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Abstract

In the real world many social and economic decisions have to be made with imperfect information and uncertainty. In the past two decades, economists and mathematicians have devoted a great deal of time and effort into the study of ambiguity and much progress has been made in modeling ambiguity. Decision models under ambiguity have been widely used in portfolio selection, asset pricing, and risk measurement. However, few studies have been done on linking ambiguity to the social welfare function, although social welfare evaluation also faces a scarcity of information and ambiguity of income distribution. In this paper I set up a framework with policy relevance for social welfare evaluation, with the help of a model that is developed to handle income distribution ambiguity. Under some reasonable conditions the relation of income distribution to social preference is identified and the social welfare function is clearly expressed. It is shown that the social welfare functions derived from the framework are robust in form and invariant up to a monotonous increasing transformation. The framework is also flexible enough to contain many thoughtful ideas about the social welfare function.

Keywords: Preference, Ambiguity, Income Distribution, Social Welfare Function

JEL Classification: C61, D31, D63

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1. Introduction

In past decades, many countries and regions including emerging Russia, India and China, and developed Hong Kong experienced rapid growth, but the benefits of economic growth are not equally shared by the people. Actually, often a major proportion of the benefit of economic development goes to a minority of people, while the majority of people do not enjoy the benefit of economic development but suffer from it instead, because high economic growth often is followed by a widening income gap and soaring consumer as well as property prices. This indicates that income distribution is a very important factor influencing social welfare and social welfare policy urgently needs to be adjusted to enhance social welfare.

Recently this issue has become a hot topic on websites and drawn more and more attention from economists and politicians all over the world. In many countries calls for income redistribution reform and demonstrations against income policies are a frequent occurrence. If this situation is not handled properly, social and political stability may be unavoidably threatened and harmed. Therefore, governments of concerned countries and regions have to take steps to meet the potential threat to social and political stability.

The social welfare function is an eternal topic in welfare economics. It is a real-value function that ranks conceivable social states (alternative complete descriptions of the society) from lowest to highest. Inputs of the function include any variables considered to affect welfare of the society (Sen, 1970), but most of the variables can be measured in monetary terms.

The social welfare function closely relates to social choice in a democratic society and capitalist markets (Arrow, 1963), social welfare evaluation, income distribution, tax policies and other issues in macroeconomic management. It also closely relates to the micro-foundation of macroeconomics and is seen as a bridge from microeconomics to macroeconomics. Without in depth understanding of how the social welfare function is formulated, one would not understand if and how a macro variable can be aggregated from different micro variables, or a social preference from different individual preferences.

Although studies on the relationship of social welfare to income distribution have a long history, economists have not reached an agreement on the framework for the social welfare function. Early development of welfare economics was pushed forward by classical utilitarianism economist Edgeworth (1881), and his followers, Marshall (1890) and Pigou (1920). Utilitarianism economists assume that individual utility or happiness from goods and services consumption can be measured and compared with the same value scale called “utils”. The classical social welfare function is defined as the sum of utilities of all the members of the society: $W = \sum_{i=1}^{n} U_i$ (or $W = \sum_{i=1}^{n} x_i$ in terms of income x_i), where each utility (U_i) receives an equal weight, implying that one extra unit of utility for a

starving person is not seen to be of any greater value than an extra unit of utility for a millionaire. Utilitarian economists also borrow some findings of psychologists as a prerequisite of economic theory. As an example, diminishing marginal utility was widely quoted as law after the marginal revolution initiated by Walras (1874) and Marshall (1890).

From the 1930s, “economists came to be persuaded by arguments presented by Robbins (1938) and others (deeply influenced by "logical positivist" philosophy) that interpersonal comparisons of utility had no scientific basis” (Sen, 1999). Robbins strongly opposed the view of utilitarianism on the interpersonal comparability of utilities. He said, "Every mind is inscrutable to every other mind and no common denominator of feelings is possible” (Robbins, 1938), and thus the epistemic foundations of utilitarian welfare economics were seen as incurably defective.

With the sweeping influence of Robbins’s standpoint and the introduction of the indifference curve, leading economists in this field became adverse to subjective concepts and hostile to cardinal and interpersonally comparable utilities (Ng Yew-Kwang, 1997). It marked a turning point of welfare economics, where old welfare economics began to decline and new welfare economics started to rise.

From the 1940s onward, economists devoted a great deal of effort to the axiomatic set-up regarding the existence of the social welfare function and the possibility of social choice. The most widely adopted frameworks were formulated by Bergson (1938) and Samuelson (1947). Bergson and Samuelson defined the social welfare function W as functional of and individual’s utilities U_i: $W = F(U_1, U_2, \cdots, U_n)$ (where W denote social welfare, and individual utility) and Arrow defined the social welfare function as a map of a group of individual’s preference orderings (R_i) to a social one (R): $R = F(R_1, R_2, \cdots, R_n)$. As a result, following the frameworks formulated by Bergson (1938) and Arrow (1951, 1963), Arrow’s impossibility theorem and its many variants were introduced (Parks, 1976; Kemp and Ng, 1976; Pollak, 1976; and Hammond, 1976; Kaplow and Shavell, 1999).

Regarding the relationship of Arrow’s impossibility theorem to Bergson-Samuelson’s social welfare function, insightful analysis has been made by Little (1950), Rothenberg (1953), Sen (1973, 1979) and Arrow himself. According to Rothenberg (1953), Arrow’s social welfare function (actually a social choice function different from the general function concept) could also be seen as a variant of the Bergson-Samuelson social welfare function, and it can be derived from the Bergson-Samuelson social welfare function. “Impossibility results can be precipitated in the Bergson-Samuelson framework in ways that are similar to the Arrow Impossibility theorem.” (Sen, 1977).

With the reemergence of the neoclassical utilitarianism in the 1970s, many social welfare functional forms were developed (see table 1). The neoclassical utilitarianism social welfare function is expressed as $W = p_1U_1 + p_2U_2 + \cdots + p_nU_n$ given a discrete probability distribution $(p_1, p_2, \cdots, p_n)$; for continuous distribution $F(x)$, the neoclassical utilitarian social welfare function is

expressed as $W = \int U(x)dF(x)$. Rawls (1973) attached more importance to the benefit of the poor and proposed to define the social welfare function to be welfare of the poorest in the society $W = Min(U_1, U_2, \cdots, U_n)$ and maximizing social welfare amounts to maximizing welfare of the poorest. Sen proposed to measure social welfare by $W_{gini} = \mu(1-G)$, where μ is the average income of a measured group, and G is the Gini coefficient. Foster (1996) proposed to use one of Atkinson's Indexes instead of the Gini index. Considering the relationship between Atkinson's Indexes and Theil indexes, he also proposed to define the social welfare function as $W_{Theil-L} = \mu e^{-T_L}$ or $W_{Theil-T} = \mu e^{-T_T}$, where, T_L and T_T are two different Theil indexes.

Each social welfare functional form has its advantages and disadvantages. The classical utilitarianism social welfare function as a sum of individual utilities has nothing to do with income distribution and inequality. It implies that different utility functions are comparable and additive. The assumption of comparability and additivity would later become a critical target of the new welfare economists.

Neoclassical utilitarianism defines the social welfare function as the expected utility function under risk, where the probability distribution is known. However, no agreement has been reached so far on whether or not preferences are interpersonally comparable, and the debate on this issue continues.

Rawls' social function captures ethical judgment of justice and equity, the key philosophical point about social welfare, but it is criticized for paying too much attention to the poorest and neglecting the coordination of the poorest with others. Rawl's rule may discourage people from working hard and probably lead to social resources being used inefficiently. In the long run, if resources are not used efficiently in a society, the society would eventually fail in allowing people to live their lives with dignity.

Sen's social welfare function and Foster's social welfare function are similar in that both are expressed as a multiplication of average income and an equality index; the difference lies in that they use different equality indexes. It is a pity that neither Sen's nor Foster's social welfare function is based on preference axioms and economic rationality. Therefore, it is not clear whether the Gini index or the Theil index better measures social welfare.

The major purposes of this paper are, firstly, to investigate the logic of impossibility theorems and to find a way to escape from impossibility; secondly, to set up a unifying framework for social welfare evaluation under imperfect information and ambiguity; and thirdly, to identify factors that influence social welfare, obtain an analytical form of the social welfare function, and then discuss characteristics of the social welfare function. The rest of the paper is organized as follows: in section 2, a literature review is made about impossibility of social choice and the relativity of preference and the importance of reference frame in social welfare evaluation is discussed. In section 3, the role of moral judgments and its relationship with social welfare evaluation is investigated, and the possibility of taking moral judgments of social welfare evaluation is considered. In section 4, the latest important theoretical

progress in modeling ambiguity is sketched and its possible applications in social welfare evaluation are analyzed. In section 5, a formal variational model is established, which consists of social objective, constraints and some boundary conditions. By solving the model, the social welfare function and the optimal distribution function are obtained, and the relation of social preference with distribution is expressed and the nature of social choice and social welfare function is clarified. Importantly, robustness and invariance of the model are also investigated. With the model, economic welfare of Mainland China and Hong Kong and other regions are estimated, and policy implications for Hong Kong and Mainland China are discussed in section 6. Finally, I make concluding remarks in section 7.

2. Relativity of Social Welfare, Reference Frame and Possibility

Since preference is a relative concept associated with moral judgments, social welfare associated with the value function of preference is also a relative concept. Therefore, when a person makes observations about social welfare, his viewpoints must relate to a reference frame regarding where he stands and what his values are with respect to social welfare.

The importance of a reference frame in social welfare evaluation was ever mentioned by (Rothenberg, 1953), Tversky and Kahneman (1986). It seems to me that the importance of reference frame to social welfare evaluation is somewhat like a reference frame to Classical Physics and Theory of Relativity by created by Einstein.

In special relativity theory in physics, both time and space are relative concepts and depend on a reference frame in which people observe the natural world. Observers see different phenomena in different reference points. However, physical rules that the motion of matter follows are the same, and independent of special reference frames. Similarly, in welfare economics, each person values social welfare according to his own preference (or utility function) and the environment he live with, as if he stands in a different reference frame. If different people own different preferences, they value social welfare with different measures and they may endow the same social state with different values. Therefore, preference differences determine differences of valuation to social states.

It may be more understandable if we make an analogy to the motion rule of object. We know that in physics object motion velocity and motion equation vary with reference frame. However, both classical physics and relativity shows that motion equations obtained from different reference frames are of a certain type of invariance. Similarly, within different reference frames (people differ in utility function), different people may see different phenomena and obtain different social welfare functions. Different people value the same social state with a different value and a different social welfare function. However, this doesn't mean that social welfare functions obtained in different reference frames are totally incomparable. Since people's preferences are assumed to be restricted by conventions, customs or laws, and follow the same preference axioms, it is possible for different people to share a similar pattern of moral judgement, and then at least partial comparability is possible. A good social welfare function model should embody such observations. Therefore, it is expected that social welfare

functions observed by different individuals in different reference frames would show some commonality or similarity, in other words, the expression of social welfare functions should be invariant and robust to some extent.

The above idea was also supported by Buchanan. He cast a doubt on the relevance of the frameworks formulated by Arrow, Bergson and Samuelson, and recommended separating collective rationality from individual rationality. He argued that "the philosophical bases of individualism in which the individual is the only entity possessing ends and value. In this case no question of social or collective rationality may be raised. A social value scale as such simply does not exist. Alternatively, we may adopt some variant of the organic philosophical assumptions in which the collectivity is an independent entity possessing its own value ordering. It is legitimate to test the rationality or irrationality of this entity only against this value ordering." (Buchannan, 1954)

Existing social welfare functions can be investigated from different aspects. They can be classified in terms of inputs, the preference measurability; they can also be classified in terms of preference comparability. Social welfare functions can be individualistic, dictatorial or collective according to their formulation. Arrow's impossibility theorem is based on individualistic assumptions, where the domain of the social welfare function is universal and unrestricted, and any social welfare function is assumed to be aggregated from different individual preferences. On the other side, social welfare may be dictatorial and evaluated by a person of special position. From Sen's point of view, a social welfare function can be aggregated from restricted and partly comparable individual preferences. The author here supports the partly comparable assumption and believes that social welfare is a relative concept that is meaningful only when it is evaluated from a special reference frame. At the same time, he also believes that the social welfare function is not only measurable but also interpersonally comparable when the profile of Bergson-Smuelson's social welfare function are restricted strongly enough.

Arrow's social welfare function is typically individualistic, but it is proven that his strong assumptions lead to inconsistency. Once individualistic and interpersonal incomparability assumptions are put together, social choice meets unconquerable difficulties, and it is impossible to make a full ranking of social states and aggregate different individual utilities into a social welfare function.

On the other hand, Bergson-Samuelson's social welfare function is criticized for being too general in form to clear policy implications and it does not tell what kinds of social welfare function are useful in empirical study and policy making. Some social welfare functions can be deduced when the domain of profiles of Bergson-Samuelson's social welfare function is restricted. In my point of view, Classical Utilitarian, Rawls Max-Min, Elite Min-Max and welfare functions proposed by Sen (1973) and Foster (1996), all can be seen as special restricted cases of Bergson-Samuelson's social welfare function.

In addition, an individualistic framework of the social welfare function raises a high quality requirement to the informational base. If the Bergson-Samuelson framework applies, it is assumed that there is a person or an agency who knows every social member's utility function, full information about income, the unit and scale of each individual's utility, and so on. A full ranking of all social states can be

completed only when all the information related to the social welfare function is available. However, this kind of information is clearly unavailable from the standpoint of new welfare economists, because they believe that "every mind is inscrutable to every other mind and no common denominator of feelings is possible." Sen (1999)

Sen (1999) paid more attention to the information base of social welfare evaluation. After a thorough investigation of Arrow's impossibility theorem, he concluded that under Arrow's axioms, "The already-limited informational base of Bentham's calculus was made to shrink even further to that of Borda (1781) and Condorcet (1785); the use of different persons' utility rankings without any interpersonal comparison is analytically quite similar to the use of voting information in making social choice" (Sen, 1999). Therefore, an Arrow-type impossibility theorem is only relevant to social choice issues such as the voting process and is silence on income distribution that is related to the social welfare function.

Although cardinal measurability is important to social welfare evaluation (Ng, 1976, 1999; Sen, 1999), in Sen's viewpoint, introduction of cardinal measurability cannot help to escape from impossibility. He pointed out "Admitting cardinality of utilities without interpersonal comparisons does not change Arrow's impossibility theorem at all, which can be readily extended to cardinal measurability." (Sen, 1977) Therefore, impossibility has little to do with cardinal measurability, but has much to do with Arrow's immoderate hypothesis. Sen claimed that impossibility can be avoided by dropping any row's assumptions (Sen, 1977).

Arrow's universal domain assumption is also irrelevant because it highlights too much about the difference of preferences between individuals and ignores their similarities. "How preferences are specifically influenced may reflect culture, social convention or custom, so that they are context dependent. But whatever the cause, this may create sufficient restrictions on the preference domain that collective rationality results as a consequence of some aggregation procedure that is democratic.[1] The independence axiom is criticized for its ruling out information that is valuable to comparison of preference intensity.

The above analysis shows that there would be no future if welfare economics continued to be restricted in Bergson-Samuelson and Arrow's frameworks. To escape from impossibility, it is necessary to set up a theoretical framework that distinguishes from Bergson-Samuelson and Arrow's.

A few ways mentioned help to escape from impossibility. The simplest way is to assume that all individuals have the same preferences, which is in contrast with Arrow's universal domain condition. This assumption is widely used in policy analysis (Mirlees, 1971; Sadka, 1976).

The second way is to assume that preferences are full cardinally measurable and interpersonal comparable. Because aggregation of cardinal utility must involve an arithmetic operation which requires every term in the utility functions to have a specific unit, scale and reference point, no one

[1] The New Palgrave Dictionary of Economics, Second Edition, p896.

has so for claimed to be able to have such information. Moreover, such a strong assumption has long been rejected by new welfare economists because full cardinal measurability and interpersonal comparability are not easily accepted by all individuals and I don't recommend adopting such assumptions. Although someone may conceive that a machine measuring people's happiness may be created someday, it is better to avoid full comparability before the machine is created.

The third way is to assume that social welfare is evaluated by an entity with a special position in a society. The entity can be either a dictator or a group of team members. A government can also be seen as such an entity. Introduction of government has clear policy implications since social welfare can be adjusted by government policies.

Generally, the social decision process is complicated and can be classified into three phases: voting, policy-making and implementation, and policy revision or improvement. Official election of the government can be either democratic or autocratic. Most democratic countries elect their leader following a certain voting procedure. The reasonableness of election procedures is challenged by the Arrow impossibility theorem and is a main theme of social choice theory. Once a government leader is elected, the social decision process goes to the second phase, policy making and implementation. In this phase, government can be seen as a dictator. Most policies in this phase are generally formulated independent of voting for the time being; at most, the policies are made after discussion by a committee – a group of team members who usually share similar preferences with its government leader (because the team members are nominated by the leader). This makes it reasonable to assume that government policy is made approximately under dictatorship. Since the policies are made under ambiguity and imperfect information in the second phases, some problems may be discovered during implementation, and it is possible that policy made in the second phase will be examined and criticized by the public. In this case, policies formulated in the second phase should be improved. This implies that public opinions have some impact on government preference and force the government to change preference to meet various situations. Public opinions can also be strong enough to cause the old government to be replaced by a new government. Therefore, in the third phase policy would be adjusted to meet the new social environment, responding to public opinions and comments. In short, social welfare valuation could be seen as a process in which the government evaluates social welfare according to its preference. The Rawlsian social welfare function and the elite welfare function can be seen as special types of social welfare functions that respectively represent the poorest and the richest. Because social welfare is, to a large extent, determined by government policies and institutional changes, compared with an individualistic social welfare function, the government preference based social welfare function obviously has strong policy implications.

The above discussion indicates that that there is a need to choose a reference frame when evaluating social welfare to escape from impossibility. Given a reference frame, everyone can value social welfare in his own reference frame, but only government valuation is policy relevant. If social welfare is evaluated by a team whose members have similar preferences, agreement on social welfare function may be easily reached.

Therefore, the government plays a role as if it were a dictator in Arrow's framework, and it is feasible and meaningful to assume that the government is an independent rational entity that represents all social members, even though social preference is imposed for some members.

3. Moral Judgement and Rationality

It is not sufficient to obtain the social welfare function if only a reference frame is chosen. To escape from impossibility, Arrow's specifications should be relaxed; on the other hand, some restrictions, which can be seen as non-utility information or essentials of rationality, should be added to the profile of the social welfare function. The restrictions reflect similarities in moral value judgement between individuals. The similarities are either due to common culture background, convention, custom, social policies or laws imposed.

Generally, three moral judgments: justice, equity and efficiency, and decision rule, are to be taken into consideration in social welfare evaluation. The first is related to the trade-off between efficiency and equality. Income inequality in terms of the Gini coefficient is seen as an important reference indicator of government policy. Experience has indicated that if income inequality is too small, it may cause inefficient resource allocation; on the other side, if income inequality is too large, it may lead many people to participate in street demonstrations and even cause social and political instability. To prevent unexpected things from happening, the government may find, according to its past and international experiences, that the Gini coefficient should be controlled within a specific interval. For example, people may agree that an ideal society should not distribute income so evenly that nobody has the motivation to work; at the same time, however, income should not be distributed so unevenly that most people are not satisfied with it. Such value judgement has a strong policy implication, that is, income distribution policy should be adjusted so that income inequality is at a moderate interval and reflects a trade-off between efficiency and fairness.

The second moral judgement is justice and inequality aversion. Justice in most theories was placed in a special position. According to Rawls (1971) "Justice is the first virtue of social institutions, as truth is of systems of thought." In a just society, social and economic inequalities are arranged so that: a) they are of the greatest benefit to the least-advantaged members of the society (difference principle), and b) offices and positions must be open to everyone with fair and equal opportunity.

In modern society, the security line is thought to be an important consideration of social welfare policy, under which whoever has income lower than the security line is entitled to subsistence. It is the government's responsibility to adjust income distribution and make sure that each individual's living standard is raised above the minimum security line. Policy should guarantee minimum wages and a living standard for the benefit of the poor. However, the principle of justice is not all that is necessary for a society to run successfully. In traditional socialist China, equality was a top priority of government; each able person was promised a job in order to meet basic living needs. However, this institutional arrangement lowers competition and reduces efficiency, and in the ends the society could not produce

enough food, clothes and other goods to satisfy the basic needs of people. It was Deng Xiaoping who recognized the weakness of the traditional mandatory economic system and the inefficiency of "big pot rice", and launched economic reform and opened door to the outside world in 1978, in order to "let some of the people to be better off first and then realize common prosperity." Deng's idea represents one of the most important value judgments of China at that time.

The third moral judgment is related to rational decision rules. Rationality in economics can be expressed in different ways depending on the nature of problems. In a world with no risk and uncertainty, rationality refers to the behavior of an economic entity that follows preference axioms, and efficiently allocates economic resources. In a world with risk, where probability distribution is given, rationality can be characterized with the von Neumann-Morgenstern utility function. In this case, the rational principle of utility maximization is naturally replaced by expected utility maximization. However, not every phenomenon in the real world can be measured in terms of probability. Knight (1921) recognized that some phenomena cannot be expressed in a single probability distribution. Allais (1953) and Ellsberg (1961) further demonstrated by experiments that inconsistency may arise between actual observed choices and predictions of expected utility theory. Ellsberg's experiments also showed that some choices cannot be rationalized by subjective expected utility (SEU) introduced by Savage (1954). In an uncertain world, where random phenomena cannot be represented by probability measures, rationality could characterize by expected utility maximization under ambiguity.

As in risky and uncertain environments where expected utility maximization under risk is regarded as the moral judgment of an economic person, expected utility maximization also applies to social welfare issues. However, perhaps because the possibility of social choice and the existence of the social welfare function dominate research in this field, no one, as far as I know, has so far developed a model linking ambiguity to social welfare evaluation. In the next section of this paper, the Neumann-Morgenstern utility function government is assumed to be an entity aiming to maximize social welfare in terms of expected utility according to its own reference frame.

4. Ambiguity and Social Welfare Evaluation

It would be ideal for a theory to be supported by perfect information. However, unfortunately perfect information is generally unavailable in the real world. For example, in financial markets, one may observe asset price but not completely observe its rate of return and its volatility; regarding the social welfare state, only a part of people's income data are available and no one can get the information about all social members; specifically, one may estimate the mean and Gini coefficients of the income of a population, and not necessarily know the income and income distribution of all the society. Even the official statistical department can do nothing to help if a population is large enough. In the real world, noise from information asymmetry and counterfeit data are also puzzling. Traditional welfare economics is silent in such cases because "The concept of an objectively measurable probability or chance is simply inapplicable" (Knight, 1921).

The classical utilitarianism social welfare function is defined as the arithmetic average (sum) of all social members. To complete full ranking of all social states, one needs to know everyone's information; otherwise comparability and arithmetic operation cannot be performed. Comparison of social states needs high-quality information about different individuals. Without perfect information, social states cannot be compared and ranked, especially in the framework of ordinal utility where the Pareto principle applies. More importantly, the classical social welfare function has nothing to do with income distribution and leaves no room for government policy to play a part. Similarly, the individualistic neoclassical social welfare function is also short of supportive data and not applicable in empirical studies. The Bergson-Samuelson social welfare function has similar drawbacks as the classical welfare function.

In fact, a government is not concerned about each citizen's information, but some indexes synthesized from individual's information; generally, a limited number of synthesized indictors are sufficient for policy making. In other words, the government can make decisions without perfect information. The most frequently used synthesized indicators relating social welfare function include mean income, the Gini coefficient, poverty line, minimum wage, social security line, and so on. These indicators can be easily obtained from government publications and private sampling surveys. It will be helpful to express the social welfare function in terms of average income, Gini coefficient, poverty line, and other relevant indexes.

With efforts by economists and mathematicians many models associated with ambiguity have been developed to cover Allais and Ellsberg paradoxes so far. Here I just mention a few of them, which I think are relevant.

Let Ω be a given set of states, S denotes the set of relevant states of the world $\omega \in \Omega$, and C a set of outcomes $C \subset R$; let H be a linear space of real functions defined on Ω such that if $X_1, X_2, \cdots, X_n \in H$, then $\varphi(X_1, X_2, \cdots, X_n) \in H$ for each $\varphi(X_1, X_2, \cdots, X_n) \in C_{poly}(R^n)$, where $C_{poly}(R)$ denotes the space of continuous functions with polynomial growth, i.e., there exists constants C and $k \geq 0$, such that $|\varphi(x)| \leq C(1 + |x|^k)$. H is considered as a space of "random variables" (Peng, 2007).

In case random phenomena cannot be described by a single linear probability measure, it is natural for a decision maker to think in terms of a set of probability measures. The multiple priors model proposed by Gilboa and Schmeidler (1989) postulates the following utility function on the set of Anscombe-Aumann (1963) (AA) acts:

$$U^{MP}(\varphi) = \min_{p \in P} \int_{\Omega} u(\varphi) dp = \min_{p \in P} \int_{R} u(\varphi(x)) dF(x) \quad (1)$$

Here, $u : C \to R$ is a von Neumann-Morgenstern utility functional. P, the set of priors p, is a set of probability measures on Ω. The multiple priors model captures the character of an ambiguous event stated in the Ellsberg paradox and gives ambiguous problems a formal mathematical representation. The MP model is sometimes criticized on the grounds that it implies extreme aversion to ambiguity and does not capture the magnitude of ambiguity aversion.

To overcome the deficiencies of the MP model, Klibanoff, Marincci and Mukerji (2005), proposed a Smooth Ambiguity Model in which a utility U^{KMM} is expressed over AA acts:

$$U^{KMM}(\varphi) = \int_{\Delta(\Omega)} \phi\left(\int_{\Omega} u(h(\omega)) dp(\omega) \right) d\mu(p) \tag{2}$$

Here, $u : \Delta(C) \to R$ is a von Neumann-Morgenstern utility function as before. μ is a probability measure over $\Delta(C)$, representing model uncertainty, and ϕ is continuous and strictly increasing on C. Epstein and Schneider (2010) pointed out that the multiple priors model is a limited case of the smooth ambiguity model, if P is supporting of μ and the degree of concavity of ϕ increases without bound. Ambiguity attitude is captured in this model by the shape of ϕ, and it is seen as an advantage of the MP model. If ϕ is concave, then the individual is ambiguity averse and greater concavity implies greater ambiguity. On the other hand, ambiguity itself seems to be captured by μ, and thus it is claimed that the KMM model separates ambiguity and attitude of aversion to ambiguity. The separation of ambiguity aversion and ambiguity is another advantage of this model (Epstein and Schneider, 2010). However, the smooth model faces difficulties since there is little information available about the measure of μ, which means that μ itself faces uncertainty.

Inspired by robust control theory, Anderson et al. (2003) introduced a functional form called multiplier utility (MU). The MU model was axiomatized and defined by Strzalecki (2010) as

$$U^{MU}(\varphi) = \min_{p \in \Delta(\Omega)} \left[\int_{\Omega} u(\varphi) dp + \theta R(p \| p^*) \right], \tag{3}$$

where $\theta \in (0, \infty)$ is a parameter, p^* is a reference measure of an individual's "best guess" of the true probability law, and $R(p \| p^*)$ is relative entropy defined by $R(p \| p^*) = \int_{\Omega} \left(\log \frac{dp}{dp^*} \right) dp$.

Maccheronni et al. (2006a) introduced and axiomatized the following utility function called the variational model:

$$U^{\text{var}}(\varphi) = \min_{p \in \Delta(\Omega)} \left[\int_{\Omega} u(\varphi) dp + c(p) \right]. \quad (4)$$

Here, $c(p) : \Delta(\Omega) \to R^{+}$, is a cost or penalty function. The variational utility model includes multiplier utility, the MP model as its special case, corresponding respectively to $C(p) = \theta R(p \| p^{*})$ and $C(p) = 0$.

However, various ambiguity models themselves don't tell where they apply. Obviously, whether or not a type of ambiguity model is adopted depends on the context of problems. Without taking account of the context, it is difficult to identify which better describes practical problems.

The above static models are studied in static situations; in past decades some progress has been made in dynamic situations. With the help of backward stochastic differential equations, the MP model was generalized to link two important concepts: g-expectation by Peng (1997), and further studied by Chen and Epstein (2002) in the context of backward stochastic differential equations. This model satisfies all properties of mathematical expectations except additivity; put in another way, they satisfy all conditions of what is called sublinear expectations (see definition 1).

Definition 1: a function $\hat{E} : \mathrm{H} \to R$ is called a sublinear expectation, if $\hat{E}$ satisfies:

1 · Monotonicity: $X \geq Y \Rightarrow \hat{E}(X) \geq \hat{E}(Y)$

2 · Constancy preserving: $\hat{E}(c) = c$

3 · Convexity: $\hat{E}(\lambda X + (1-\lambda)Y) \leq \lambda \hat{E}(X) + (1-\lambda)\hat{E}(Y)$

4. Positive Homogeneity: $\lambda \geq 0 \Rightarrow \hat{E}(\lambda X) = \lambda \hat{E}(X)$

Under some moderate assumptions it is shown that sublinear expectations can be represented by the MP model. Artzner et al. (1999) establish the links between sublinear expectation with coherent risk measure.

5. Modeling and Solving Social Welfare Functional

5.1 Objective Function and Constraints

Assume that individual income of a society is represented by a random variable X distributed on $R^+ = (0,+\infty)$. It is well know that there generally are an infinite number of distributions corresponding to a given Gini coefficient, so a natural question that one would ask is which of the distributions is the best from the perspective of society, in other words, which of the distributions with given Gini coefficient G maximizes social welfare. Pushing this issue a step further, the government may conceive of an interval for G such as $G \in (\underline{G}, \overline{G})$. Once G is outside of the interval, some policy will be adopted to let G return inside of the interval. This problem is associated with ambiguity and moral judgments stated previously. The moral judgments constitute constraints of the social objective, and they thus are elements of collective rationality.

Hypothesis 1: Taking justice as the moral judgment of a society, the government is obligated to consider the basic needs of every person of the society and ensure everyone lives with a dignity life; a reasonable assumption is that the minimum income or security line x_m (>0) is a major consideration of policy making. This policy is expected to be equally shared by people whether they are poor or rich at present. Therefore, the random income is restricted by such a policy to take value only from interval $[x_m, \infty)$ (people earning income less than $x_m > 0$ will be entitled to subsistence).

Hypothesis 2: Within the multiple priors framework, X is distributed with a set of continuously differentiable probability distributions denoted by $F(x \mid x_m, \mu, G \in (\underline{G}, \overline{G}))$, where, μ is the expectation of income, G denotes the Gini coefficient of income, x_m is variable minimum income. Let P denote the set of income distributions with given expectation μ and Gini coefficient $G \in (\underline{G}, \overline{G})$, then the distribution set P can be expressed as:

$$P = \left\{ F(x) \middle| x \in (x_m, +\infty), \int_{x_m}^{\infty} xF'(x)dx = \mu, 2\int_{x_m}^{\infty} x(1-F(x))F'(x)dx = \mu(1-G), G \in (\underline{G}, \overline{G}) \right\}$$

The condition $G \in (\underline{G}, \overline{G})$ captures the “trade-off” between equity and efficiency as an important consideration of the government, meaning that the income gap should be large enough to guarantee efficiency and should not be too large so as to cause social instability.

Hypothesis 3: value function (utility function) $U = U(x)$ is increasing and bounded satisfying with $U'(x) > 0$, $U(x) < +\infty$, and $\lim_{x \to \infty} U'(x) = 0$.

The boundedness of utility function ensures the existence of expected utility or value function. This assumption is typically adopted in many dynamic stochastic studies (see, Aiyagari 1993; Huggett 1997; Huggett and Ospina 2001; and Miao 2006), although relaxation is possible for some specific case.

Hypothesis 4: Social welfare is evaluated and adjusted to maximize expected utility:

$$W(F) = E_F\left[U(X)\right] = \int_{y_m}^{\infty} U(x)F'(x)dx \tag{5}$$

Under multiple priors, the social objective is to maximize social welfare of the society through income redistribution. Therefore, the purpose of solving the maximization problem of the social welfare function is to find distribution $F^* \in P$ so that social welfare is maximized:

$$W(F^*) = \underset{F\in \mathrm{P}}{Max}\, E_F\left[U(X)\right] = \underset{F\in P}{Max} \int_{x_m}^{\infty} U(x)F'(x)dx \tag{6}$$

If the income distribution F^* defined in the objective function (5) exists, then F^* is called the optimal income distribution function and $W(F^*)$ is the optimal social welfare function.

5.2 Modeling Social Welfare Function

According to hypothesis stated above, the social welfare problem can be rewritten as a standard variational problem with objective function (7) integral constraints (8) and (9), and boundary conditions (11) and (12).

$$W(F) = \underset{F\in \mathrm{P}}{Max}\, E_F\left[U(X)\right] = \int_{x_m}^{\infty} U(x)F'(x)dx \tag{7}$$

Subject to

$$Ex = \int_{x_m}^{\infty} xF'(x)dx = \mu < +\infty \tag{8}$$

and

$$2\int_{x_m}^{\infty} x\left(1-F(x)\right)F'(x)dx = \left(\mu(1-G)\right) \tag{9}$$

and boundary conditions

$$F(y_m) = 0\,, \tag{11}$$

$$F(\infty) = 1\,. \tag{12}$$

The Lagrangian is

$$L(x, F(x), F'(x)) = U(x)F'(x) - \lambda_1\left[xF'(x)\right] - \lambda_2\left[x\left(1 - F(x)\right)F'(x)\right] \tag{13}$$

The first order condition of the problem is Euler-Lagrange equation

$$\frac{\partial L}{\partial F} - \frac{d}{dy}\left[\frac{\partial L}{\partial F'}\right] = 0\,. \tag{14}$$

Since

$$\frac{\partial L}{\partial F} = \lambda_2 xF'(x)\,,$$

$$\frac{\partial L}{\partial F'} = U(x) - \lambda_1 x - \lambda_2 x(1 - F(x))$$

$$\frac{d}{dy}\left[\frac{\partial L}{\partial F'}\right] = U'(x) - \lambda_1 - \lambda_2(1 - F(x)) + \lambda_2 xF'(x)$$

We have

$$U'(x) - \lambda_1 - \lambda_2\left[1 - F(x)\right] = 0\,. \tag{15}$$

Substituting $F(y_m) = 0$ and $F(\infty) = 1$ into (15), and applying $U'(+\infty) = 0$ and bounded condition of $U(x)$, we derive the optimal distribution function:

$$F^*(x) = 1 - \frac{U'(x)}{U'(x_m)} \tag{16}$$

where x_m is solved from (9) and $F^*(x) = 1 - \dfrac{U'(x)}{U'(x_m)}$ obviously satisfies properties of distribution function $F'(x) > 0$, $F(x_m) = 0$ and $F(\infty) = 1$.

To ascertain whether the distribution obtained from first order condition is optimal, we need to check if the second order condition of the problem is satisfied. The second order conditions are discussed by

Chiang, Alpha C. (1999) and Diogo Aguiar Gomes (Calculus of Variations and Partial Differential Equations). Denote $\overline{W}(F)=\int_{x_m}^{\infty}L(x,F(x),F'(x))dx$ and $\xi(x)$ any continuous function in $(x_m,+\infty)$ subject to $\xi(x_m)=0$ and $\lim_{x\to\infty}\xi(x)=0$, then, for any $\varepsilon>0$, $F(x)=F^*(x)+\varepsilon\xi(x)$ represents a perturbation to $F^*(x)$. By computing $\frac{\partial^2 L}{\partial F^2}=0$, $\frac{\partial^2 L}{\partial F'^2}=0$, $\frac{\partial^2 L}{\partial F\partial F'}=\frac{\partial^2 L}{\partial F'\partial F}=\lambda_2 x$, $\frac{d}{dt}\left(\frac{\partial^2 L}{\partial F\partial F'}\right)=\lambda_2=U'(x_m)>0$, we have

$$\frac{d^2\overline{W}(F^*+\varepsilon\xi(x))}{d\varepsilon^2}=\int_{y_m}^{\infty}\left[\left(\frac{\partial^2 L}{\partial F^2}-\frac{d}{dx}\frac{\partial^2 L}{\partial F\partial F'}\right)\xi^2(x)+\frac{\partial^2 L}{\partial F'^2}\xi'^2(x)\right]dx$$
$$=-\lambda_2\int_{y_m}^{\infty}\xi^2(x)dx<0$$

This shows that the second order condition for $F^*(x)$ is satisfied (Chiang, Alpha C. 1999), and we can conclude that $F^*(x)=1-\frac{U'(x)}{U'(x_m)}$ maximizes $W(F)$.

Theorem 1 (Representation theorem of the optimal income distribution): if utility function $U=U(x)$ in the above variational problem is continuously increasing, differentiable, bounded and concave, then there must be a distribution $F^*(x)=1-\frac{U'(x)}{U'(x_m)}$ in P that maximizes social welfare.

Theorem 1 tells us that the optimal distribution is determined by social preference or the utility function. Once the utility function is known, a specific expression of the optimal distribution function can be obtained. The distribution function is a decreasing function of marginal utility. On the other hand, by taking the first derivative on both sides of the optimal distribution, we have

Corollary 2: social preference and income distribution obtained in theorem 1 is satisfied with the second ordinary differential equation: $U''(x)=-U'(x_m)f^*(x)$, where $f^*(x)$ is density function of optimal distribution.

Theorem 1 establishes the relationship between the utility function and distribution function. An important implication of theorem 1 is that social preference can be estimated through income distribution, since social preference is embodied in income distribution. The estimation procedure include two steps: the first is to estimate the income distribution function $F^*(x)$ using data available

by nonparametric statistical methods; the second step is to compute the integral $U(x)=\int_{x_m}^{x}(1-F^*(x))dx$.

If the evaluator is constant relative risk averse (CRRA), without losing generality, suppose $U(x)=\frac{x^{1-\sigma}-1}{1-\sigma}, \sigma>1$[2], then optimal income distribution $F(x)=1-\left(\frac{x}{x_m}\right)^{-\sigma}$ is a standard Pareto distribution, and then we have corollary 3.

Corollary 3: if the utility function in theorem 1 is constant relative risk averse, then the optimal income distribution is a Pareto distribution.

If the evaluator is constant absolute risk averse (CARA), it can be supposed that his utility function is $U(x)=1-e^{-ax}$, $a>0$, then the optimal income distribution is exponential distribution $F(x)=1-e^{-a(x-x_m)}$. This can be restated as corollary 4.

Corollary 4: if the utility function is constant absolute risk averse, the optimal income distribution is an exponential distribution.

Theorem 5 (Representation of the social welfare function): distribution function $F^*(x)=1-\frac{U'(x)}{U'(x_m)}$ derived in theorem 1 maximizes the social welfare and the social welfare function $W(F^*)$ is expressed as a function of minimum income x_m, expected income μ and supremum $\overline{G}$.

$$W(F^*)=R(x_m)+U'(x_m)\left(\mu(1-\overline{G})\right) \tag{17}$$

where $R(x_m)=U(x_m)-U'(x_m)x_m$, x_m is not independent but endogenously determined by expected income μ and the Gini coefficient $\overline{G}$.

Their relation can be written as:

Proof: substitute $F(x)=F^*(x)$ into (5), the social welfare function can be obtained.

[2] This is requirement for existence of Gini coefficient.

$$W(F)=\int_{x_m}^{\infty}U(x)F'(x)dx=\int_{x_m}^{\infty}U(x)\left(\frac{-U''(x)}{U'(x_m)}\right)dx \\ =U(x_m)-U'(x_m)x_m+U'(x_m)\big(\mu(1-G)\big) \quad (18)$$

Noticing that in the derivation process of social welfare function, the boundedness of $U(x)$ and $\lim_{x\to\infty}U'(x)=0$, and (19) is applied.

$$\mu(1-G)=x_m+\frac{1}{U'(x_m)}\int_{y_m}^{\infty}\big(U'(x)\big)^2dx \quad (19)$$

The social welfare function is actually transformed to be a function of μ and $\overline{G}$, it is clear that only when $G=\overline{G}$, the social welfare function reaches its minimum value, and thus (17) is proved.

Theorem 5 tells us that there are three ways to promote social welfare levels: to increase the consumer surplus of the poorest, the average of the social wealth, or decrease inequality. Since the wealth of the poorest is positively associated with expected wealth μ and the Gini coefficient G, social welfare must be a function of $\mu(1-\overline{G})$, the product of average income and (1- Gini coefficient).

Next two corollaries show Rawls's social welfare function and Sen's welfare function can be seen as two special cases of Theorem 5.

Corollary 6: if all income is distributed equally ($G=0$), then $x_m=\mu$ and the social welfare function $W(F^*)=U(x_m)=U(\mu)$. This is absolute equivalent of distribution and can be seen as a special case of Rawls's social welfare function.

Corollary 7: assume $U(0)=0$, $U'(0)=1$, $x_m=0$, and $G=\overline{G}$, then according to (17) Sen's social welfare function $W_{sen}(F^*)=\mu(1-G)$ is derived.

Corollary 7 indicates that Sen' binary social welfare function can be derived from the framework of maximization under ambiguity and has its rational foundations. However, Sen's social welfare function was originally not based on such foundations (see Deaton, 1980).

The above discussion is based on four hypotheses and the law of diminishing marginal utility. However, although diminishing marginal utility has been claimed as law and is supported by many empirical studies (Greene & Baron, 2001), some modern behavioral economists are skeptical about the concavity of the utility function being applicable anywhere of an interval considered. Kahneman and Tversky (1979) who developed prospect theory argued that the shape of the value function varies

with asset position. There may be a reference point and the positive or negative change from that point is represented by the payoff in question. The value function that passes through the reference point is S-shaped and, as its asymmetry implies, there is a bigger impact of losses than of gains (loss aversion) on valuation given the same variation in absolute value. They concluded that over a quite broad range, the value function is concave in the domain of gain and convex in the domain of loss.

In the following theorem 8, I list results about the social welfare function when the utility function is concave. The S-shaped utility related social welfare function is more complicated and I will discuss them in another paper.

Theorem 8: if $U = U(x)$ on $(-\infty, x_m)$ is a continuously differentiable, increasing bounded and convex utility function, and $\lim_{x\to-\infty} U(x) = 0$, then there must be a probability distribution of income

$$F^*(x) = \frac{U'(x)}{U'(x_m)}, \tag{20}$$

which maximizes social welfare, and the social welfare function can be represented by

$$W(F^*) = U(x_m) - U'(x_m)x_m + U'(x_m)\left(\mu(1+\overline{G})\right). \tag{21}$$

5.3 Implication of the Minimum Income and Feature of Welfare Function

In the discussion below, I return to the assumption that the utility function is concave and the other assumptions do not change. Let $h = \mu(1-\overline{G})$, according to (19), x_m and $h = \mu(1-\overline{G})$ are not independent from each other, and their relation is determined by

$$h = x_m + \frac{1}{U'(x_m)}\int_{x_m}^{\infty}\left(U'(x)\right)^2 dx \tag{22}$$

It is easy to show by simple operations that h and x_m are positively related, that is the following relations are true:

$$\frac{dh}{dx_m} = \frac{-2U''(x_m)\left[h - x_m\right]}{U'(x_m)} > 0, \tag{23}$$

$$\frac{dW}{dx_m} = U''(x_m)\left[x_m - h\right] > 0. \tag{24}$$

$$\frac{dW}{dh} = 2U'(x_m) > 0 \tag{25}$$

$$\frac{d^2W}{dh^2} = \frac{-U'(x_m)}{h - x_m} < 0 \tag{26}$$

Intuitively, equation (22) and (23) have two implications: first, h is a function of x_m and x_m is an inverse function of h; second, at the optimal condition, the minimum income x_m should be increasing with the product of average income with the difference between one and the Gini coefficient, which is called real income or Sen's social welfare function. Equations (24) and (25) imply that social welfare is positively related with the living conditions of the poorest and an effective way to promote social welfare is by improving the conditions of the poorest. Equation (26) demonstrates that the welfare function is concave with respect to $h = \mu(1-\overline{G})$.

If we compare the optimal social welfare function $W = R(x_m) + U'(x_m)\left(\mu(1-\overline{G})\right)$ with the other existing one, we find that the optimal social welfare function in this paper better expresses the meaning of social welfare in that the optimal social welfare function is determined by the minimum income, average income and inequality measures, while others take only one or two of the factors into consideration. Moreover, the most important and attractive characteristic of the optimal social welfare function is that it takes both level variable sand structural variables into consideration.

5.4 Relativity and Invariance

The relativity of the social welfare function is based on the relativity of preferences. Relativity of preferences implies that each individual has his or her own preferences that can be described by a special utility function. When an individual measures social welfare based on his own standard or measure, the social welfare function he obtains is only meaningful from his own standpoint. Therefore, any social welfare function that is derived from a special reference frame under certain conditions is logically right, nevertheless, when the reference frame changes, the outcome of social welfare evaluation will change too. Therefore, any social welfare function is a relative truth, and no social welfare function is meaningful for every one. This is the relativity of the social welfare function.

However, since any individual preference follows preference axioms and certain moral norms and conventions, making different individual preference has some similarities; social welfare functions obtained in different reference frames must have some commonalities, indicating that there is something interpersonally comparable, not totally incomparable. From their commonalities some general laws can be found; mathematically they are invariant under a specific kind of transformation. This is invariance of the social welfare function. If relativity states the differences between valuations for social welfare, invariance reflects their similarities.

First, each person values social welfare in his own preferences and reference frame as if one observes the locus of the moon with different coordinates from the earth and sun; what he observes must be different. Assume a rational person whose preferences are represented by the utility function

$U(y)$, then according to the hypotheses of the theorems stated above, he values the social welfare function by a probability measure with distribution function $F^*(y) = 1 - \frac{U'(y)}{U'(y_m)}$. Different preferences ($U(y)$) correspond to a different measure of the social welfare function. Correspondingly, for a different measure of the social welfare function ($F^*(y)$), a different social welfare function is denoted, which is called a sublinear expectation by Peng (2007): $\hat{E}U(x) = W(F^*) = U(x_m) - U'(x_m)x_m + U'(x_m)\left(\mu(1-\overline{G})\right)$.

Since there has been no machine invented to measure utility, there is no reason to force the belief of interpersonal comparability. If we can find a transformation by which a social welfare function can be transformed so that it can be expressed in terms of money, we could conclude that the social welfare function is measurable and interpersonally comparable.

To obtain a comparable social welfare function, let us first standardize the utility function. Given utility function $U(x)$, define the standardized utility function of $U(x)$ as $\overline{U}(x) = \left(U(x) - U(x_m)\right)/U(x_m)$, then from (21) we have

$$\overline{W}(F^*) = \hat{\mathrm{E}}\left[\overline{U}(x)\right] = \mu(1-G) - x_m \quad (27)$$

Now let us turn to comparability and additivity. Since the standardized social welfare function is expressed in terms of money, given any two different utility functions $\overline{U}_1(x)$ and $\overline{U}_2(x)$, we conclude that two social welfare functions $\hat{\mathrm{E}}\left[\overline{U}_1(x)\right] = (1-G)\mu - x_{m1}$ and $\hat{\mathrm{E}}\left[\overline{U}_2(x)\right] = (1-G)\mu - x_{m2}$ could be added and compared, and then we have $\hat{\mathrm{E}}\left[\overline{U}_1(x)\right] - \hat{\mathrm{E}}\left[\overline{U}_2(x)\right] = x_{m1} - x_{m2}$. This demonstrates that the same income distribution is valued differently by different people; differences between two social welfare functions equals the difference between their minimum incomes defined respectively by the two persons.

From the left side of equation (27), we see that the social welfare function is a sublinear expectation of a value function; from the right side of the equation, we see that the social welfare function is expressed as the difference between x_m and h and this relationship is irrelevant with respect to the specific utility function. Therefore, the standardized social welfare function is robust regarding the standardized utility function.

[3] One could not simply say that the social welfare function is a decreasing function of x_m because x_m is not dependent from h. Actually, the social welfare function is an increasing function of x_m.

Invariance of the social welfare function here has two implications: on one hand, the invariance of the social welfare function is related to robustness and it refers to a kind of rule or law of social preferences expressed in a mathematical expression. Whatever the standardized utility function $U(x)$ is, the social welfare function $\hat{\mathrm{E}}\left[\overline{U}(x)\right]=\mu(1-\overline{G})-x_m$ is invariant to a change in the specific form of a utility function, that is, the right side of the formula of the social welfare function is independent of $\overline{U}(x)$; on the other hand, the invariance of the social welfare function refers to the idea that the social welfare functions are invariant to a group of monotonous increasing transformations. In section 3 we have already proved that x_m is an increasing function of $h=\mu(1-\overline{G})$, thus, for any standardized utility function $\overline{U}_1(x)$ and $\overline{U}_2(x)$, there is a transformation $g_{\overline{U}}$ transforming the social welfare function $\hat{\mathrm{E}}\left[\overline{U}_1(x)\right]$ to $\hat{\mathrm{E}}\left[\overline{U}_2(x)\right]$, that is, $\hat{\mathrm{E}}\left[\overline{U}_1(x)\right]=g\left(\hat{\mathrm{E}}\left[\overline{U}_2(x)\right]\right)$. Similarly, since Sen's social welfare function is a special case of the social welfare function obtained in theorem 5, any social welfare function form can be transformed into Sen's function by g: $\hat{\mathrm{E}}\left[\overline{U}(x)\right]=g_U(h)=g_U\left(\mu(1-\overline{G})\right)$. Equivalently, for any standardized utility function $\overline{U}(x)$ there is an increasing function transformation $\overline{g}=g^{-1}$ making $\overline{g}\left\{\hat{\mathrm{E}}\left[\overline{U}(x)\right]\right\}=\mu(1-\overline{G})$. This is a generalized invariance principle of the social welfare function.

6. Economic Welfare and its Policy Implications to Hong Kong and Mainland China

The above theoretical discussion indicates that the social welfare function is not necessarily individualistic; it may be consistent with opinions of all of the society, it may be oligarchy, or dictatorial irrespective of opinions of others, and it also may be determined by some specific rules. In short, any function that follows preference axioms and ranks social states rationally can be regarded as a social welfare function. Mathematically, the social welfare function is a projection of social states on a specific reference frame that represents a specific preference and restriction. We have demonstrated that under proper assumptions the social welfare function can be represented by the function $\mu(1-G)$ (per capita income multiplied by the Gini coefficient).

Generally speaking, per capita GNI can be considered as the first representative indicator of economic welfare. China's nominal per capita GNI in 2010 was $4,382, ranked 94th in the world, while Hong Kong's nominal per capita GNI was $31,591, and ranked 25th in the world. However, goods are priced differently across countries and regions and purchasing power of money varies across time and place. Therefore, any measurement of GNI in a specific currency is not a good measure of social welfare. It is suggested that per capita GNI be adjusted by purchasing power parity (PPP) before it is used to compare social economic states. After adjustment by PPP, it is estimated that per capita GNI in 2010 in mainland China and Hong Kong would have respectively reached $7,519 and $45,736.

Economic performance of some major developing and developed countries is listed in table 2. From the table, we can see that the US was in a leading position and all other developed economies including Japan, Germany, France and UK were very close to each other in GNI per capita. Low growth is thus a common feature of developed economies. India and China' performance are much better than other developing economies.

Although per capita GNI in PPP offers a partial remedy for the deficiency of nominal GNI, it fails to take into account income inequality and it is not a good indicator of welfare. Our model analysis has demonstrated that equality plays an important role in social welfare and social welfare should be measured with per capita GNI discounted by the Gini coefficient. The United Nations' Human Development database, which provides data on per capita GNI, the Gini coefficient and the economic welfare index of mainland China, Hong Kong, and some other selected countries, shows that the inequality in terms of Gini coefficients in China has experienced a significant rise, and the Gini coefficients in Hong Kong has long been the highest in the developed economies.

Table 2 demonstrates that social welfares are not as large as that indicated by GNI. In the case of Russia, its economy has recovered from the collapse of the Soviet Union, but the welfare level is still lower than 20 years ago. China, Brazil, and India's social welfare improved greatly, but it seems not perform so well as their GNI.

Table 2 also demonstrates that the Gini coefficient has a significant impact on the social welfare index and the relative order of the countries may be changed after GNI per capita is adjusted by the Gini. For example, per capita GNI of the United States in 2008 ranked first place of the six developed economies, The order of the six economies is US>Hong K>UK>Germany>Japan>France, but the social welfare index of the United States has no longer been the first and the order of the six economies has been changed to Germany> US>France>UK>Japan>Hong Kong. It is clear that Germany's welfare state ranked in first place and Hong Kong was last.

In past decades, China' economy has benefited from economic reform and opening up policies, but the major benefit of economic growth goes mainly to land and real estate developers, foreign venture enterprises, monopolistic state owned enterprises and the government sector; only a minor part of the benefit is shared by farmers, ordinary workers and employees of non-foreign funded enterprises. A considerable number of people feel that they are marginalized by economic growth, their life quality is worsening and their feeling of happiness is declining.

In terms of PPP, Hong Kong's GNI per capita ranks second to the United States, but its welfare index is the lowest among the developed economies because of its highest income inequality. Therefore, Hong Kong's economy is not so successful from the angle of welfare states. This conclusion is also supported by data from the Hong Kong family survey. According to a recent report of Oxfam Hong Kong (http://www.oxfam.org.hk/en/wageneeds.aspx), poverty has worsened in the past five and a half years among families of low-income workers, and the gap between poor and rich families is at its highest ever. In 2010 Q2, one in every 10 households with at least one working member (10.2%) was

living in poverty, with a monthly income less than half the median among families of comparable size. From 2005 to 2010 Q2, the number of households of employed poor increased from HK$172,600 to a record high of HK$192,500 or 12%, despite the economic growth.

From 2006 to 2010Q1, the monthly median wage in Hong Kong experienced only a minor change, from HK$10,000 to HK$11,500 or increased 15%, compared with the 21% increase in consumer prices from 2005 to 2009 and soaring property prices in recent years. Comparing the monthly household median income of the richest 10% of Hong Kong households with the poorest 10% in 2010 Q1, it is found that the former is 27 times that of the latter. Moreover, the median monthly income of the poorest 10% and 20% of households has remained the same as in 2005, which are HK$3,000 and HK$6,000 respectively. However, an increase of 16% is noted in the highest 10% of households, from HK$70,000 in 2005 to HK$80,900 in 2010 Q1. If the rising consumer price index is considered, middle and low income families are facing a worsening situation.

Many measures can be taken to remedy social welfare loss. The first is to raise minimum wage. So far, most developed economies have set up a minimum wage standard and a minimum living standard. Perhaps to maintain the freest economy in the world, Hong Kong lags behind other developed economies in implementing minimum wage. Fortunately, Hong Kong passed a statutory minimum wage standard in 2010. It is advised that the minimum wage standard is adjusted with economic growth and inflation to ensure ordinary people have a decent life. Mainland China introduced its minimum wage system in 2004 and per month minimum wage standards in different regions are stipulated separately by local governments according to local economic development. China's minimum wage in different regions ranges in 2010 between 460 and 1,180 yuan, which is clearly lower than the $1 per day poverty line of the United Nations. According to the $1 per day standard, it is reported that there are still over one million people living in poverty.

The raising minimum wage standard may cause other problems. One of the potential problems is to worsen unemployment. Mainland China and Hong Kong are also advised to take concrete steps to adjust the income distribution relationship between government and enterprises, and let more people share the fruit of economic growth. More effort should be made to strengthen medical care, health care and increase social insurance and unemployment insurance coverage.

The Chinese government is also advised to take concrete step to adjust the distribution relationship between the local government and central government, personal income tax, eliminate private use of public resources, rule out government entertainment consumption, and many other corruption actions. More bonuses of state owned enterprises should be collected to put into social security funds and enable ordinary people to share the benefits of state owned enterprises. A wage ceiling on senior managers of state owned enterprises should be implemented.

7. Concluding Remarks

The social welfare function is a real-value function associated with preference that ranks conceivable social states. Since preference is a relative concept varying with the reference frame of evaluation and associated with moral judgments. The social welfare function is also a relative concept. In welfare economics, each person values social welfare according to his own preferences. The impossibility theorem is mainly caused by putting different individualistic assumptions together with interpersonal incomparability and other conflicting assumptions. Nevertheless, unconquerable difficulties of social choice under Arrow's framework can be conquered in other frameworks. It is possible to obtain a social welfare function by aggregating different individual preferences.

Inspired by the relativity theory in physics, the author of this paper thinks that to escape from impossibility, it is necessary to choose a given reference frame in which social welfare is evaluated. If one's reference frame is given according to his preference, then social welfare can be evaluated. Although social welfare can be evaluated in any preference reference frame, only the social welfare function evaluated from the government standpoint has clear implications for policy. In the case in which social welfare is evaluated by a team with similar preferences, agreement on social welfare evaluation may be possible.

Moral judgments are important in social welfare evaluation. This paper takes expected utility maximization under ambiguity as a social welfare objective, and justice, equity and efficiency condition as its constraints. In a risky environment, expected utility maximization is regarded as the moral judgment of a rational economic person; in social welfare evaluation under ambiguity, expected utility maximization also applies. The difference is that decision makers face different contexts.

The paper contributes to economic literature in five aspects. First, it sets up a unifying framework of social welfare evaluation using a decision model under ambiguity, provides a methodology of preference estimation and establishes the relationship between preference and income distribution.

Second, it is shown that some existing social welfare functions such as Sen's social welfare function, Elite's and Rawls's are special cases of the social welfare function obtained here. This indicates that although some other social welfare functions are derived from conditions that are different from this paper, they can be explained by a rational framework.

Third, the social welfare functions obtained here is invariant up to a group of monotonous increasing transformations, and it is proven that social welfare functions obtained from different preferences are at least partly compared because of similarity of preference. Its robustness provides empirical studies with a good alternative model.

Fourth, the social welfare functions obtained here satisfy all axioms of sublinear expectations, and then it can be thought as a generalization of traditional expectations.

Finally, some empirical studies with regard to mainland China and Hong Kong are carried out and policy implications are discussed and recommended. In addition, the framework is found to be flexible enough to contain various variational models and can be extended to involve tax policy analysis. Therefore, it can be seen as a starting point for later research on the social welfare function. Many problems are left to discuss in a separate paper.

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Table 1. Definitions of Various Social Welfare Functions (SWF)

Function Name	Advocators	Expressions
Classical Utilitarian SWF	Mashall, Pigou,	$W = U_1 + U_2 + \cdots + U_n$
Neoclassical SWF		$W = p_1 U_1 + p_2 U_2 + \cdots + p_n U_n$,or
		$W = \int U(x) dF(x)$
Nash SWF	John Nash Jr.	$W = U_1 * U_2 * \ldots * U_n$
Elite SWF		$W = Max\{U_1, U_2, \cdots U_n\}$
Rawls' SWF	John Rawls	$W = Min\{U_1, U_2, \cdots U_n\}$
Sen's SWF	Amartya K. Sen	$W_{gini} = \mu(1-G)$
Foster's SWF	James E. Foster	$W_{Theil-L} = \mu e^{-T_L}$

Table 2. Economic Welfare Index

Regions	Index	1990	2000	2005	2008
	GINI	27	25	28	28
Germany	GNI	27296.1	31797.6	33373.1	35949.5
	WELFARE	19926.2	23848.2	24028.6	25883.6
	GINI	42.8	46.2	46.9	46.7
US	GNI	34405.6	43079.1	45894.1	46788.7
	WELFARE	19680	23176.6	24369.8	24938.4
	GINI	30	28.8	28	28
FR	GNI	27026.2	32011.2	33396.6	34294.9
	WELFARE	18918.3	22792	24045.6	24692.3
	GINI	34	32	32	32
UK	GNI	24192.6	30675.5	34466.4	36237.4
	WELFARE	15967.1	20859.34	23437.2	24641.4
	GINI	36.43	38	31.9	32.7
Japan	GNI	28213.3	31433	33654.6	35188.2
	WELFARE	17935.2	19488.46	22918.8	23681.7
	GINI	43.44	52.3	53.3	53.3
HK	GNI	24954.7	32463.1	38675.2	43998
	WELFARE	14114.4	15484.9	18061.3	20547.1
	GINI	14	39.5	37.5	42.2
Russia	GNI	13646.9	9086.1	12523	15455.4
	WELFARE	11736.3	5497.091	7826.9	8933.2
	GINI	60.6		56.4	56.7
Brazil	GNI	7565.8	8337.1	8981.9	10076.6
	WELFARE	2980.9	8337.1	3916.1	4363.2
	GINI	34.8	43.2	47	47
China	GNI	1193.8	2849.3	4434.5	6014.2
	WELFARE	778.4	1618.4	2350.3	3187.5
	GINI	32	32.5	36.8	36.8
India	GNI	1290.2	1840	2401.1	2931.5
	WELFARE	877.3	1242	1517.5	1852.7

Source: http://en.wikipedia.org/wiki/List_of_countries_by_income_equality, United Nations Development Report and the Central Intelligence Agency (CIA) World Fact Book.