} =& \frac{B_{t+1}}{x(\varepsilon_{t+1})y_{t+1} + \mathbb{E}_{t+1}\sum_{s=2}^{\infty}\frac{z_{t+s}}{z_{t+1}}\tau_{t+s}} \\ =& \frac{B_{t+1}}{y_{t+1}\left[x(\varepsilon_{t+1}) + \mathbb{E}_{t+1}\sum_{s=2}^{\infty}\beta^{s-1}\left(\frac{y_{t+s}}{y_{t+1}}\right)^{1-\gamma}x(\varepsilon_{t+s})\right]}. \end{aligned} \tag{37}$$

In equation (37), the assumption of i.i.d. growth and that the primary surplus/GDP ratio is only a function of the current shock implies that $\mathbb{E}_{t+1}\sum_{s=2}^{\infty}\beta^{s-1}\left[\left(\frac{y_{t+s}}{y_{t+1}}\right)^{1-\gamma}x(\varepsilon_{t+s})\right]$ is a constant, which we define as ρ.[25] It is straightforward to prove that ρ must be positive for government debt to also be positive in the future: intuitively, the present value of future taxes must remain positive. From equation (1), B_{t+1} is predetermined (that is, time-t measurable). Hence, P_{t+1} is time-t measurable if and only if

$$x(\varepsilon_{t+1}) = \bar{x}e^{-\varepsilon_{t+1}} - \rho \tag{38}$$

for some constant $\bar{x} > 0$. Iterating on the definition of the present value of taxes ρ, we can recover how it is related to $\bar{x}$:

$$\begin{aligned} \rho =& \mathbb{E}_{t+1}\left[\beta x(\varepsilon_{t+2})\left(\frac{y_{t+2}}{y_{t+1}}\right)^{1-\gamma} + \sum_{s=3}^{\infty}\beta^{s-1}\left(\frac{y_{t+s}}{y_{t+1}}\right)^{1-\gamma}x(\varepsilon_{t+s})\right] \\ =& \beta\mathbb{E}_{t+1}\left[\left(\bar{x}e^{-\varepsilon_{t+2}} - \rho\right)\left(\frac{y_{t+2}}{y_{t+1}}\right)^{1-\gamma} + \left(\frac{y_{t+2}}{y_{t+1}}\right)^{1-\gamma}\rho\right] = \frac{\bar{x}\beta\Delta^{1-\gamma}\lambda}{\lambda+\gamma}. \end{aligned} \tag{39}$$

Combining (38) and (39) and taking expected values, we obtain

$$E_t(x_{t+1}) = \frac{\bar{x}\lambda}{\lambda+1}\left[1 - \frac{\beta\Delta^{1-\gamma}(\lambda+1)}{\lambda+\gamma}\right] < 0,$$

where the last inequality follows from assuming that the interest rate is low, more precisely, that (5) and hence (35) hold. Hence, in this i.i.d. economy, when parameters are such that (5) holds and fiscal policy stabilizes the debt/GDP ratio, expected taxes one period ahead are always negative.

Appendix B A Model with Government Debt and Money

We reconsider the economy of Section 5, but we now explicitly introduce money, so that money and government bonds circulate at the same time. The environment is the same as in Section 5, except

[25] Note however that ρ depends on the function x(.), so we need to solve for both of them jointly in what follows.

that government debt is no longer accepted with probability one in bilateral meetings, but only with probability $\zeta \in (0,1)$. In contrast, the central bank issues "money," which is perfectly durable, divisible, and intrinsically useless, and yields a zero nominal return. Money is always recognized in bilateral meetings and can therefore be used with probability one. We again assume that buyers make a take-it-or-leave-it offer to the sellers, and they make their offer knowing whether the seller is able to accept only money or both money and bonds in exchange for goods.

In the centralized market, government debt is now a commitment by the government to deliver the face value in money at maturity, justifying the assumption that it is nominal debt. The central bank policy of setting an interest rate R implies that there is an infinitely elastic supply of new one-period bonds vs. money, at a relative price $1/(1+R)$.[26]

The government budget constraint is now modified to

$$\frac{B_{t+1}}{1+R} + M_{t+1} = B_t + M_t - P_t\tau_t. \tag{40}$$

B.1 Characterizing the Economy

As before, denote with W and V the value functions when entering the centralized and decentralized market, respectively. They now depend on money holdings m, in addition to government debt holdings and private claims. We have

$$W_t(m,b,a) = \max_{c,y,b',a'} \{c - y + \beta \mathbb{E}_t V_{t+1}(m',b',a')\} \text{ s.t.} \tag{41}$$

$$P_t c + \frac{a'}{1+R_t^p} + \frac{b'}{1+R} + m' \leq P_t(y-\tau_t) + a + b + m, \tag{42}$$

(23), (24), and

$$m' \geq 0. \tag{43}$$

As government debt, money cannot be sold short.

[26] See Bassetto and Phelan [4] for a discussion of the consequences of imposing limits on the central bank's ability to convert new bonds into money and vice versa.

The solution to the maximization problem above yields

$$W_t(m,b,a) = \hat{W}_t + \frac{a+b+m}{P_t}, \tag{44}$$

where $\hat{W}_t$ is independent of the current state. When entering the centralized market, private claims and government bonds are both nominal, so they are a commitment to deliver money in that market: hence, they are perfect substitutes for money at that stage. This implies that, when able to accept government bonds, sellers are indifferent between receiving bonds or money.

When buyers and sellers meet in the decentralized market, let $\tilde{l}$ be the amount of nominal claims that sellers receive in exchange for $\tilde{q}$ units of goods. Depending on the meeting, this can take the form of money only, if the seller cannot accept government bonds or both bonds and money. Equation (44) then implies that the sellers' participation constraint is

$$-\tilde{q} + \tilde{l}/P_t \geq 0. \tag{45}$$

Buyers face again a linear price P_t when making their take-it-or-leave-it offer.

Using equation (26), and exploiting the fact that all the surplus from decentralized trade is appropriated by the buyers, the value function at the beginning of the period, before it is known who will be a seller, who will be a buyer, and who will not trade, is given by

$$\begin{aligned} V_t(m,b,a) =& W_t(m,b,a) + \chi\zeta \max_q [u(q) - q] + \chi(1-\zeta) \max_{\hat{q}} [u(\hat{q}) - \hat{q}] \\ =& \hat{W}_t + \frac{a+b+m}{P_t} + \chi\zeta \max_q [u(q) - q] + \chi(1-\zeta) \max_{\hat{q}} [u(\hat{q}) - \hat{q}], \end{aligned} \tag{46}$$

subject to

$$P_t q \leq m + b \tag{47}$$

and

$$P_t \hat{q} \leq m. \tag{48}$$

We thus get $\frac{\partial V_t(m,b,a)}{\partial a} = \frac{1}{P_t}$,

$$\frac{\partial V_t(m,b,a)}{\partial b} = \begin{cases} \frac{1}{P_t} & \text{if } m + b > P_t q^* \\ \chi\zeta u'\left(\frac{m+b}{P_t}\right) + (1 - \chi\zeta)(1/P_t) & \text{otherwise,} \end{cases}$$

and

$$\frac{\partial V_t(m,b,a)}{\partial m} = \begin{cases} \frac{1}{P_t} & \text{if } m > P_t q^* \\ \chi(1-\zeta)u'\left(\frac{m}{P_t}\right) + (1-\chi(1-\zeta))(1/P_t) & \text{if } m \in (P_t q^* - b, P_t q^*) \\ \chi\zeta u'\left(\frac{m+b}{P_t}\right) + \chi(1-\zeta)u'\left(\frac{m}{P_t}\right) + (1-\chi)(1/P_t) & \text{otherwise.} \end{cases}$$

Taking this into account, the necessary and sufficient conditions for optimality in maximizing (41) subject to (42), (23), (24), and (43) are given by (29),

$$\frac{P_{t+1}}{\beta(1+R)P_t} - 1 = \chi\zeta\left[\max\left\{u'\left(\frac{M_{t+1}+B_{t+1}}{P_{t+1}}\right), 1\right\} - 1\right], \tag{49}$$

and

$$\frac{P_{t+1}}{\beta P_t} - 1 = \chi\zeta\left[\max\left\{u'\left(\frac{M_{t+1}+B_{t+1}}{P_{t+1}}\right), 1\right\} - 1\right] + \chi(1-\zeta)\left[\max\left\{u'\left(\frac{M_{t+1}}{P_{t+1}}\right), 1\right\} - 1\right]. \tag{50}$$

We will assume that $R > 0$, for otherwise money and government bonds would have the same opportunity cost and households would only hold bonds if they were satiated with liquidity. In this case, the rate of return on money is always lower than the rate of government bonds; equations (49) and (50) then imply that households will never find it optimal to accumulate enough money so as to buy q^* with money only. Government bonds will have a lower rate of return than private assets when liquidity is scarce even in meetings where money and bonds are traded ($(M_{t+1}+B_{t+1})/P_{t+1} \leq q^*$).

We obtain an excess demand for money relative to government bonds

$$\frac{P_{t+1}}{\beta(1+R)P_t} - \frac{P_{t+1}}{\beta P_t} = -\chi(1-\zeta)\left[u'\left(\frac{M_{t+1}}{P_{t+1}}\right) - 1\right], \tag{51}$$

which means that money carries a liquidity premium over government bonds.

We invert equation (49) to obtain the demand for total liquidity (money plus bonds), which is given by

$$\frac{B_{t+1}+M_{t+1}}{P_t(1+R)} = f(1+r_{t+1}) := \frac{1}{1+r_{t+1}}(u')^{-1}\left[\frac{\frac{1}{\beta(1+r_{t+1})} - (1-\chi\zeta)}{\chi\zeta}\right]. \tag{52}$$

Using this equation and (40), we obtain a difference equation

$$f(1+r_{t+1}) = (1+r_t)f(1+r_t) - \frac{R}{1+R}\frac{M_{t+1}}{P_t} - \tau_t \tag{53}$$

with an initial condition

$$f(1+r_1) = \frac{B_0 + M_0 - RM_1/(1+R)}{P_0} - \tau_0.$$

Equation (53) differs from equation (16) due to the presence of seigniorage revenues. If taxes are set so as to offset these revenues, i.e., if

$$\tau_t = \tau - \frac{R}{1+R}\frac{M_{t+1}}{P_t}, \tag{54}$$

the difference equation coincides with the cashless economy and the same results apply.[27] In this case, given an equilibrium path for the real interest rate $\{r_{t+1}\}_{t=0}^{\infty}$ that satisfies the difference equation

$$f(1+r_{t+1}) = (1+r_t)f(1+r_t) - \tau, \quad t \geq 1$$

along with the initial condition

$$f(1+r_1) = \frac{B_0 + M_0}{P_0} - \tau,$$

the additional equilibrium condition, which is equation (51), can be used to determine real (and nominal) money balances.[28]

[27] Equation (54) assumes that the fiscal authority observes and can react to M_{t+1} and P_t. These variables are observed in the centralized (evening) market of period t, where taxes are also levied. The issue of joint determination of macroeconomic aggregates and policy variables is at the heart of complications discussed in Bassetto [2, 3]. This is a side issue from the perspective of the current paper. However, for completeness, (54) is a well-specified government strategy under the following description of the centralized market. First, the government repays maturing bonds in money. Second, households participate in the goods market according to some Walrasian mechanism (e.g., an auction), where money is used as numeraire. Households then purchase new government bonds with money at the set nominal rate R. At this stage, households in the aggregate are left with $B_t + M_t - B_{t+1}/(1+R)$ units of money. The government sets and collects nominal taxes equal to $P_t\tau_t = P_t\tau\,(1+R) - R[B_t + M_t - B_{t+1}/(1+R)]$, to be settled in money. Households are left in the aggregate with $M_{t+1} = B_t + M_t - B_{t+1}/(1+R) - P_t\tau_t$, and simple algebra shows that (54) holds. Finally, an auction opens for private borrowing and lending.

[28] Note that knowing r_{t+1} is equivalent to knowing π_{t+1}, given the nominal interest rate R and the definition of real rate (14).