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Can Exchange Rate Dynamics in Krugman's Target-zone Model be Directly Tested?

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Abstract

Despite Krugman's (1991) model being a benchmark for modelling target zones, empirical support has been sparse due to the subtle non-linear relationship between the observable exchange rate and underlying unobservable fundamental. This paper provides an alternative approach to derive explicit exchange rate dynamics by approximating a quadratic relationship between the exchange rate and fundamental through a power-series method. The exchange rate dynamics with a parametric class of drift terms of the stochastic fundamental, including zero-trend (Krugman's model), symmetric and asymmetric mean-reverting forces regarding how central banks intervene are ready for direct empirical tests. The empirical results demonstrate that the derived dynamics following a square-root process (in Krugman's model), or mean-reverting square-root process, adequately fit the exchange rate data of various target-zone systems including the Exchange Rate Mechanism and the Linked Exchange Rate System of the Hong Kong dollar. The model parameters of the exchange rate dynamics under the asymmetric mean-reverting fundamental are shown to be associated with realignment of the currencies' target zones.

Keywords: Exchange rate dynamics, target zones, interventions, stochastic processes

JEL classification: F31, G13

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1. Introduction

Based on the standard monetary model, Krugman (1991) develops a target-zone model with a stochastic differential equation that links the exchange rate between two currencies to underlying fundamentals. The exchange rate is bounded in the target zone as a result of market dynamics and central bank interventions. In the Krugman model, central banks intervene only when the fundamentals hit the boundaries of a band (so called marginal intervention), and a Brownian motion with a reflecting boundary condition at the upper and lower limit can be used as the driving process to keep the exchange rate in the zone. The fundamentals will therefore be regulated to remain in the band and the smooth-pasting condition holds at the boundaries. However, it is difficult to capture the non-linear relationship in the model between the observable exchange rate and unobservable fundamental value empirically. In addition, there is no explicit analytical (mathematical) dynamics of the exchange rates for empirical tests. Because of such difficulties, many empirical studies indirectly attempt to investigate this theoretical prediction by examining the mean-reverting properties of the European currencies' exchange rates in the Exchange Rate Mechanism (ERM) induced by the reflecting boundary condition. The results are mixed, perhaps reflecting a violation of the assumption of a fully credible target zone, as some countries' exchange rate bands were abandoned or realigned during the ERM crisis. Some currencies follow a random-walk process and do not display reversion towards the central parity.¹ There was no convincing evidence that the assumptions of perfect credibility and marginal interventions hold true.²

To improve the empirical performance of the basic Krugman model, extensions of the basic model were developed to capture features of intra-marginal interventions and imperfect credibility. Froot and Obstfeld (1991) and Delgado and Dumas (1992) extend the model to incorporate mean reversion of fundamentals. The driving forces behind this mean-reverting property are central bank intervention in the target zone, or "stability speculation" by market participants, producing forces that pull the exchange rate back to its central parity whenever it drifts too far from it. Their effect on the expected

¹ See Ball and Roma (1993, 1994), Svensson (1993), Rose and Svensson (1994), Nieuwland *et al.* (1994), Anthony and MacDonald (1998, 1999), and Kanas (1998) for the empirical tests of the ERM currencies on the target-zone model.

² See Duarte *et al.* (2013) and Lera and Sornette (2016) for the reviews of empirical tests on the Krugman model.

exchange rate is captured by the mean-reverting drift of the fundamentals towards central parity. Bertola and Svensson (1993) extend the basic target-zone model by including a time-varying realignment risk with stochastic jumps in the central parity. This suggests the exchange rate is not necessary mean-reverting to the original central parity.³ However, no explicit exchange rate dynamics are derived from these extended models for empirical tests.

Despite Krugman's model being a benchmark for modelling target zones, empirical support has been sparse and empirical tests have most often led to rejections of the model due to the subtle non-linear relationship between the observable exchange rate and underlying unobservable fundamental value. To overcome this deficiency of the basic Krugman model, this paper employs an alternative approach to derive analytical solutions for the stochastic exchange rate dynamics with a parametric class of drift terms of the fundamental by approximating a quadratic relationship between the exchange rate and fundamentals through a power-series method, in which the key feature of the smooth-pasting condition holds. Explicit stochastic differential equations for the exchange rate dynamics and the associated probability density functions (p.d.f.) that are derived accordingly can be readily estimated empirically using market exchange rate data. The parametric class of the drift terms in the fundamental dynamics regarding how the central banks intervene, including zero-trend, symmetric and asymmetric mean reversion, are incorporated into the target-zone model, suggesting marginal and intra-marginal interventions respectively. The exchange rate dynamics is shown to follow a square-root process under the zero-trend fundamental shock (Krugman's model); and a generalised double square-root process under the symmetric mean-reverting fundamental shock; and a mean-reverting square-root process under the asymmetric mean-reverting fundamental shock. As shown in Lo *et al.* (2015) and Hui *et al.* (2016), the asymmetric mean-reverting fundamental dynamics also allow the exchange rate to breach a currency band (i.e., realignment or abandoning the target zone) and a breakdown of the smooth-pasting boundary condition under restricted conditions of the relationship between the parameters of the drift term and stochastic part of the process. The proposed analytical approach enables direct empirical tests for exchange rate dynamics under the target-zone model with a parametric class of drift terms of the fundamental representing different central banks' intervention and realignment policy.

³ Also see Svensson (1992), Christiansen *et al.* (1998) and De Jong *et al.* (2001) for reviews on modelling of exchange rates in target zones.

To test the exchange rate dynamics in a target zone, we use the currencies of the countries (Belgium, Denmark, France, Ireland, Italy, Spain and the UK)⁴ under the ERM. We also employ the Hong Kong dollar (HKD), which is under a linked exchange rate system with the US dollar. The empirical results demonstrate that the derived exchange rate dynamics adequately fit the exchange rate data on the ERM currencies before the realignment or abandoning of the target-zone regime in the ERM crisis, and the HKD under speculative attacks during the Asian financial crisis in 1997-8 when the monetary authority intervened to prevent its depreciation against the US dollar across the weak-side limit.

The paper is organised as follows. We discuss the approximation of a quadratic relationship between the exchange rate and fundamental in the following section. The exchange rate dynamics and its associated p.d.f. under different fundamental shocks are derived in section 3. Empirical estimates of the exchange rate dynamics and p.d.f. for the currencies under the ERM and the HKD are presented in section 4. The final section of the paper concludes.

2. Analytical relationship between exchange rate and fundamental

In this section, we illustrate how a quadratic relationship between the exchange rate and fundamental is established by solving the differential equation for the exchange rate in a target zone with a smooth-pasting boundary condition using a power-series approximation method. We consider the exchange rate S defined as a domestic currency value of a unit of a foreign currency, and let S_L and S_U be the strong-side and weak-side limits respectively. With no loss of generality, the normalised log exchange rate s is defined by:

$$s = \ln \left(\frac{S - S_L}{S_U - S_L} \right) \quad (1)$$

where $-\infty < s \leq 0$, $s = -\infty$ and $s = 0$ correspond to the strong-side and weak-side limits respectively. Regarding a one-sided target zone, S_L is set as zero and S_U is either the strong-side or

⁴ The exchange rate of the Portuguese escudo at the ERM period is not available.

weak-side limit depending on the establishment of the exchange rate system.⁵ To establish the relationship between the exchange rate and fundamental in the target-zone model, we use the basic log linear model of the exchange rate on which most of the target-zone literature is based for a small open economy. The log exchange rate at time t , $s(t)$, is equal to a “fundamental”, $f(t)$, plus a term proportional to the expected change in the log exchange rate:

$$s(t) = f(t) + \alpha \frac{E[ds(t)]}{dt} \quad (2)$$

where α is the absolute value of semi-elasticity of the exchange rate with respect to its expected rate of change, and E is the expectation operator.

The Krugman model assumes a monetary process of exchange rate determination, in which the “fundamental” (f) is the source of uncertainty. The “fundamental” could be a combination of the foreign and domestic money supplies, real incomes, money demand disturbance and real exchange rate movements. Along with this combination, we assume the fundamental is the sum of two components:

$$f(t) = m + v(t), \quad (3)$$

where m is the logarithm of the constant money supply and $v(t)$, which follows a stochastic process, is the logarithm of a general-purpose term encompassing changes in real output, money demand, and anything else other than the money supply and expected currency depreciation or appreciation. The variable $v(t)$ also includes exogenous determinants of the exchange rate that the authority cannot influence. The central bank is prepared to change m by reducing the money supply with selling its foreign reserves to prevent s from rising above the weak-side limit in the case of capital outflows.

The fundamental is assumed to follow a stochastic process with drift μ_v and instantaneous standard deviation σ_v :

⁵ For example, on 6 September 2011, the Swiss National Bank (SNB) put a ceiling on the value of the Swiss franc by imposing an explicit upper bound on the exchange rate of the Swiss franc at CHF 1.2 per euro (1/1.2 EUR/CHF) and vowed to enforce this limit “with utmost determination” and to buy foreign currency “in unlimited quantities”. This measure implemented by the SNB effectively established a one-sided target-zone system for the Swiss franc with a strong-side convertibility undertaking at 1/1.2 EUR/CHF.

$$df = dv = \mu_v dt + \sigma_v dZ, \quad (4)$$

where dZ is a Wiener process with $E[dZ] = 0$ and $E[dZ^2] = dt$. In the Krugman (1991) model, the drift term μ_v of the fundamentals in Eq.(4) is zero. We apply Ito's lemma to Eqs.(2) and (4), and have

$$\frac{E[ds(t)]}{dt} = \mu_v \frac{ds}{dv} + \frac{1}{2} \sigma_v^2 \frac{d^2s}{dv^2}. \quad (5)$$

Then substituting Eqs.(2) and (3) into Eq.(5) yields

$$\frac{1}{2} \alpha \sigma_v^2 \frac{d^2s}{dv^2} + \alpha \mu_v \frac{ds}{dv} - s = -v - m, \quad (6)$$

which is a second-order linear ordinary differential equation. Given a constant-drift fundamental, including the zero drift in the Krugman model, the general solution for Eq.(6) is as follows:

$$s = A_+ e^{\lambda_+ v} + A_- e^{\lambda_- v} + v + m + \alpha \mu_v, \quad (7)$$

where

$$\lambda_{\pm} = \frac{\mu_v}{\sigma_v^2} \left[\pm \sqrt{1 + \frac{2}{\alpha} \left(\frac{\sigma_v}{\mu_v} \right)^2} - 1 \right]. \quad (8)$$

Here A_+ and A_- are arbitrary constants to be determined by the prescribed boundary condition at the fundamental of $v = 0$ corresponding to the weak-side limit:

$$s(0) = 0$$

$$\left. \frac{ds(v)}{dv} \right|_{v=0} = 0, \quad (9)$$

where the former condition ensures a proper normalisation of the exchange rate and the latter is the smooth-pasting boundary condition suggesting an optimal boundary condition for the process. The two constants can be easily found to be:

$$A_{\pm} = \pm \frac{1}{\lambda_{\pm}} \left(\frac{\lambda_{\mp} - C}{\lambda_{+} - \lambda_{-}} \right) \quad (10)$$

where

$$C = \frac{2m}{\alpha\sigma_v^2}. \quad (11)$$

Since $\lambda_{+} < 0$, $\lambda_{-} > 0$ and $\lambda_{+} - \lambda_{-} < 0$, it is not difficult to show that $A_{-} < 0$, which in turn suggests that $e^s \rightarrow 0$ as $v \rightarrow \infty$ for the strong-side limit. An explicit inaccessible boundary at the strong-side limit ensures the exchange rate will not breach the limit.⁶ If the central bank's intervention against speculative attacks only occurs at the weak-side limit, the inaccessible boundary shares the same property of a one-sided target zone in which there is no upper limit on v . At both boundaries, there is no foreseeable jump in the exchange rate, i.e., no arbitrage condition.

By applying the Taylor expansion to Eq.(7), the leading-order term of the solution s is simply given by:

$$s = -\frac{1}{2}Cv^2 + A_{+} \sum_{n=3}^{\infty} \frac{(\lambda_{+}v)^n}{n!} + A_{-} \sum_{n=3}^{\infty} \frac{(\lambda_{-}v)^n}{n!} \approx -\frac{m}{\alpha\sigma_v^2}v^2 \quad (12)$$

As shown in Appendix A, exchange rate s can be approximated as a quadratic function of the fundamental v . In other words, v exhibits a square-root dependence of s . The solution satisfies the smooth-pasting boundary condition at the weak-side limit and the inaccessible boundary at the strong-side limit. The volatility of s can be obtained by applying Ito's lemma to Eq.(12) and is expressed as

$$\sigma_s = \sigma_v \frac{ds}{dv} = \frac{2m}{\alpha\sigma_v}v \quad (13)$$

Given that v is proportional to the square-root of s , Eq.(13) illustrates that σ_s also exhibits a square-root dependence of s and suggests that the exchange rate dynamics follow a square-root process. The exchange rate volatility vanishes at the weak-side boundary, which is a property associated with

⁶ It also assumes that intervention policy always functions at the strong-side limit by simply selling the domestic currency to the market.

the smooth-pasting boundary condition. This finding is consistent with the analysis of the Swiss franc under a strong-side target zone during 2011-2015 done by Lera and Sornette (2016). They show that testing the exchange rate volatility for the square-root dependence on the exchange rate close to the band barrier is an important component of Krugman's model. However, our analysis is different from theirs in two aspects. First, the exchange rate dynamics expressed in Eqs.(12) and (13) are applied to the entire target zone, i.e., not only close to the band barrier. Second, the smooth-pasting boundary condition is applied to the weak-side limit rather than the strong-side limit in the case of the Swiss franc.

3. Derivation of exchange rate dynamics for a parametric class of drift terms of the fundamental

Given the quadratic relationship between the exchange rate and fundamental, in this section we derive the exchange rate dynamics explicitly from a parametric class of drift terms of the stochastic fundamental, including zero-trend and mean-reverting forces, regarding how central banks intervene. A zero drift for the fundamental v used in the basic Krugman model as defined in Eq.(4) suggests central banks only conduct marginal intervention in a target zone. Extensions of the basic model were then developed to capture features of intra-marginal interventions. Froot and Obstfeld (1991) and Delgado and Dumas (1992) incorporate a simple way to model such interventions with imperfect credibility by specifying the fundamental following a constant trend and the drift term of the fundamental towards central parity proportional to the deviation from central parity, i.e., mean reversion. The constant trend suggests the domestic currency is inherently weak (strong) with a negative (positive) μ_v , if there is no intervention. The driving force behind a mean-reverting property is central bank intervention in the band (intra-marginal intervention), reflecting a policy of "leaning against the wind". "Stability speculation" by market participants would also produce forces to pull the exchange rate back to its long-run equilibrium whenever it drifted too far from it. The mean-reverting fundamental can represent an error-correction policy on the part of the authorities by conducting interventions in the foreign exchange market.

During the ERM crisis, speculations in the foreign exchange market and their induced interventions were more intensive on the weak side than the strong side of the band with 85% of all European Monetary System interventions done intra-marginally. The corresponding mean-reverting fundamental shock due to interventions is likely to be asymmetric with stronger force at the weak side. Regarding the Swiss franc's one-sided target zone during 2011–2015, the Swiss National Bank was reported to have engaged in intra-marginal intervention in order to defend its currency whenever the currency appreciated towards the strong-side limit, suggesting that the mean-reverting fundamental was asymmetric.⁷ These observations of the central banks' interventions allow us to extend the analysis to generalise the fundamental shock in Eq.(4) as:

$$\begin{aligned} dv &= \mu_v dt + \sigma_v dZ \\ &= \frac{1}{2} \left(-\lambda v - \kappa + \frac{4\beta - \sigma^2}{4v} \right) dt + \frac{\sigma}{2} dZ \end{aligned} \quad (14)$$

for β, κ and $\sigma \geq 0$, and $\sigma_v = \sigma/2$. Eq.(14) generates a class of parametric drifts, including a zero drift ($\kappa=0, \beta = \sigma^2/4$ and $\lambda=0$), symmetric mean-reverting drift ($\beta = \sigma^2/4$) and asymmetric mean-reverting drift ($\kappa=0$). The detailed exchange rate dynamics and p.d.f. associated with these fundamental shocks are studied analytically in this section and used for empirical tests in the next section.

While the asymmetric mean-reverting fundamental keeps a smooth-pasting boundary condition on the weak (or strong)-side limit of the target zone, Lo *et al.* (2015) and Hui *et al.* (2016) show the exchange rate can breach the limit (i.e., realignment or abandoning the target zone) and therefore evoke a breakdown of the smooth-pasting boundary condition under restricted conditions of the relationship between the parameters of the drift term and stochastic part of the process. The exchange rate is thus quasi-bounded in the target zone.^{8,9}

⁷ See footnote 5 above.

⁸ Hui *et al.* (2016) find empirical evidence that the quasi-bounded process can describe the exchange rate dynamics and interest rate differential of the Swiss franc against the euro during the target zone regime of September 2011 to January 2015. While the exchange rate was bounded below the strong-side limit during most of the time, as indicated by its dynamics, the condition for breaching the limit was met in November 2014 using only information until that point, i.e., about two months before abandoning the limit. The asymmetric mean-reverting fundamental therefore incorporates both the characteristics of intervention and realignment.

3.1 Zero-trend fundamentals – Krugman model

If we put $\beta = \sigma^2 / 4$ and $\lambda = 0$, the drift term in Eq.(14) of the fundamental dynamics becomes a constant:

$$\mu_v = -\frac{1}{2} \kappa. \quad (15)$$

In the Krugman model, κ is set to be zero.¹⁰ The approximate solution in Eq.(12) for the exchange rate x is used to derive the exchange rate dynamics. It is convenient to use the notation $x \equiv -\alpha\sigma_v^2 s / m$, such that the exchange rate is positive with $0 \leq x < \infty$ and $x = 0$ corresponds to the weak-side limit. Alternatively, by scaling v , we can choose the factor $\alpha\sigma_v^2 / m$ in Eq.(12) equal to one. By applying Ito's lemma to Eqs.(14) and (15) with Eq.(12) (i.e., $x = v^2$), the dynamics of x is shown to follow a square-root (SR) process:

$$dx = \frac{\sigma^2}{4} dt + \sigma\sqrt{x}dZ \quad (16)$$

where $\sigma^2 x$ is the variance that depends upon the level of x . An analysis of the boundary condition indicates that it is an entrance boundary at the origin such that the exchange rate can never be lower than zero. Regarding the boundary at $x = \infty$, as the instantaneous variance σ_x^2 / x of the fractional change in x (i.e., dx/x) is a decreasing function of x , the strong-side limit is also inaccessible.¹¹

The zero drift of the fundamental in Eq.(15) provides implications for the exchange rate dynamics in Eq.(16). When the exchange rate is close to the origin (the weak-side limit), the standard deviation

⁹ Such a property is similar to the bounded exchange rate dynamics in Ingersoll (1996) and Larsen and Sørensen (2007) in which the variance of the exchange rate vanishes at both the weak-side and strong-side limits in a two-sided target zone. In their models the exchange rate is completely bounded under all circumstances determined by the model parameters. However, the exchange rate following the quasi-bounded process can breach the limit under particular conditions.

¹⁰ The exchange rate dynamics under a non-zero constant-trend fundamental are presented in Appendix B.

¹¹ The boundary behaviour is determined by the values of the parameters of the process. See Karlin and Taylor (1981, ch. 15).

$\sigma\sqrt{x}$ also becomes very small, which dampens the effect of the stochastic shock on the exchange rate. The exchange rate dynamics become dominated by the drift factor. Because $\sigma^2/4 > 0$, the drift is positive and pushes the exchange rate away from the origin if x reaches zero. There is therefore an inaccessible reflecting boundary at $x = 0$, suggesting that the corresponding target zone is always credible. We interpret such reflecting dynamics on the weak-side limit as one in which intervention occurs at the margin only. The exchange rate dynamics share the same stochastic property with a reflecting boundary condition proposed in Krugman (1991). However, no explicit exchange rate dynamics are presented in his work because of the complicated nonlinear equations between the exchange rate and fundamental. Therefore, it is not possible to have direct empirical tests on his model specifications using the market exchange rate data.

The exchange rate dynamics under the SR process demonstrates that there may be forces or incentives for market participants to drive the exchange rate away from its weak-side limit, not least that the probability of making money by holding a long position in the currency is almost zero when the exchange rate depreciates very close to its weak-side limit. This behaviour leads to a result such that the exchange rate dynamics in a target zone appear less sensitive to changes in the fundamental than the corresponding free-floating exchange rate. The effect of the fundamental on the exchange rate decreases as the exchange rate moves toward the limits of the band. This feature is called the “honeymoon effect”, which suggests there is an inherent stabilising mechanism in a perfectly credible exchange rate target zone, as in the Krugman model.

The p.d.f. of the exchange rate dynamics in Eq.(16) at time t' , x' , conditional on its current value x , is obtained from the density of the square of a reflected Brownian motion:¹²

$$\Phi(x, t; x', t') = \frac{1}{\sqrt{2\pi x \sigma^2 (t-t')}} \left[\exp\left(\frac{-2(\sqrt{x} - \sqrt{x'})^2}{\sigma^2 (t-t')}\right) + \exp\left(\frac{-2(\sqrt{x} + \sqrt{x'})^2}{\sigma^2 (t-t')}\right) \right]. \quad (17)$$

where $N(\cdot)$ is the cumulative standard normal distribution function.

¹² The p.d.f. is obtained from Eq.(B2) in Appendix B by setting $\kappa = 0$.

3.2 Mean-reverting fundamentals

The general solution in Eq.(7) cannot be applied to Eq.(6) with the generalized drift term of the fundamental in Eq.(14), as the drift term depends on v . To solve Eq.(6), we apply the power series method used in Lo *et al.* (2015) and Hui *et al.* (2016) as presented in Appendix A. By keeping the leading-order term in Eq.(A2), the power series solution is reduced to a simple quadratic relationship between the exchange rate s and the fundamentals v :

$$s(v) = A_0 v^2. \quad (18)$$

To illustrate the exchange rate dynamics, it is convenient to use the notation $x \equiv -s$. By applying Ito's lemma to Eq.(14) with Eq.(18), the corresponding dynamics of the exchange rate x follows a generalised double square-root (DSR) process:

$$dx = (\tilde{\beta} - \tilde{\kappa}\sqrt{x} - \lambda x)dt + \tilde{\sigma}\sqrt{x}dZ \quad (19)$$

where

$$\tilde{\beta} = \beta|A_0| = \frac{m}{\alpha}, \quad (20)$$

$$\tilde{\kappa} = \kappa\sqrt{|A_0|} = \kappa\sqrt{\frac{m}{\alpha\beta}}, \quad (21)$$

$$\tilde{\sigma} = \sigma\sqrt{|A_0|} = \sigma\sqrt{\frac{m}{\alpha\beta}}. \quad (22)$$

It is noted that the exchange rate dynamics also follow the generalised DSR process when the mean-reversion of the fundamental is symmetric with $\beta = \sigma^2 / 4$ in Eq.(14). However, there is no analytical p.d.f. for Eq.(19). Given that currency interventions are usually only carried out on either the weak-

side or strong-side of a currency band, we assume $\kappa = 0$ such that an asymmetric mean-reverting shock in the fundamental is used as a simplified case of Eq.(14):¹³

$$dv = \frac{1}{2} \left(-\lambda v + \frac{4\beta - \sigma^2}{4v} \right) + \frac{\sigma^2}{2} dZ. \quad (23)$$

where λ and β determines the drift speed and the level of the fundamental respectively.¹⁴ We will then show that these two parameters are explicitly related to the mean-reverting process of the exchange rate. The drift term in Eq.(23) exhibits a mean-reverting property for the fundamental similar to the mean reversion considered by Delgado and Dumas (1992). When the term $(4\beta - \sigma^2) > 0$ and v is large (far away from the origin), the first term of the drift will push v down and towards the origin, such that the domestic currency will depreciate accordingly. Conversely, when v is small (near zero), the second term of the drift in Eq.(23) will push v upward and away from zero. The corresponding domestic currency will appreciate and the exchange rate will move away from its weak-side limit. However, the mean-reverting force in Eq.(23) is not symmetric. The restoring force (domestic currency appreciation) given by the second term with v close to zero is stronger than the force (domestic currency depreciation) provided by the first term. Such an asymmetric mean-reverting property with a strong force pushing the exchange rate away from the origin (the weak-side limit) is consistent with the idea that a central bank's intervention is more intensive when the currency depreciates toward the weak-side limit in order to maintain the credibility of the target-zone regime.

By applying Ito's lemma to Eq.(23) with Eq.(18), the corresponding dynamics of the exchange rate x follows a mean-reverting square-root (MRSR) process:

$$dx = \lambda \left(\frac{\tilde{\beta}}{\lambda} - x \right) dt + \tilde{\sigma} \sqrt{x} dZ \quad (24)$$

¹³ A symmetric mean-reverting shock in the fundamental with $\beta = \sigma^2 / 4$ gives the same exchange rate dynamics as in Eq.(25)

with $\tilde{\kappa} = \frac{2\kappa}{\sigma} \sqrt{\frac{m}{\alpha}}$ and $\tilde{\sigma} = 2\sqrt{\frac{m}{\alpha}}$. There is, however, no analytical p.d.f. for this dynamics.

¹⁴ Eq.(23) is a well-known generalisation of the Rayleigh process. The generalised Rayleigh process is a diffusion process with mean reversion, of which some stochastic processes, such as the Ornstein-Uhlenbeck process, are special cases. It has been considered in the context of the path-dependent options models used in economics and stochastic finance studies.

where λ determines the speed of the mean-reverting drift towards the long-term mean $\tilde{\beta}/\lambda$. When the exchange rate is close to zero, the standard deviation $\tilde{\sigma}\sqrt{x}$ also becomes very small. The exchange rate dynamics become dominated by the mean-reverting drift, which pushes the exchange rate towards the equilibrium. The presence of mean reversion will help to move the exchange rate away from its limit, enhancing the “honeymoon effect”. The long-term mean $\tilde{\beta}/\lambda$ associated with the exchange rate dynamics is a time-varying equilibrium level, which can be determined either by the monetary authority through intra-marginal intervention, or through action (or incentives) by market participants to drive the exchange rate towards its mean level. Such effects increase the mean-reverting force, determined by the size of λ . Conversely, if market participants believe the target zone is not credible, their speculative attacks will weaken the restoring force (i.e., smaller λ) towards its mean level.

The p.d.f. of x under the MRSR process is given by:

$$G(x, t; x', t') = \frac{2}{\tilde{\sigma}^2 C_1(t-t')} \left(\frac{x}{x'}\right)^{\omega/2} \exp\left[-\frac{\omega+2}{2} C_2(t-t')\right] \times \exp\left\{-\frac{2x'+2x \exp[-C_2(t-t')]}{\tilde{\sigma}^2 C_1(t-t')}\right\} \times I_{\omega} \left\{ \frac{4x^{1/2} x'^{1/2} \exp[-C_2(t-t')/2]}{\tilde{\sigma}^2 C_1(t-t')} \right\} \quad (25)$$

where $\omega = 2\tilde{\beta}/\tilde{\sigma}^2 - 1$, $C_1(\tau) = [\exp(\lambda\tau) - 1]/\lambda$, $C_2(\tau) = -\lambda\tau$, I_{ω} is the modified Bessel function of the first kind of order ω . Using Feller's classification of boundary points, it can be inferred that there is a non-attractive natural boundary at infinity and that the one at the origin is a boundary of no probability leakage for $(4\beta - \sigma^2) > 0$ in Eq.(23) and $(4\tilde{\beta} - \tilde{\sigma}^2) > 0$ in Eq.(24), and it is not otherwise.¹⁵ The no-leakage condition ensures the exchange rate will not breach the origin (the weak-side limit) and the target zone is credible; otherwise, the exchange rate may pass through the boundary, i.e., the target zone is quasi-bounded at the origin. If the no-leakage condition does not

¹⁵ For the definitions of the boundary conditions, see Karlin and Taylor (1981).

hold at the boundary, the smooth-pasting condition of Eq.(9) may break down in the proposed model, and therefore realignment or abandoning of the target zone could occur. Given the p.d.f. in Eq.(25), the parameters of the MRSR process for the exchange rate dynamics can be tested using market exchange rate data.¹⁶

In the following section, we test the SR and MRSR processes of the exchange rate dynamics empirically using the currencies of the countries (Belgium, Denmark, France, Ireland, Italy, Spain and the UK) under the ERM in which their exchange rates eventually breached the weak-side limit, and the HKD during the Asian financial crisis in 1997-8, when the monetary authority successfully intervened in the exchange rate market to prevent currency depreciation against the US dollar across the peg rate.

4. Empirical test of exchange rate dynamics

4.1 Data

In this section, we investigate whether the SR and MRSR processes for exchange rate dynamics presented in the previous section can describe the movements of exchange rates in currency bands. We use the currencies under the ERM in the early 1990s and the HKD. The maximum likelihood estimation (MLE) using daily data is employed to estimate the model parameters of the SR process specified in Eq.(16) and the MRSR process in Eq.(24) based on a log-likelihood function that is constructed by the analytical p.d.f of Eqs.(17) and (25) respectively.¹⁷

The European Economic Community adopted a two-sided target zone system, which is also known as the ERM, in March 1979, and it was replaced with the ERM II on 1 January 1999. Under the ERM of 1979, each member country was required to maintain its exchange rate with the European Currency Unit (ECU) within certain bands. In September 1992, Italy and the UK were required to leave the ERM

¹⁶ Similar to the DSR process, the p.d.f. of the MRSR process will eventually approach the steady-state limit. See Lo *et al.* (2015).

¹⁷ The data used in this section are from Bloomberg.

because they could not maintain their currencies within the bands. There were realignments of the bands of the depreciated Portuguese escudo and Spanish peseta in September and November 1992. In August 1993, the bands for five member countries (Belgium, France, Ireland, Portugal and Spain) were relaxed to $\pm 15\%$.

We study seven countries under the ERM: Belgium, Denmark, France, Ireland, Italy, Spain and the UK. The currencies had bands of $\pm 2.25\%$ (Belgium, Denmark, France, Ireland and Italy) and $\pm 6\%$ (Spain and the UK) around parity with the Deutsche mark or ECU. In practice, the Deutsche mark is used as the reference currency for their exchange rates to determine the bands. The estimation periods are 1 January 1990 - 29 July 1993 for Belgium; 1 January 1990 - 31 July 1993 for Denmark; 1 January 1990 - 29 July 1993 for France; 30 March 1990 - 24 December 1992 for Ireland; 1 January 1990 - 21 August 1992 for Italy; 1 January 1990 - 15 September 1992 for Spain; and 17 May 1990 - 15 September 1992 for the UK. The sample periods end when either the countries' exchange rate bands were abandoned or realignment occurred. We use a rolling one-year window to estimate the parameters of the model. The corresponding bands of the currencies against the Deutsche mark are listed in Table 1.

Since 1983, Hong Kong has operated the Linked Exchange Rate System (LERS) whereby the HKD is fixed at a rate of 7.8 per US dollar (USD), i.e., a one-sided target zone.¹⁸ The estimation period of the HKD is 1 March 1995 - 31 March 1999 which includes the Asian financial crisis of 1997-98 when the HKD was under speculative attacks. Given that the monetary authority's intervention was targeted at the rate of 7.75, the normalised log exchange rate x of the HKD is scaled according to the lower limit at 7.75.¹⁹ The estimations use a rolling two-year window. The spot exchange rates S , corresponding normalized log exchange rates x and their bands of the currencies of Belgium, Denmark, France, Ireland, Italy, Spain and the UK under the ERM, and the HKD are shown in Figure 1.

¹⁸ The LERS was refined into a two-sided target zone system in May 2005 when a symmetric convertibility zone was introduced with a strong-side convertibility undertaking at 7.75 HKD/USD and the weak-side convertibility undertaking at 7.85. See Genberg and Hui (2011) about the development of the LERS.

¹⁹ The normalized log exchange rate x of the HKD scaled according to the lower limit at 7.8 (the official fixed rate) is also used for the estimations. The results are qualitatively the same as those based on 7.75 and available upon request.

4.2 Estimation results of SR process

The estimation results for the ERM currencies under the SR process are presented in Figure 2. The estimated volatilities σ of the currencies shown in Figure 2 range between 0.02 and 0.5, where the Spanish peso and Belgian franc have the highest and lowest volatilities respectively. The corresponding z-statistic is much higher than 1.96 (i.e., at 5% significance level), indicating that the estimated σ is highly significant. This suggests the estimation of the SR dynamics is robust. The empirical result indicates that the ERM currencies' exchange rates follow the SR, as derived from the Krugman model in the previous section. The estimated σ of the currencies of Belgium, Denmark, France and Ireland increased over time, reflecting that their exchange rates became more volatile before their realignments. On the other hand, σ of the currencies of Italy, Spain and the UK decreased during the estimation periods. Their exchange rate dynamics could be influenced by the central banks' interventions that reduced exchange rate volatility.

The estimated σ for the HKD shown in Figure 2 is significant at the level of 0.004-0.013 with the corresponding z statistic much higher than 1.96 (i.e., at 5% significance level). The results are similar to those for the ERM currencies, demonstrating that the HKD follows the SR process with the zero-trend fundamental as suggested in the Krugman model. It is noted that the HKD exchange rate volatility increased in August 1997 when the Asian financial crisis intensified and became flat after September 1998.

In summary, the estimation results presented in Figure 2 based on the MLE for the p.d.f. in Eq.(17) provide evidence the square-root process adequately fits the data on the ERM currencies in two-sided target zones and the HKD in a one-sided target zone. Given that the ERM currencies were required either to leave the ERM because they were not able to maintain their currencies within the bands, or to realign their bands at the end of the estimation periods, the fully credible target-zone model can adequately describe their exchange rate dynamics. However, it does not give any information about the creditability of their target-zone regimes. The figures show that the estimated model parameters are time varying. The zero-trend fundamental, as suggested in the Krugman (1991) model, is shown to be valid in the empirical tests.

4.3 Estimation results for MRSR process

The estimation results for the ERM currencies under the MRSR process are presented in Figures 3-5. The estimated volatilities $\tilde{\sigma}$ of the currencies shown in Figure 3 range between 0.02 and 0.6, which are similar to those estimates under the SR process in Figure 2. The corresponding z-statistic is also much higher than 1.96 (i.e., at 5% significance level), indicating that the estimated $\tilde{\sigma}$ is highly significant under the MRSR process. For individual currencies, the values of $\tilde{\sigma}$ under the MRSR and their changes over time are similar to those under the SR process. This demonstrates that the estimations of the square-root process are consistently robust under the SR and MRSR models, and provides evidence that the ERM currencies' exchange rate dynamics followed the square-root process in the entire target zones (i.e., not only close to their band barriers).

Figure 4 shows the estimates of the drift term λ are significant in terms of the z-statistic (higher than the 5% significance level) for the currencies of Belgium, Denmark, Ireland, Italy and Spain, and for the French franc and British pound with λ higher than 0.05. The drift term estimations for the MRSR process show a significant mean-reverting force in the exchange rate dynamics. The changes in λ over time reflect the effects of central bank intervention in the band on the exchange rate dynamics, in particular, stronger mean reversion associated with higher λ suggests a more credible target-zone system. The drift in the French franc and British pound was sometimes weaker and even insignificant when λ was not significantly different from zero. As all the ERM currencies depreciated and breached the weak-side limit due to their exit from the ERM or realignment, their λ fell sharply at the end of the estimation periods.

Figure 5 demonstrates that the estimated $\tilde{\beta}/\lambda$ of all the currencies are statistically significant at the 5% significance level and in general trended towards the weak-side limit during the estimation periods. For the Belgian franc and British pound, their $\tilde{\beta}/\lambda$ fell sharply at the end of the estimation periods. The estimations of κ and $\tilde{\beta}/\lambda$ show mean reversion is present for the ERM currencies before realignment or abandoning their target-zone regimes. The estimation results shown in Figures 3-5 provide evidence that the MRSR process adequately fits the data on the currencies under the target-zone system of the ERM.

Figures 4 and 5 show that both λ and $\tilde{\beta}/\lambda$ dropped at the end of the estimation periods. This suggests realignment pressure building at the weak-side limit with the weakening mean-reverting force. Such pressure, indicated by the critical condition $(\tilde{\sigma}^2/4\tilde{\beta}) > 1$ of probability leakage at the weak-side limit for the currencies, is presented in Figure 6. A high value of the measure of $(\tilde{\sigma}^2/4\tilde{\beta})$ reflects deteriorating credibility of a target zone. While the measures for the ERM currencies were below one during the estimation periods, they surged at the end of the estimation periods. The results show that the currencies were bounded in their bands but their likelihood of breaching the weak-side limit increased significantly before the realignment or abandoning of the target-zone regimes. Regarding the currencies of Denmark, France and Ireland, their measures rose to the level of 0.4 several months before realignment. Such changes coincided with increases in their volatilities $\tilde{\sigma}$, and declines in the two parameters λ and $\tilde{\beta}/\lambda$ as the mean-reverting force weakened. Their exchange rate dynamics suggest that risk of probability leakage increased and the exchange rates would breach the weak-side limit. The deterioration of the credibility of their target zones measured by the leakage condition is consistent with the realignment of these ERM currencies. For the Belgian franc and British pound, while their measures of probability leakage stayed low at 0.1, they jumped higher than one at the end of the estimation periods, i.e., showing probability leakage at the weak-side limit. This illustrates that the central banks' intervention could maintain the credibility of their target zones as demonstrated by the estimated exchange rate dynamics. However, the credibility of their target zones deteriorated substantially when the central banks changed their monetary policy.

During the Asian financial crisis, the HKD was under speculative attack near the weak-side limit at the rate of 7.8 HKD/USD and the intervention was conducted at 7.75. The estimations of $\tilde{\sigma}$, λ and $\tilde{\beta}/\lambda$, presented in Panels A-C of Figure 7, are statistically significant at the 5% confidence level, suggesting that the MRSR process can adequately describe the dynamics of the HKD under a one-sided target-zone system. It is noted that both $\tilde{\sigma}$ and κ increased in late 1997 and 1998 when interventions were operated vigorously to defend the currency from depreciation. In contrast, the mean $\tilde{\beta}/\lambda$ declined during the period.

Panel D of Figure 7 shows the measure $(\tilde{\sigma}^2 / 4\tilde{\beta})$ to study the credibility of the one-sided target zone of the HKD. The measure was quite steady at the level of 0.7 before March 1998. This suggests that, during the first speculative attack on the currency in October 1997 to January 1998, the exchange rate was well bounded at the rate of 7.75 (the intervention's target rate) and the one-sided target zone was adequately credible. However, the measure began to rise in April 1998 and surged at the 1.0 level in June 1998 and, subsequently, stayed at that level until October 1998. The exchange rate dynamics suggest that, during June-October 1998, probability leakage was possible and there was a risk that the exchange rate would breach the 7.75 level. The exchange rate dynamics indicated an erosion of the credibility of the target zone under the second speculative attack from June to September 1998. However, the probability leakage measure dropped in November 1998 and fell to 0.8 after the introduction of the "seven technical measures" in September 1998.²⁰ This demonstrates that such measures were successful in enhancing the creditability of the target-zone system for the HKD. This finding is consistent with that in Genberg and Hui (2011), who demonstrate that the credibility of the target-zone system, measured by information extracted from financial asset prices about market expectations, increased with the various changes made to the system. It should be noted that the Hong Kong government's operations in the stock market in August 1998 could also be reason for the change in exchange rate dynamics in September 1998 and after.

In summary, the estimation results based on the MLE for the p.d.f. in Eq.(25) for the ERM currencies with two-sided target zones and the HKD with a one-sided target zone provide evidence that the MRSR process adequately fits their exchange rates. Therefore, their exchange rate dynamics are consistent with asymmetric mean-reverting fundamental shocks. The figures show the estimated model parameters are time varying. While the empirical results show the exchange rates of the ERM currencies were bounded in the target zones during most of the period as indicated by its dynamics, the surge of the leakage condition (i.e., $\tilde{\sigma}^2 / 4\tilde{\beta}$) at the end of the estimation periods indicate that the likelihoods of their exchange rates breaching the weak-side limits increased significantly before the realignment or abandoning of the target-zone regimes. Regarding the HKD, the condition for

²⁰ The so-called seven technical measures were introduced in September 1998 to strengthen the Linked Exchange Rate System. For our purposes, the two most significant features were (i) the introduction of the weak-side convertibility undertaking, and (ii) the introduction of a discount window facility that made it possible for banks to borrow from the Hong Kong Monetary Authority against collateral. The objective of the latter was to dampen interest rate volatility that was very high during the crisis.

breaching the 7.75 level was met in June-October 1998 using only information until that point. The dynamics of the exchange rate suggests an erosion of the credibility of the target zone under the severe second speculative attack. However, the leakage measure dropped in November 1998 and the credibility of the target-zone system resumed.

5. Conclusion

This paper provides an alternative approach to derive explicit exchange rate dynamics under Krugman's target-zone model by approximating a quadratic relationship between the exchange rate and fundamental through a power-series method. The explicit stochastic processes for the exchange rate dynamics and the associated probability density functions are derived accordingly for a parametric class of the drift terms in the fundamental dynamics, including zero-trend, symmetric and asymmetric mean-reverting stochastic fundamental, regarding how central banks intervene. The zero-trend fundamental in the Krugman model, assuming marginal intervention, is a case in the framework. This analytical approach overcomes the difficulty of capturing the non-linear relationship in the Krugman model between the observable exchange rate and unobservable fundamental value empirically, such that the exchange rate dynamics can be tested directly.

The empirical results demonstrate that the stochastic exchange rate, following the square-root processes under the Krugman model (zero-trend fundamental) and asymmetric mean-reverting fundamental, adequately fits the exchange rate data on the ERM currencies in two-sided target zones and the HKD in a one-sided target zone during the Asian financial crisis. Furthermore, the results show that the mean-reverting square-root process of the exchange rate dynamics associated with an asymmetric mean-reverting fundamental, assuming intra-marginal intervention at the weak side of a band, adequately fits the exchange rates of the ERM currencies and the HKD. Given that the exchange rate is quasi-bounded under this process, the measure for the probability leakage condition of the exchange rate breaching the weak-side limit surged before the realignment or abandoning of the target-zone regimes for the ERM currencies and during the speculative attack on the HKD. The exchange rate dynamics under the asymmetric mean-reverting fundamental shock captures the realignment risk of the currencies' target-zone regimes.

Given that the fundamental dynamics, in particular its drift term, determine the associated exchange rate dynamics, and how central banks operate their interventions in the foreign exchange market, the mechanism of central banks' operations is "mechanical" in a stochastic system. In view of this property, future research can study the efficiency of a target zone system as a stochastic mechanical system.

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Appendix A

To solve Eq.(6), we apply the power series method which is used in Lo *et al.* (2015) and Hui *et al.* (2016), namely

$$s(v) = \sum_{n=0}^{\infty} a_n v^n . \quad (\text{A1})$$

In order to have a suitable normalisation of the exchange rate and satisfy the smooth-pasting boundary condition, we require that $a_0 = a_1 = 0$. Then the power series solution can be re-written as:

$$s(v) = \sum_{n=0}^{\infty} A_n v^{n+2} . \quad (\text{A2})$$

Substituting this power series into Eq.(6), we can easily obtain

$$A_0 = -\frac{m}{\alpha\beta} \quad (\text{A3})$$

$$A_1 = -\frac{2}{3\alpha} \left(\frac{1 - 2\kappa A_0}{\beta + \sigma_v^2} \right) \quad (\text{A4})$$

$$A_n = \frac{8\kappa(n+1)}{\alpha(n+2)(n+4\beta)} A_{n-1} + \frac{4(2 + \alpha\lambda n)}{\alpha(n+2)(n+4\beta)} A_{n-2} \quad \text{for } n \geq 2 . \quad (\text{A5})$$

It is noted that A_0 is independent of κ , σ , and λ , and all the coefficients $\{A_n\}$ are negative. The series solution can be shown to be a convergent series for all v by means of the ratio test as $\lim_{n \rightarrow \infty} |A_{n+1} / A_n| \rightarrow 0$. By keeping the leading-order term in Eq.(A2), the power series solution is reduced to a simple quadratic relationship between the exchange rate s and the fundamentals v as in Eq.(18).

The series solution in Eqs.(12) and (A2) can be shown to be a convergent series for all v by means of the ratio test as

$$\lim_{n \rightarrow \infty} \left| \frac{(A_+ \lambda_+^{n+1} + A_- \lambda_-^{n+1}) / (n+1)!}{(A_+ \lambda_+^n + A_- \lambda_-^n) / n!} \right| \rightarrow 0. \quad (\text{A6})$$

This leading order term provides an upper bound to the exact solution of Eq.(7) in the case of a zero-drift fundamental in the Krugman model.

Figure A1 shows the relationship between the exchange rate S and the fundamental v using different numbers of terms ($A_0, A_0 + A_1, \dots, A_0 + \dots + A_9$) of the series solution in Eq.(A2) and parameters $\kappa = 0.1, \beta = 0.05, \lambda = 0.1, \alpha = 0.2, \sigma^2 = 0.3, m = 1$, which are broadly consistent with the estimations in section 4. With a constant-trend drift ($\beta = \sigma^2 / 4$ and $\lambda = 0$) of the fundamental, Eq.(A2) is equivalent to Eq.(12). The result demonstrates that the convergence of the series solution is fast and only its leading term contributes significantly. Using higher-order terms does not have material impact on the results, in particular on the weak side of a currency band where $\exp[s(v)] = 1$ is the weak-side limit. It is also noted that the leading term is the upper bound of the series solution. Accordingly, we can conclude that $s(v) = A_0 v^2$ in Eq.(18) is a good approximation of the exact relationship between s and v with extremely small errors.

Appendix B

In the case of constant-trend fundamentals with non-zero κ in Eq.(15), the dynamics of x is shown to follow a double square-root (DSR) process:

$$dx = \left(\frac{\sigma^2}{4} - \kappa \sqrt{x} \right) dt + \sigma \sqrt{x} dZ \quad (\text{B1})$$

The term $\kappa \sqrt{x}$ in the drift of Eq.(B1) is a nonlinear restoring force that makes the dynamics different from those in the ordinary mean-reverting process, which has a linear restoring force. It is noted that only two parameters σ^2 and κ are required to describe the exchange rate dynamics. This suggests that the long-run fundamental exchange rate cannot be specified independently of σ^2 and κ . For

instance, if σ^2 is higher then the mean exchange rate is also higher. The boundary condition is a regular (attainable) boundary at the origin when $\sigma^2, \kappa > 0$.

The constant-trend drift of the fundamental provides implications for the exchange rate dynamics in Eq.(B1), which shares some properties of the exchange rate under the SR process in section 3.1 above. Given the mean-reverting property in Eq.(B1), when the exchange rate is well above the long-term mean $\sigma^2/4$, it reverts downwards. The exchange rate dynamics share the same stochastic property with a reflecting boundary condition in Delgado and Dumas (1992) who propose a constant-trend stochastic fundamental.

The p.d.f. of the exchange rate dynamics in Eq.(B1) at time t' , x' , conditional on its current value x , is obtained from the density of the square of a reflected Brownian motion:²¹

$$\begin{aligned} \Phi(x, t; x', t') = & \frac{1}{\sqrt{2\pi\kappa\sigma^2(t-t')}} \left[\exp\left(\frac{-2(\sqrt{x} - \sqrt{x'} + \kappa(t-t')/2)^2}{\sigma^2(t-t')}\right) \right. \\ & + \exp\left(\frac{4\kappa\sqrt{x}}{\sigma^2}\right) \exp\left(\frac{-2(\sqrt{x} + \sqrt{x'} + \kappa(t-t')/2)^2}{\sigma^2(t-t')}\right) \left. \right] \cdot \quad (B2) \\ & + \frac{2\kappa}{\sigma^2\sqrt{x}} \exp\left(\frac{-4\kappa\sqrt{x}}{\sigma^2}\right) \left(1 - N\left(\frac{2(\sqrt{x} + \sqrt{x'} - \kappa(t-t')/2)}{\sqrt{\sigma^2(t-t')}}\right) \right) \end{aligned}$$

where $N(\cdot)$ is the cumulative standard normal distribution function. The distribution of the exchange rate approaches a steady-state density as $t \rightarrow \infty$. The stationary density is

$$\frac{2\kappa}{\sigma^2\sqrt{x}} \exp\left(\frac{-4\kappa\sqrt{x}}{\sigma^2}\right). \quad (B3)$$

This is the Weibull distribution with mean $\sigma^4/8\kappa^2$ and variance $5\sigma^8/64\kappa^4$.²²

²¹ The p.d.f. is given in Cox and Miller (1970, ch. 5).

²² See Johnson and Kotz (1970, ch. 20) about the Weibull distribution and its properties.

Table 1. Currency bands of ERM currencies

		Strong side limit S_U	Weak side limit S_L
Belgium	DEM/BEP	20.166	21.095
Denmark	DEM/DKK	373	390.16
France	DEM/FRF	3.2792	3.4305
Ireland	DEM/IEP	0.364964	0.381825
Italy	DEM/ITL	730.32	763.9409
Spain	DEM/ESP	61.1	68.9
UK	GBP/DEM	3.127	2.773

Figure 1: Spot rate S and log of exchange rate x of ERM currencies and Hong Kong dollar

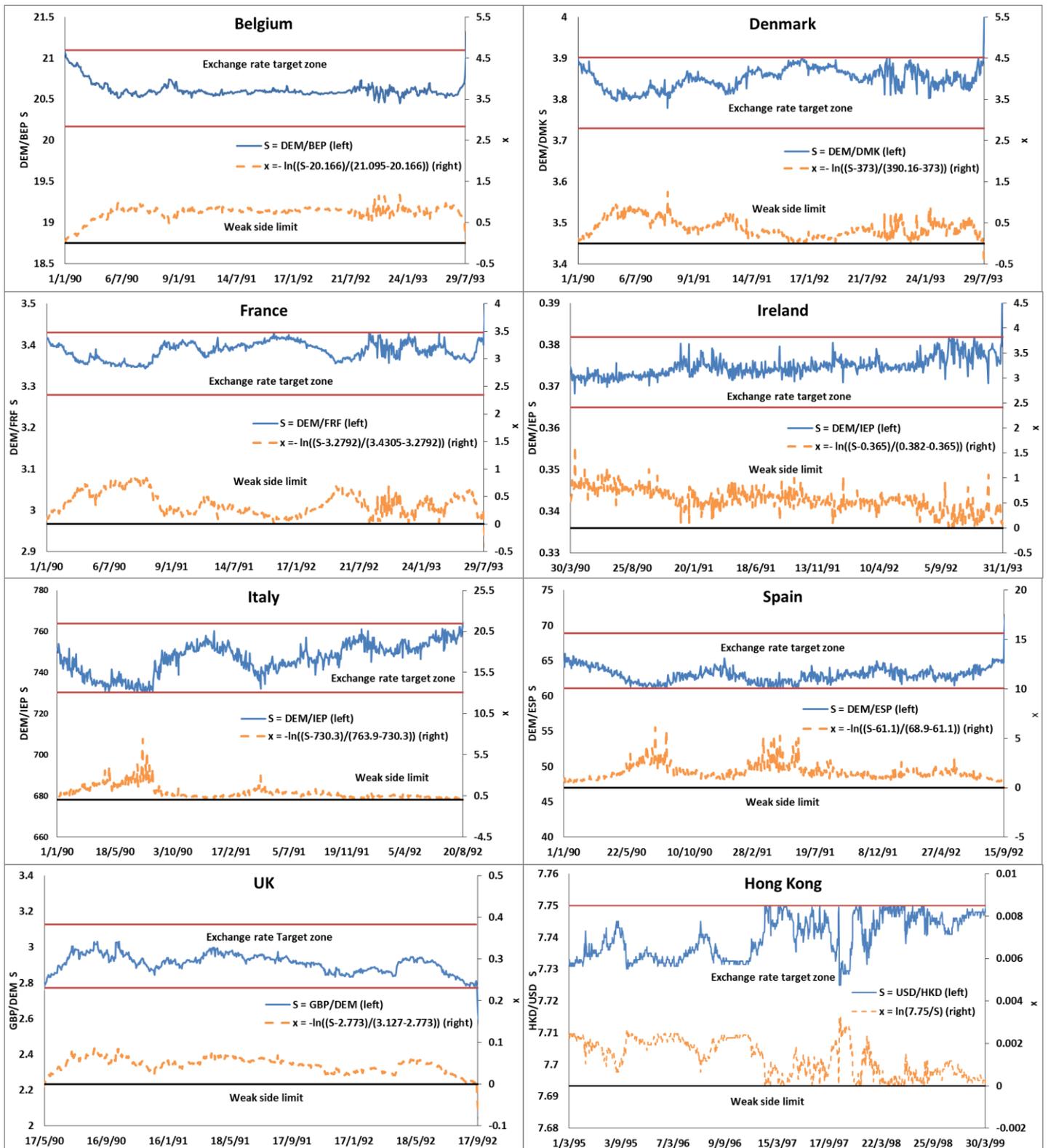


Figure 2: Estimated σ and z-statistics of exchange rates of ERM currencies and HKD under SR process

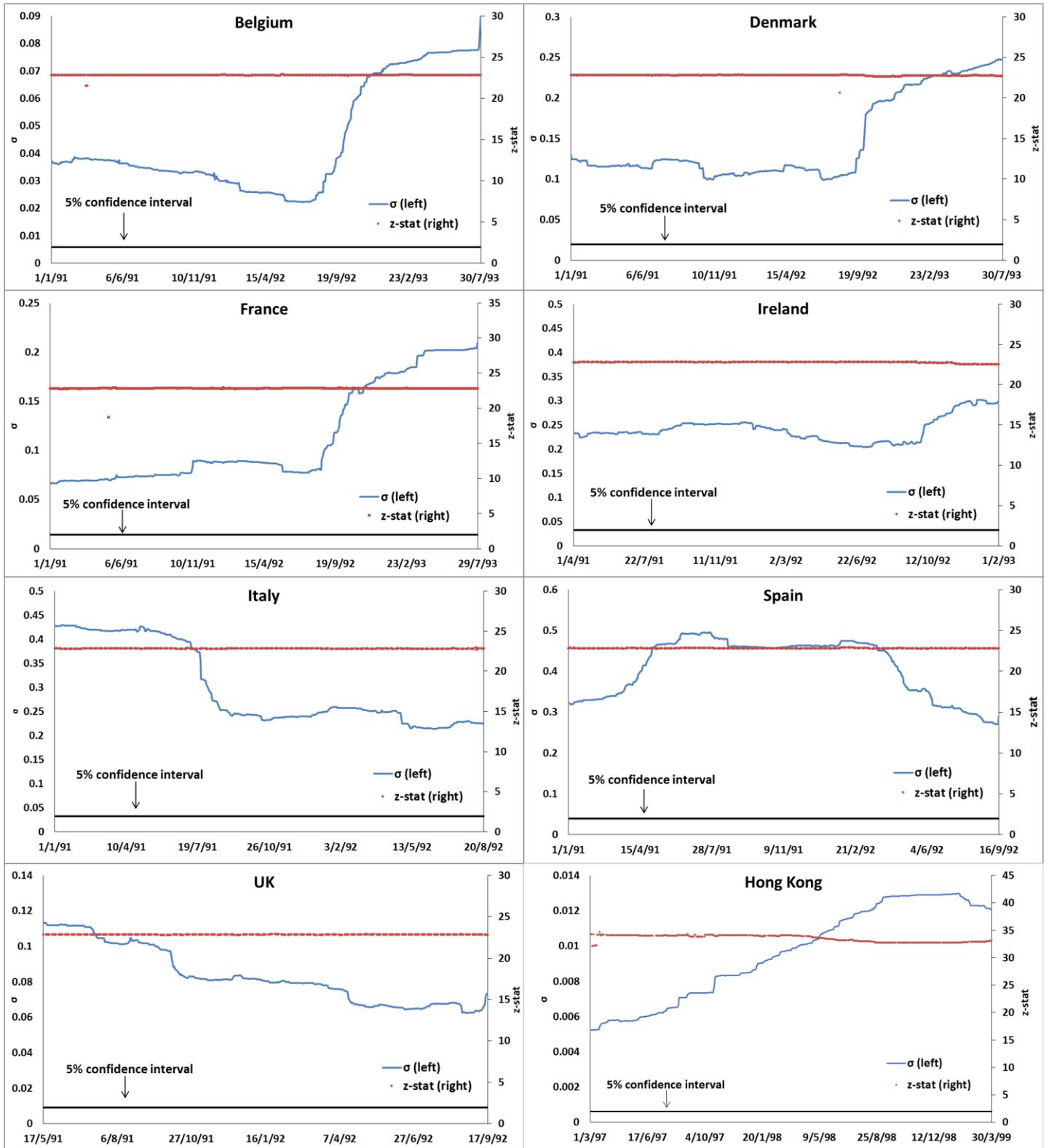


Figure 3: Estimated $\tilde{\sigma}$ and z-statistics of exchange rates of ERM currencies under MRSR process

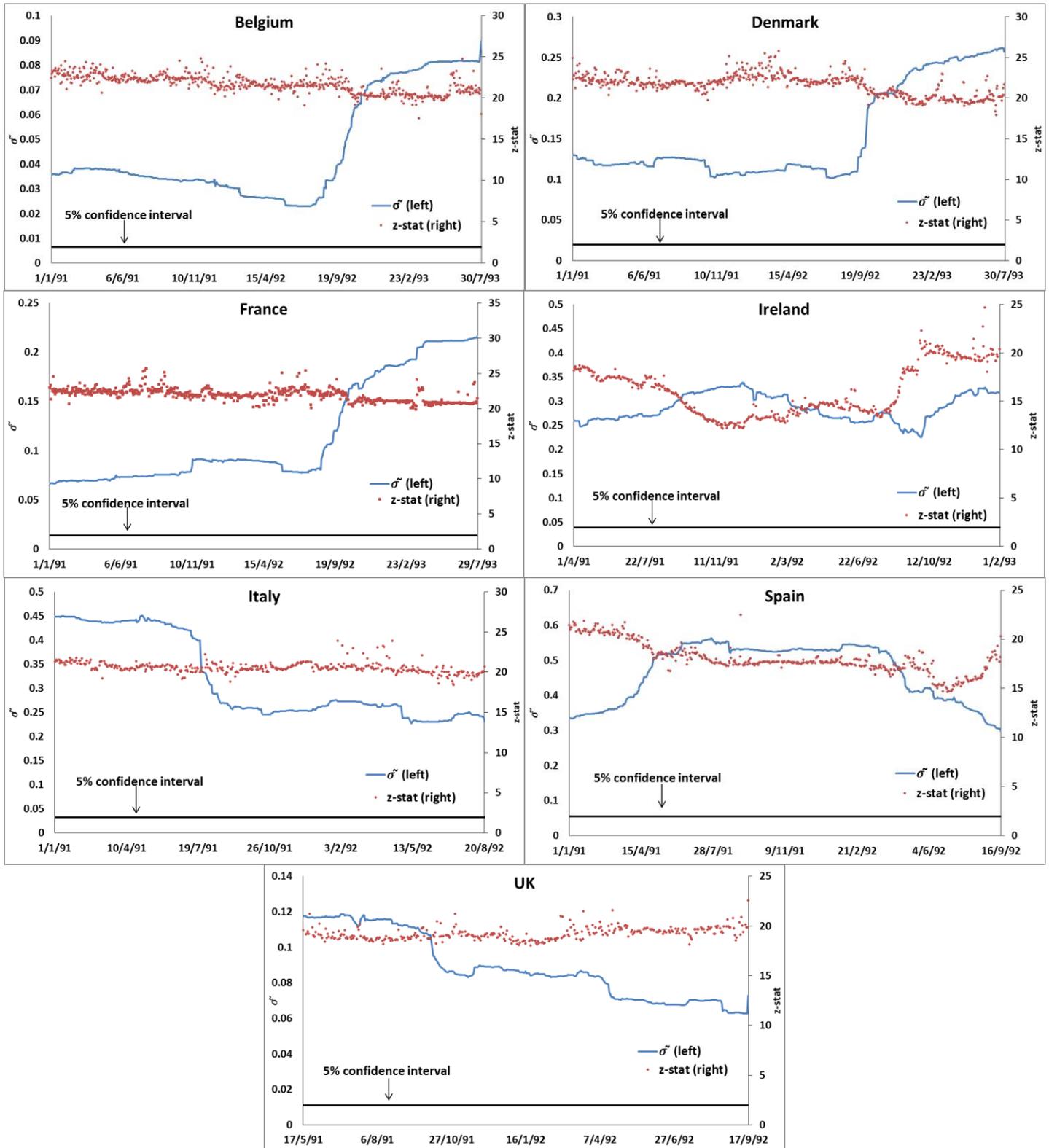


Figure 4: Estimated λ and z-statistics of exchange rates of ERM currencies under MRSR process

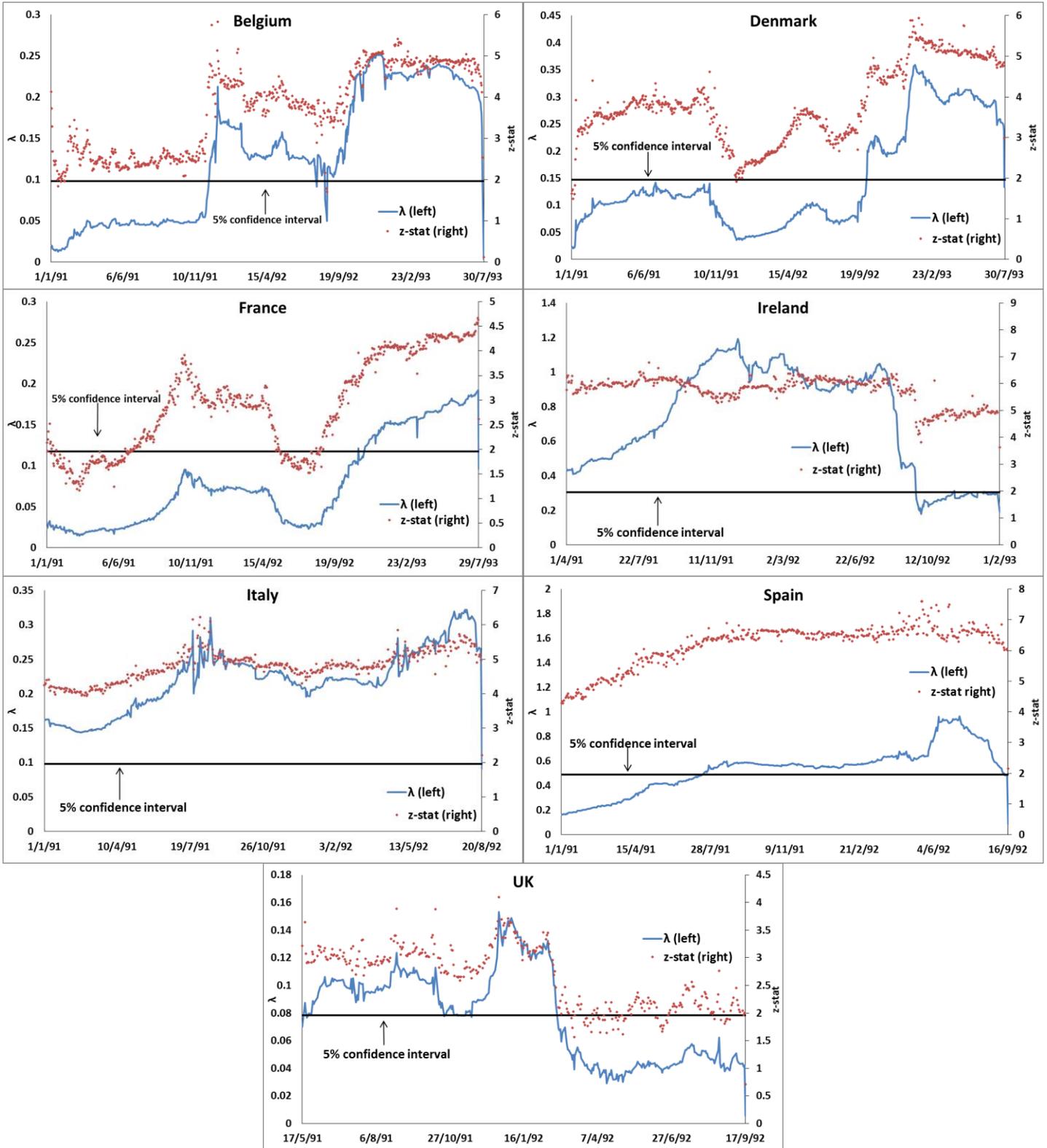


Figure 5: Estimated $\tilde{\beta}/\lambda$ and z-statistics of exchange rates of ERM currencies under MRSR process

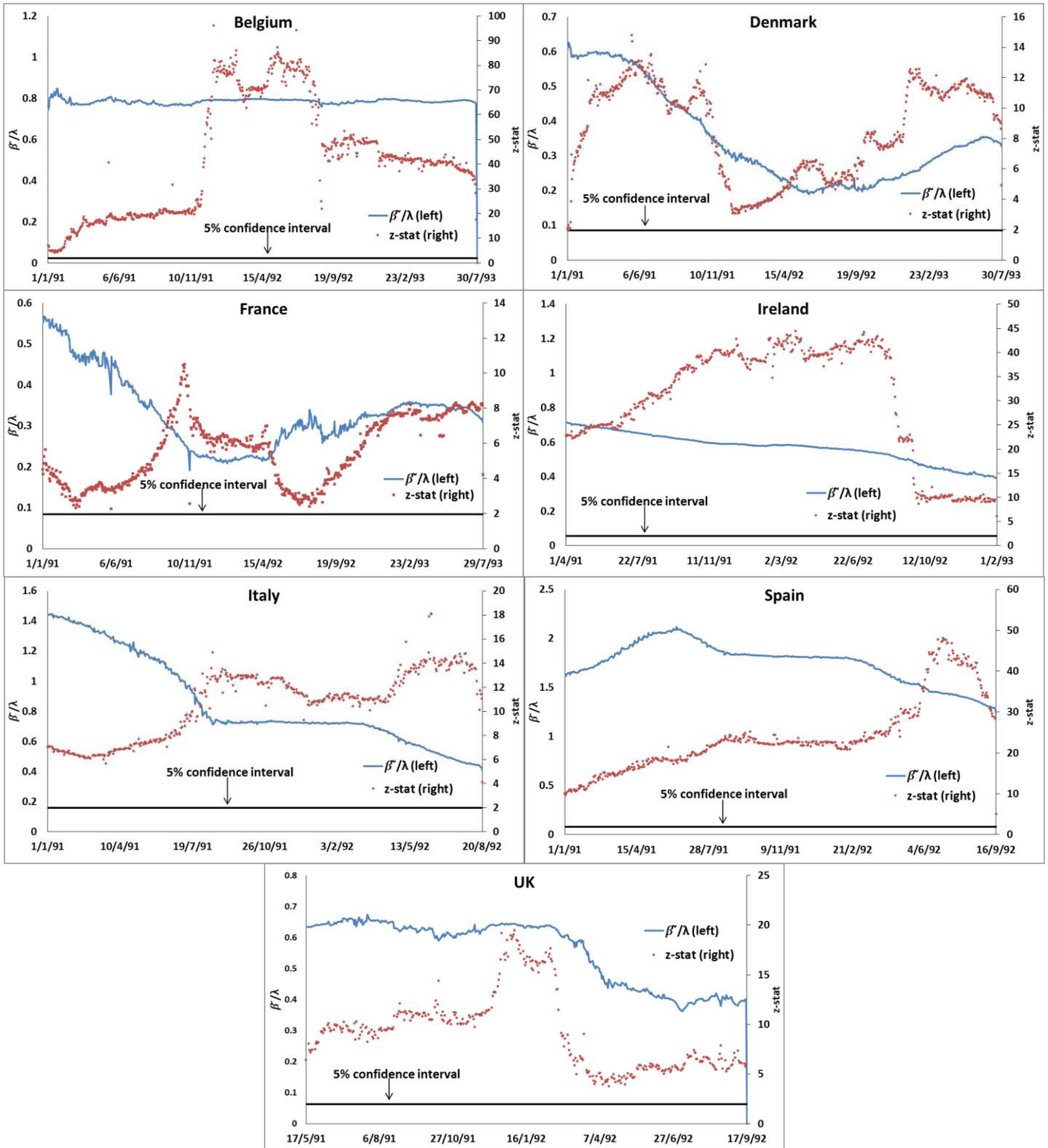


Figure 6: Estimated probability leakage condition $\tilde{\sigma}^2 / 4\tilde{\beta}$ at weak-side limit and exchange rates x of ERM

currencies under MRSR process

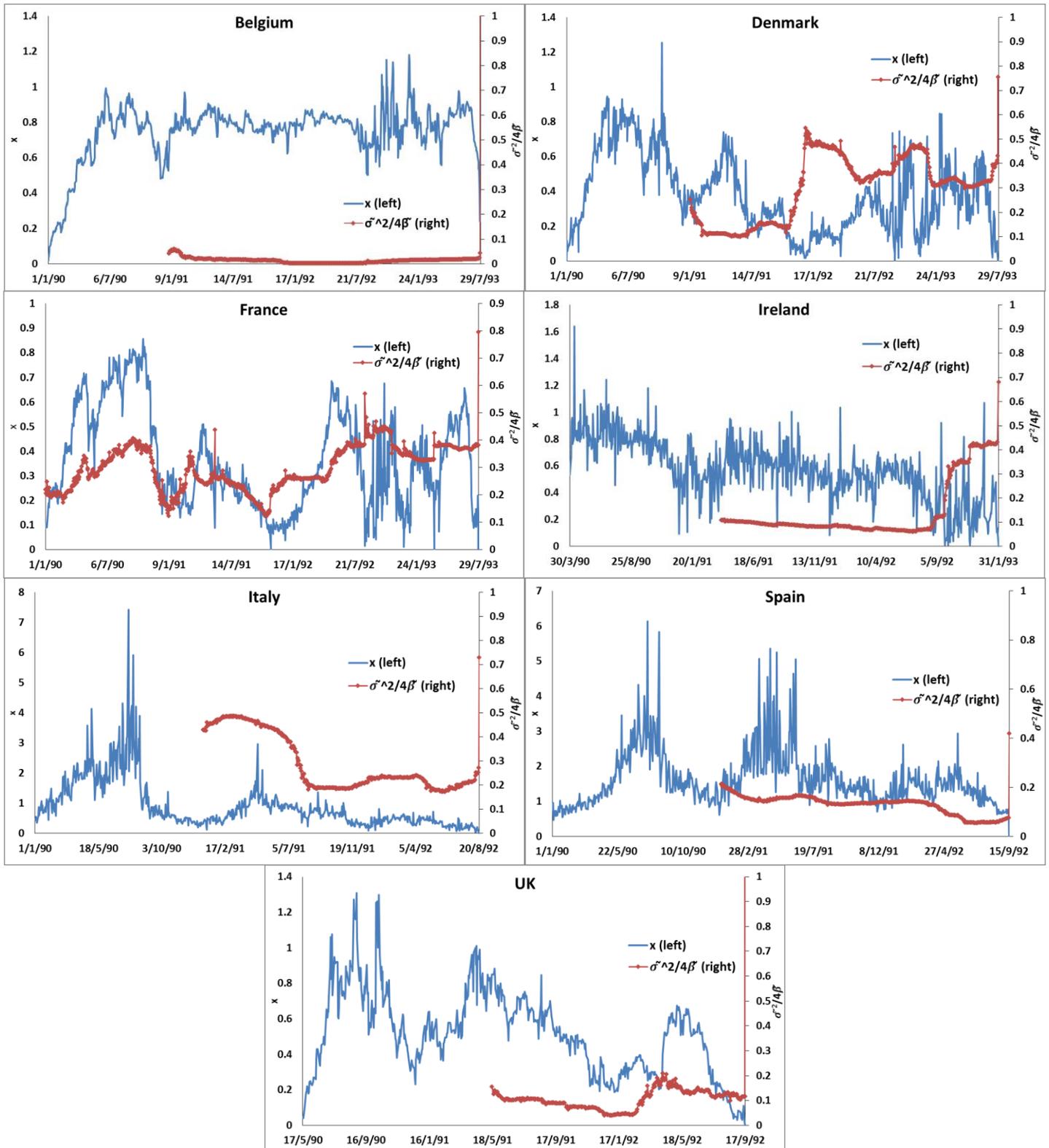
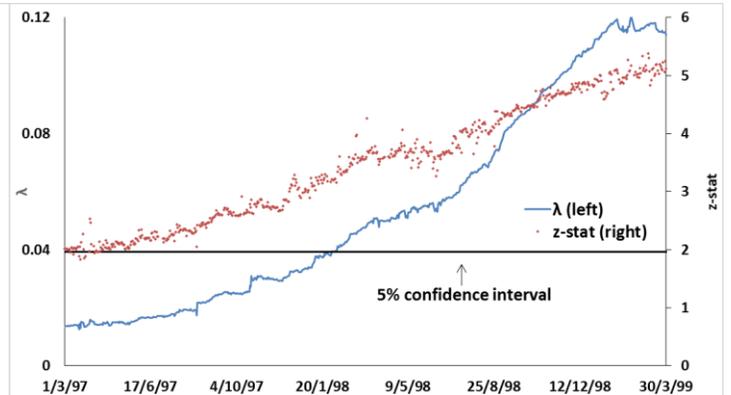
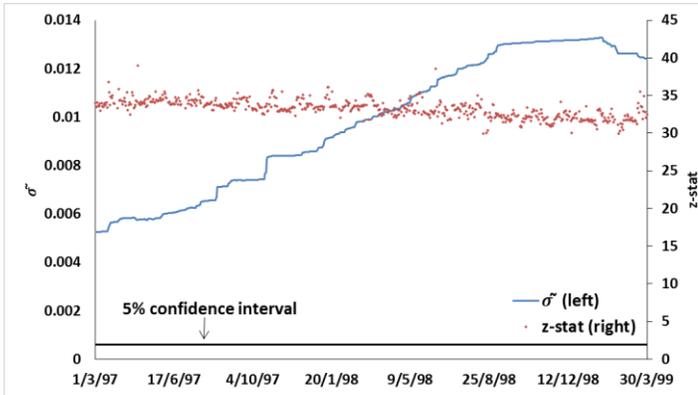


Figure 7: Estimated model parameters and probability leakage condition $\tilde{\sigma}^2 / 4\tilde{\beta}$ of HKD exchange rate under MRSR process

(Panel A: $\tilde{\sigma}$)

(Panel B: λ)



(Panel C: $\tilde{\beta} / \lambda$)

(Panel D: $\tilde{\sigma}^2 / 4\tilde{\beta}$)

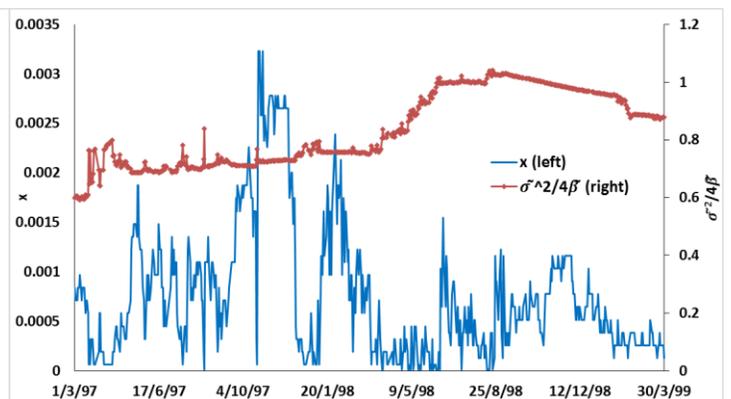
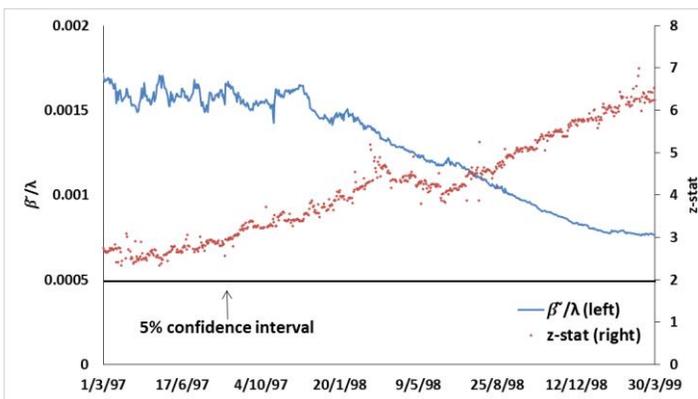
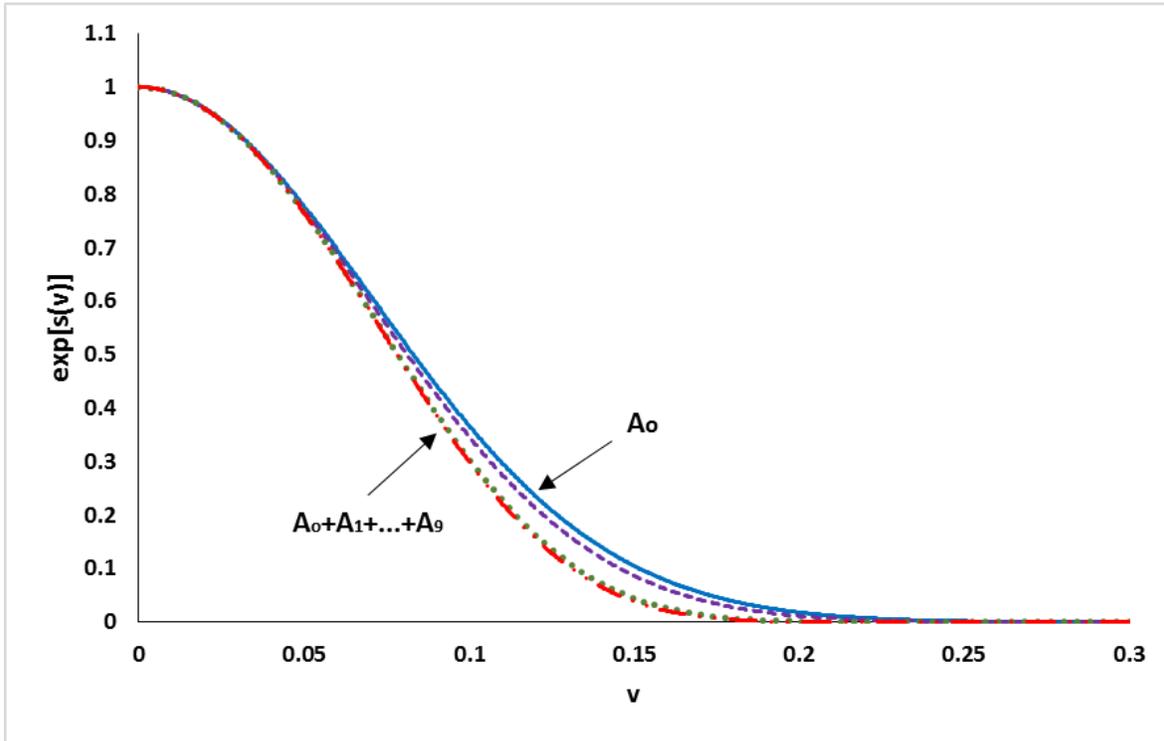


Figure A1. Relationship between the normalized exchange rate $s(v)$ and the component v of the fundamental

Note: $s(v)$ is calculated from the drift $\mu_v = \frac{1}{2}(-\lambda v - \kappa + \frac{4\beta - \sigma^2}{4v})$. Different curves in the graph represent $e^{s(v)}$ summing up to different terms $\{A_0, A_0 + A_1, \dots, A_0 + A_1 + \dots + A_9\}$ in series $s(v)$, with parameters $\kappa = 0.1, \beta = 0.05, \lambda = 0.1, \alpha = 0.2, \sigma^2 = 0.3, m = 1$.