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A Quasi-Bounded Target Zone Model – Theory and Application to Hong Kong Dollar

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November 2012

Abstract

The exchange rate target zone literature has often suggested that mean reversion in an exchange rate within a zone can be taken as evidence that the system is credible. While the exchange rate system in Hong Kong is perceived as having a high degree of credibility, there is mixed empirical evidence to suggest that the HKD shows mean reversion. This paper proposes a quasi-bounded process for exchange rate dynamics within a target zone, consistent with a fully credible exchange rate band in which the exchange rate cannot breach the strong-side limit while the weak-side limit is only accessible under restricted conditions of the relationship between the parameters of the drift term and stochastic part of the process. Our empirical results suggest that this model can describe the dynamics of the Hong Kong dollar where the drifting force is an increasing function of foreign reserves.

Keywords: Stochastics Process, Target Zone, Bounded Process

JEL Classification: F31, G13

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1.   Introduction

It has often been argued in the policy literature that within an exchange rate target zone, the mean-reverting properties of the currency can be taken as a sign that the market judges the zone to be credible. According to the target-zone model in Krugman (1991), a Brownian motion with a reflecting boundary condition at the upper and lower limit can be used as the driving process to keep the exchange rate within the zone under a mean-reverting process.[1] The driving force behind this mean-reverting property has been widely debated.[2] Some attribute this to central bank intervention within the target zone, suggesting that the exchange rate may not exhibit such a property if intervention occurs only at the limits of the target zone. Others argue that credibility induces "stability speculation" by market participants, producing forces to pull the exchange rate back to the central parity whenever it drifts too far from it.

Many empirical studies attempt to investigate this theoretical prediction by examining the time-series properties of the European currencies within the Exchange Rate Mechanism (ERM).[3] The results are mixed, perhaps reflecting that the assumption of a fully credible target zone is violated. Some of the currencies are found to follow a random-walk process.[4] Since 1983 Hong Kong has operated a Linked Exchange Rate System (LERS) whereby the Hong Kong dollar (HKD) is fixed at a rate of 7.8 per US dollar (USD). The LERS was refined into a target zone system in May 2005 when a symmetric convertibility zone was introduced with a strong-side convertibility undertaking at 7.75 HKD/USD and the weak-side convertibility undertaking at 7.85. During the past three decades many fixed or target-zone exchange rate systems have come and gone, succumbing to shocks and/or speculative attacks. Yet the LERS has survived suggesting it enjoys a higher degree of credibility in financial markets. However, the empirical evidence on whether the HKD shows mean reversion is mixed. In view of these findings, this paper proposes and examines a new, simple and analytically tractable model of exchange rate dynamics in a fully credible target zone, and applies the proposed model to the HKD.

The proposed model is similar to the conventional target zone arrangement in which the exchange rate is defined as domestic currency value per unit of foreign currency. The central bank can always keep the exchange rate within the zone by selling the domestic currency to maintain the exchange rate at or higher than the strong-side limit. At the weak-side limit the central bank needs to buy

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[1]   See Christiansen et al. (1998) and De Jong et al. (2001) for reviews on modelling of exchange rates in target zones.

[2]   For example, Svensson (1993), Rose and Svensson (1994), and Anthony and MacDonald (1998).

[3]   The European Economic Community adopted this system which also known as the ERM in March 1979, that was replaced with the ERM II on 1 January 1999. Under the ERM of 1979 each member country was required to maintain its exchange rate with the European Currency Unity (ECU) within certain bands. After several realignments of the currencies in the early 1980s, the ERM stabilised with bands of $\pm 2.25\%$ of parity with the ECU for each currency. In September 1992, Italy and the UK were required to leave the ERM because they could not maintain their currencies within the bands. In August 1993, the bands for six member countries (Belgium, France, Ireland, Portugal and Spain) were relaxed to $\pm 15\%$.

[4]   For example, Ball and Roma (1994), Svensson (1993), Rose and Svensson (1994), Nieuwland et al. (1994), Anthony and MacDonald (1998, 1999), and Kanas (1998).

domestic currency using foreign reserves to prevent the exchange rate from breaching its weak-side limit. Thus the exchange rate can be bounded within the zone as long as the central bank has adequate foreign reserves. When the exchange rate is "well within" the band, market participants behave as if they are in a comfort zone and do not feel particularly compelled or encouraged to pull the exchange rate towards its central parity. However, when the exchange rate moves closer to the boundary, the market may anticipate an intervention and act to stabile the exchange rate or even to push it back to its central parity. Hence, we have a bounded process with a stopping/reversion effect that only occurs close to the boundaries of the zone but not necessarily in or around the middle of it.

The proposed exchange rate dynamics allow different boundary conditions at the weak and strong-side limits because a central bank's ability to keep the exchange rate within the zone at these two limits differs. The intervention policy of the central bank and the behaviour of market participants is modelled implicitly by specifying drift and diffusion coefficients of the exchange rate dynamics that determine the boundary conditions at the two limits. There is an explicit inaccessible boundary at the strong-side limit where the intervention policy always functions by simply selling the domestic currency to the market. The variance of the exchange rate also vanishes at the strong-side limit. Such a property is similar to the bounded exchange rate dynamics in Ingersoll (1996) and Larsen and Sørensen (2007) in which the variance of the exchange rate vanishes at both the weak-side and strong-side limit under the assumption that the central bank will intervene to stabilise the foreign exchange market and keep the exchange rate well within its zone.

However, the proposed dynamics are different to the Ingersoll model and Larsen-Sørensen model in that, while the variance of the exchange rate declines towards the weak-side limit, the exchange rate can breach the limit under particular conditions, i.e. realignment could occur. The exchange rate is thus quasi-bounded at the weak-side limit. The possible realignment conditions are determined by the value of the drift and diffusion coefficients on the exchange rate dynamics. The drift coefficient can be made a function of foreign reserves or the monetary base. Conversely, in both the Ingersoll model and Larsen-Sørensen model, the exchange rate is completely bounded under all circumstances determined by the model parameters. The specification in the proposed quasi-bounded process is consistent with empirical evidence in Ball and Roma (1993) that most realignments occur when the exchange rate is close to the weak-side limit of the target zone. The exchange rate dynamics are therefore quasi-bounded and different from the Krugman-type models in that the associated realignment risk takes place only at the weak-side limit.[5]

We discuss the quasi-bounded process of the exchange rate dynamics in a target zone and its properties in the following section. The probability density function of the process and the conditions under which the exchange rate is no longer bounded at the weak-side limit are illustrated. A brief introduction of the HKD under a target-zone regime and its exchange rate movements are presented

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[5]   The quasi-bounded stochastic process, which was first presented by Pesz (2002) and then further generalized by Silva et al. (2006) and Cardeal et al. (2007), has found applications in various systems with spatial confinement, e.g. a Brownian walker trapped between fixed plates (see Lo (2003, 2010a) and references therein). Lo (2010b) applies the process to model the tumour cell growth too.

in Section 3. Empirical estimates of the quasi-bounded process for the HKD after the introduction of a convertibility zone in May 2005, and the relationship between the model parameters and capital flows into the HKD, are presented in Section 4. The final section of the paper concludes.

2.   Quasi-Bounded Process for Target Zone[6]

To illustrate the quasi-bounded process of the exchange rate dynamics in a target zone, we consider the exchange rate *S* defined as a domestic currency value (i.e., HKD) of a unit of a foreign currency (i.e., USD), and let $S_L = 7.75$, $S_U = 7.85$ and $S_{CP} = 7.8$ be the strong-side limit, weak-side limit and central parity respectively. With no loss of generality, the normalized dimensionless exchange rate $0 \leq R \leq 1$ is defined by:

$$R = \frac{S - S_L}{S_U - S_L} \tag{1}$$

where $R = 0$ corresponds to the strong-side limit and $R = 1$ corresponds to the weak-side limit.

Then we assume that the normalized dimensionless exchange rate R within the target zone obeys the stochastic differential equation:

$$dR = \left[\kappa\left(\ln R_\theta - \ln R\right) + \frac{1}{2}\sigma^2\left(-\ln R\right) \right]R + \sigma R\left(-\ln R\right)^{1/2}dZ \tag{2}$$

where $R_\theta \equiv \exp\left(-\theta\right)$. The variance $\sigma^2 R^2\left(-\ln R\right)$ achieves its maximum at $R = \exp\left(-1/2\right)$ and vanishes at the zone boundaries. The first derivative of the variance $\sigma^2 R^2\left(-\ln R\right)$ with respect to R goes towards zero as $R \rightarrow 0$ and towards $-\sigma^2$ as $R \rightarrow 1$. The larger change in the variance with respect to R at the weak-side limit shows that the stochastic process of R is manifestly asymmetrical about the central parity at $R \approx 0.5$, reflecting the different nature of the strong-side and weak-side limits and the corresponding intervention measures.

By Ito's lemma, Eq. (2) implies that the exponent or transformed exchange rate:

$$X = -\ln R \tag{3}$$

follows the mean-reverting square-root process defined by:

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[6]   The presentation here is based upon Lo (2010a).

$$dX = \kappa(\theta - X)dt + \sigma X^{1/2}dZ \qquad (4)$$

where $\sigma^2 X$ is the variance that depends upon the level of X, dZ is a standard Weiner process and κ determines the speed of the mean-reverting drift towards the long-term mean θ. It is not difficult to see that $X = \infty$ corresponds to the strong-side limit and $X = 0$ corresponds to the weak-side limit. Figures 1 and 2 show the HKD exchange rate in S and the transformed exchange rate in X respectively from 18 May 2005 to 31 October 2012.

When the process approaches the boundary at $X = 0$, the variance becomes small and the mean-reverting drift drives the process away from the boundary. Under a fully credible target-zone regime, this boundary condition implies that the central bank has sufficient foreign reserves to prevent its currency from breaching the weak-side limit. The ability of the central bank to defend the exchange rate regime is reflected from the parameter κ. A central bank with sufficient foreign reserves may engage in intra-marginal intervention in order to defend its currency whenever the currency depreciates towards the weak-side limit. In addition to central bank interventions, market participants who believe that the exchange-rate band is credible engage in 'stabilizing speculation' which helps to push the exchange rate towards its mean level. Such effects increase the mean-reverting force, determined by the size of κ. Conversely, if market participants believe that the central bank has inadequate foreign reserves their speculative selling of the currency will weaken the restoring force (i.e., smaller κ) towards its mean level.

As the instantaneous variance σ^2 / X of the fractional change in X (i.e., dX / X) is a decreasing function of X, the boundary at $X = \infty$ is inaccessible, especially in the presence of the mean-reverting drift. There may be forces or incentives for market participants to drive the exchange rate away from its strong-side limit, not least that the probability of making money by holding a long position in the currency is almost zero when the exchange rate appreciates very close to its strong-side limit, usually corresponding to capital inflows into the currency. This implies that the parameter κ of the mean-reverting drift will increase to push the currency away from its limit.

In order to facilitate a better understanding of the boundedness of the stochastic process of X, we make use of the change of variable $\xi = X^{1/2}$, which can be easily shown via Ito's lemma to satisfy the stochastic differential equation:

$$d\xi = \frac{1}{2}\left(-\kappa\xi + \frac{4\kappa\theta - \sigma^2}{4\xi}\right)dt + \frac{1}{2}\sigma dZ \qquad (5)$$

where $0 \le \xi < \infty$. This is a well-known generalization of the Rayleigh process. Following Feller's classification of boundary points, it can be inferred that there is a non-attractive natural boundary at infinity and that the one at the origin is a boundary of no leakage for $\sigma^2 \le 4\kappa\theta$ whereas it is not otherwise.[7]

The probability density function (p.d.f.) of ξ is found to be:

$$K(\xi,t;\xi',t') = \frac{4\xi}{\sigma^2 C_1(t-t')}\left(\frac{\xi}{\xi'}\right)^{\omega} \exp\left[-\frac{\omega+2}{2}C_2(t-t')\right] \times$$
$$\exp\left\{-\frac{2\xi'^2+2\xi^2\exp[-C_2(t-t')]}{\sigma^2 C_1(t-t')}\right\} \times \quad (6)$$
$$I_{\omega}\left\{\frac{4\xi\xi'\exp[-C_2(t-t')/2]}{\sigma^2 C_1(t-t')}\right\}$$

where $\omega = 2\kappa\theta/\sigma^2 - 1$, $C_1(\tau) = [\exp(\kappa\tau)-1]/\kappa$, $C_2(\tau) = -\kappa\tau$, I_{ω} is the modified Bessel function of the first kind of order ω. The corresponding p.d.f. of X is then given by:

$$G(X,t;X',t') = \frac{4}{\sigma^2 C_1(t-t')}\left(\frac{X}{X'}\right)^{\omega/2} \exp\left[-\frac{\omega+2}{2}C_2(t-t')\right] \times$$
$$\exp\left\{-\frac{2X'+2X\exp[-C_2(t-t')]}{\sigma^2 C_1(t-t')}\right\} \times \quad (7)$$
$$I_{\omega}\left\{\frac{4X^{1/2}X'^{1/2}\exp[-C_2(t-t')/2]}{\sigma^2 C_1(t-t')}\right\}$$

It is also not difficult to show that when $\sigma^2 > 4\kappa\theta$, we observe probability leakage through the boundary at the origin. For $\sigma^2 \le 4\kappa\theta$, there is no probability leakage and the total probability is preserved. Hence, it is a quasi-bounded process.

For illustration, in Figure 3 we plot the time evolution of the p.d.f. $K(\xi,t;\xi',t')$ for a typical case of the quasi-bounded process in which $\theta = -\ln(0.5)$, $\xi' = 5\theta$, $t' = 0$, $\sigma = 0.5$ and $\kappa = 0.45$. In this case no probability leakage exists at the origin (the weak-side limit) according to the aforesaid criterion. It is observed that the bell-shaped p.d.f. drifts from the initial position towards the long-term mean steadily and that the noise-induced spreading of the p.d.f. is fairly moderate. The probability that

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[7]   See Karlin and Taylor (1981).

the stochastic variable ξ is found close to the long-term mean thus tends to accumulate gradually, and the p.d.f. will eventually approach the steady-state limit:

$$K\left(\xi,t\rightarrow\infty;\xi',t'\right)=\frac{2\xi^{2\omega+1}}{\Gamma(\omega+1)}\left(\frac{2\kappa}{\sigma^2}\right)^{\omega+1}\exp\left(-\frac{2\kappa}{\sigma^2}\xi^2\right) \tag{9}$$

where Γ is the gamma function. It is also not difficult to realize that this steady-state p.d.f. corresponds to the well-known gamma distribution in the stochastic variable ξ^2 and has a peak located at $\xi=\sqrt{\theta-\sigma^2/(4\kappa)}$.

3.   HKD in Target Zone

The LERS for HKD at the fixed rate of 7.8 HKD/USD has been in operation since 1983. The LERS is in essence a Currency Board system, which requires both the stock and the flow of the Monetary Base to be fully backed by foreign reserves. Any change in the size of the Monetary Base has to be fully matched by a corresponding change in the foreign reserves. Although in theory an exchange rate commitment involving currency only might lead to convergence between the exchange rate in the interbank market and the fixed rate for currency, through arbitrage, this did not happen in practice.[8] Thus during the Asia financial crisis in September 1998, a wider exchange rate commitment for reserve balances of banks was introduced at a rate of 7.75 HKD/USD. This was a weak-side commitment in that the Hong Kong Monetary Authority (HKMA) stood ready to purchase unlimited amounts of HKDs using USDs to prevent the currency weakening beyond that rate.[9] The weak-side commitment was gradually moved from 7.75 to 7.8 between April 1999 and July 2000.

Starting in the fall of 2003 the renminbi came under pressure to appreciate. The HKD also came under pressure, perhaps due to expectations in the market that the Hong Kong authorities would follow any move by their counterparts on the Mainland to allow the currency to appreciate with respect to the USD. The combination of very low interest rates, which risked creating overheating in the economy in general and in asset markets in particular, and increased uncertainty about the exchange rate level, which could call into question the credibility of the LERS, suggested the need for an official response. It came on 18 May 2005 in the form of "Three Refinements" to the LERS. These were (i)

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[8]   The arbitrage is a mechanism through which economic agents seek to profit from the deviation between the official exchange rate applicable to the issue and redemption of banknotes and the market exchange rate. For example, when the market exchange rate (say 7.76 for HKD) is stronger than the official exchange rate, banks can buy foreign currency (i.e., USD) in the foreign exchange market, surrender it to the monetary authority in exchange for domestic currency at the fixed exchange rate (i.e., 7.8), and thereby make a profit from the differential between the two rates.

[9]   The Hong Kong Monetary Authority was established on 1 April 1993, by merging the Office of the Exchange Fund (which was responsible for the LERS) with the Office of the Commissioner of Banking. Its main functions and responsibilities are governed by the Exchange Fund Ordinance and the Banking Ordinance and it reports to the Financial Secretary. One of the functions regarding the exchange rate of the HKD is to maintain currency stability, within the framework of the LERS, through sound management of the exchange fund, monetary policy operations, and other means deemed necessary. See Genberg and Hui (2011) about the development of the LERS.

the introduction of a strong-side Convertibility Undertaking at 7.75 HKD/USD, (ii) moving the weak-side Convertibility Undertaking to 7.85 (in small steps over a five week period) thus creating a symmetric convertibility zone around 7.8, i.e., a target zone, and (iii) giving the HKMA the possibility to intervene inside the convertibility zone. Genberg and Hui (2011) find that the LERS has become more credible over time, and their sample period includes the introduction of the convertibility zone. Their assessment is based on extracting information about exchange rate expectations from market prices, in part by investigating exchange rate and interest rate volatility, as well as studying the dynamics of the exchange rate itself.

As shown in Figure 1, after trading within a relatively wide range between May 2005 and August 2008, the HKD exchange rate strengthened towards the strong-side Convertibility Undertaking of 7.75 after September 2008 due to capital inflows, with the strong-side Convertibility Undertaking triggered repeatedly until early December 2009, prompting the HKMA to passively inject liquidity into the banking system.[10] With the capital inflows, the Monetary Base of the HKD expanded notably, by more than two times, and the exchange rate stayed close to the 7.75 as shown in Figure 1.[11] Subsequently, the HKD exchange rate weakened but stayed within the area of the strong-side Convertibility Undertaking for most of the time.

As mentioned, under a target zone regime, mean-reversion of the exchange rate has been interpreted as a sign that the system is credible. We examine the dynamics of the daily HKD exchange rate between 18 May 2005 and 31 October 2012 using the Dickey-Fuller (DF) and the variance-ratio (VR) test. The DF test examines whether the time series is mean-stationary and always returns to its long-term mean within a very short period of time.[12] The rejection of mean stationarity under the DF test does not necessarily imply rejection of mean reversion over a longer period of time.[13] Therefore, the

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[10]   The capital inflows into the HKD after the global financial crisis intensified in September 2008 is discussed in "Latest Analysis of Fund Inflows Into the Hong Kong Dollar" by Norman Chan at http://www.hkma.gov.hk/eng/key-information/insight/20100201.shtml

[11]   The Monetary Base (a part of the monetary liabilities of a central bank) is defined, at the minimum, as the sum of the currency in circulation (banknotes and coins) and the balance of the banking system held with the central bank (the reserve balance or the clearing balance). In Hong Kong, the Monetary Base comprises Certificates of Indebtedness (for backing the banknotes issued by the note-issuing banks), government-issued currency in circulation, the balance of the clearing accounts of banks kept with the HKMA (the Aggregate Balance), and Exchange Fund Bills and Notes.

[12]   The test is done by Dickey-Fuller test with GLS detrending. The Dickey Fuller test constructs a parametric correction for higher-order correlation by assuming that a time series process (y) follows an AR(p) process and adds p lagged differences of y to the test regression of:

$$\Delta y_t = \gamma + \alpha y_{t-1} + \beta_1 \Delta y_{t-1} + ... + \beta_t \Delta y_{t-p} + \varepsilon_t$$

We test the null hypothesis (the presence of a unit root) of $\alpha = 0$ against the alternative hypothesis of $\alpha < 0$. The test is evaluated using the conventional t-ratio for α (t_α) such that:

$$t_\alpha = \hat{\alpha} / (se(\hat{\alpha})),$$

where $\hat{\alpha}$ is the estimate of α and $se(\hat{\alpha})$ is the coefficient standard error.

[13]   For instance, a time series process which is a sine-wave signal with white noise may not pass the test of stationarity. But it may pass the test of mean reversion because the time series process always goes back to the mean regularly.

VR test is used to test whether the variance of multi-period returns decreases with time.[14] In order to see how the significance of these two properties evolves over time, we employ a moving-window approach. Specifically, the two tests are conducted on each of the 989 windows of the HKD, with the first window covering the period from 18 May 2005 to 16 May 2008 (i.e., 783 observations) and the last window covering the period from 2 November 2009 to 31 October 2012 (i.e., 783 observations).

Figure 4 shows the two test statistics over time. At the 5% level of significance, most of the DF test statistics indicate the presence of a unit root, suggesting that the HKD exchange rate is not mean-stationary. However, the VR test accepts the mean reversion hypothesis for a five-day horizon up to April 2011, suggesting that the HKD exchange rate could be mean-reverting over a five day period. The mean reversion hypothesis is also accepted by the VR test for a ten day horizon from November 2010 and a twenty day horizon from November 2011, which suggests that the duration of mean reversion has shifted gradually from a five day to a 10 day or more period. While the VR test results suggest mean-reversion of the HKD, the mixed results from the DF test and VR test results taken together provide more limited evidence.

4.   Empirical Test of Presence of Quasi-Bounded Process for HKD

4.1   Estimations of Model Parameters

In this section, we investigate whether a quasi-bounded process for exchange rate dynamics within a target zone can describe movements in the HKD. To do so, the maximum likelihood estimation (MLE) is used to estimate the model parameters of the process specified in Eq. (3) based on a log-likelihood function which is constructed by the analytical p.d.f. of Eq. (7). The MLE on the daily exchange rate uses time series data from 18 May 2005 to 31 October 2012 to do two sets of estimation. The first set of estimation uses a rolling three-year window with the initial window covering the period between the end of May 2005 to the end of May 2008. The second set of estimation uses an expanding window starting over the same period. For the two sets of estimation, we use three different specifications for the mean-reverting term in Eq. (3): (i) the mean θ is fixed at the middle of the strong-side convertibility zone, i.e., $S = 7.775$, $X = -\ln(0.25)$; (ii) the mean is fixed at the central parity, i.e., $S = 7.8$, $X = -\ln(0.5)$; and (iii) the mean is estimated by the MLE.

The estimated σ using the three-year window is shown in Figure 5 and ranges between 0.13 and 0.24 (see panel A), and are similar for the three specifications of the mean-reverting term. The corresponding z-statistic is much higher than 1.96 (i.e., at 5% significance level) indicating that the

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[14]   The VR at lag K is defined as the ratio of the variance of the K-period return to the variance of the one-period return divided by K. A unity VR means that the time series is a random process. The time series is considered to be mean reverting if the VR is significantly smaller than one. If the VR is significantly larger than one, the time series is considered to be under a mean aversion. Since the VR test statistic asymptotically follows a normal distribution, a standardized VR test statistic is reported in this study for ease of comparison. Further details are in Kim et al. (1991).

estimation of σ is highly significant (see panel B). The results suggest that estimation of the square-root-process part of the quasi-bounded dynamics is robust as it is not affected by the detailed specification of the mean-reverting dynamics. The results for volatility expanding σ are similar in Panel A and C using an expanding and rolling window respectively. With a longer time series, the estimation is statistically more significant (see panel D).

Figure 6 shows that the estimates of the drift term κ are more significant in terms of the z-statistic (higher than the 5% significance level) when the mean is either fixed at the middle of the strong-side convertibility zone or is estimated by the MLE using both the three-year rolling window (panels A and B) and the expanding window (panels C and D). In other words, the drift becomes weaker and less significant when the mean is fixed at the central parity. As the exchange rate moves in the strong-side convertibility zone during most of time of the estimation period (see Figure 1), it is not surprising to have a stronger drift in the strong-side convertibility zone. The drift term κ estimated by using the expanding window (panel C) has less variation than that estimated using a rolling window (panel A), but is qualitatively similar. With a longer time series, the estimates using the expanding window are statistically more significant than those using the rolling window, especially in the case where the mean is fixed at the central parity.

Consistent with the findings in Figure 6, panel A of Figure 7 shows that the estimated mean θ under the specification of the mean-reverting term in (iii) is significant and ranges from $X = 1.37$ (7.7775 in S-scale) to $X = 3.05$ (7.7547 in S-scale), using the three-year rolling window which falls into the strong-side convertibility zone. The estimated θ using an expanding window, shown in panel B, is also significant and ranges between $X = 1.42$ ($S = 7.7774$) and $X = 2.77$ ($S = 7.7506$), which is narrower than, yet still close to, the estimations in panel A using the rolling window.

In summary, the estimation results based on the MLE shown in Figures 5-7 provide evidence that the quasi-bounded process adequately fits the data on the HKD exchange rate in the convertibility zone. The figures also show that the estimated model parameters are time varying, especially in the period between September 2008 and the end of 2009 when there were significant capital inflows into the HKD (i.e., increases in the Monetary Base, see Figure 1). Below we study how the estimated model parameters are related to capital inflows.

4.2.   Relationship between the Model Parameters and the Monetary Base

To identify the relationship between changes in the Monetary Base arising from capital flows and the speed of the drift (κ) towards the long-term mean, we re-estimate all model parameters in Eq. (3) based on the log-likelihood function specified in Eq. (7) with the following additional assumption on κ:

$$\ln \kappa_{t+1} = \ln\left(\overline{\kappa}\right) + \beta \ln(MB_t / MB_{t-1}) \tag{10}$$

where *MB* is the Monetary Base, β is the coefficient to be estimated, and $\overline{\kappa} = 0.01945$ is the average κ estimated in section 4.1. Figure 8 shows the estimated κ and β using the three-year rolling (panel A) and expanding window (panel B), in which the model parameters are estimated by the MLE. The coefficient β in both panels is generally positive suggesting a positive relationship between κ and the changes in the Monetary Base. Thus when there are capital inflows, κ will increase, indicating an increase in the restoring force towards the long-term mean. When the exchange rate moves towards its strong-side limit as a result of capital inflows, the tendency to mean-reversion can act as a stabilising force limiting the upward movement of the exchange rate. Comparing with Figure 6, in which κ is estimated as an independent parameter, the estimates based on Eq. (10) are higher, but by less than 0.01 which is small.

Figure 9 shows the stochastic movement of the exchange rate which is measured by the variance $\sigma^2 X$ using the expanding window. The variance increased following significant capital inflows into the HKD (i.e., increases in the Monetary Base) during the period between September 2008 and November 2009 which pushed the exchange rate towards the strong-side Convertibility Undertaking limit. It is noted that as the strong-side boundary at $X = \infty$ is inaccessible, the higher variance does not affect the boundary condition. With transforming the exchange rate from X back to S, the movement of the exchange rate in S is substantially limited near the strong-side boundary. During the period after November 2009 with no more capital inflows, the variance decreased as X moved away from the strong-side boundary as shown in Figure 2.

4.3.   Steady-State Property and Weak-Side Boundary Condition

Figures 5-7 show that the estimated model parameters have been relatively stable since the end of 2009 since when there have been no significant capital inflows. Subsequently, the exchange rate has traded in the range of 7.75 and 7.8070 (see Figure 1). We examine whether the steady-state p.d.f. of the quasi-bounded process specified in Eq. (9) adequately fits the data on the HKD exchange rate, given no significant external disturbance arising from capital flows. Figure 10 plots the steady-state p.d.f. and the corresponding cumulative distribution function:

$$Cf\left(\xi, \sigma, \kappa, \theta\right) = \int_{\xi}^{\infty} K\left(\lambda, t \to \infty; \lambda', t'\right) d\lambda \tag{11}$$

based on the estimated parameters at September 2009 and September 2011, where $Cf\left(\xi = 0, \sigma, \kappa, \theta\right) = 1$ and $Cf\left(\xi \to \infty, \sigma, \kappa, \theta\right) = 0$. As the p.d.f. has the peak located at $\xi = \sqrt{\theta - \sigma^2 /(4\kappa)}$, the peaks of the p.d.f. at September 2009 and September 2011 are at

$\xi_P = 1.4015$ ($S_P = 7.7640$) and $\xi_P = 1.5353$ ($S_P = 7.7595$) respectively. Consistent with the change in the peaks, Figure 10 shows that the p.d.f. shifts towards the strong-side limit ($\xi = \infty$) over time. Using Eq. (11) and given the cumulative distribution function $Cf(\xi_L, \sigma, \kappa, \theta) = 0.95$, the lower limit ξ_L is found to be 0.6392 ($S_L = 7.8165$) and 0.7992 ($S_P = 7.8028$) for the p.d.f. at September 2009 and September 2011 respectively. This result implies a probability of 95% that the exchange rate in S will move in the range of 7.75-7.8165 and 7.75-7.8028 during the period around September 2009 and September 2011 respectively according to the estimated steady-state p.d.f.. The exchange rate moved between 7.75 and 7.8070 between September 2009 and October 2012, close to our model estimates. This suggests a degree of consistency between the observed dynamics of the HKD exchange rate and the theoretical quasi-bounded process.

Although there has been no pressure on the weak-side limit during the estimation period, the dynamics of the exchange rate may still be informative about the credibility of the target zone if the same dynamics occur on the weak-side of the convertibility zone. To check whether the weak-side limit is bounded given the estimated parameters, the value of $\sigma^2 / 4\kappa\theta$ is presented in Figure 11 with the three specifications of the mean-reverting term. As the condition of $\sigma^2 / 4\kappa\theta < 1$ indicates no probability leakage, the results show that, when the mean θ is fixed at the middle of the strong-side convertibility zone or estimated by the MLE, the exchange rate is bounded at weak-side limit. But when the mean is fixed at the central parity the exchange rate is not bounded. It is noted that θ tends to decrease towards the weak-side limit when the exchange rate is in the weak-side convertibility zone. Given a robust system, κ is expected to increase in order to keep the exchange rate bounded below the weak-side limit.

5.   Concluding Remarks

After the introduction of the three refinements to the LERS in May 2005, the HKD showed no strong tendency to revert towards the centre of the convertibility zone. This is perhaps not surprising as there have been no active interventions in the foreign exchange market notwithstanding passive triggering of the strong-side Convertibility Undertaking.

The results of our empirical analysis suggests that the HKD follows a quasi-bounded process, that is consistent with a fully credible exchange rate band. The estimated model parameters and the corresponding p.d.f. of the process adequately fits the data on the HKD exchange rate in the convertibility zone. The process will limit movements of the exchange rate close to the strong and weak-side limits because its instantaneous variance vanishes at the boundaries of the convertibility zone making it almost inaccessible to the limits. When the HKD exchange rate is "well within" the band, market participants do not feel compelled or encouraged to pull the exchange rate towards the central parity. However, under the quasi-bounded process, when the exchange rate moves closer to

the boundary, a stopping/reversion effect occurs. These forces are not necessarily present in or around the middle of the zone, therefore, relatively large movements of the exchange rate could happen.

The speed of the mean-reverting drift is estimated as an increasing function of foreign reserves. As the Monetary Base increases due to capital inflows into the HKD the speed of the mean-reverting drift increases, which pushes the exchange rate towards its long-term mean and limits the movement of the exchange rate. This suggests that the LERS is a robust target-zone system. The relationship between the restoring force and changes in foreign reserves that we have uncovered implies that if foreign reserves are used to defend the currency at the weak-side limit, the consequent fall in the reserves will weaken the restoring force towards the long-term mean. This property is consistent with the fact that the credibility at the weak-side limit is determined by the adequacy of foreign reserves.

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Figure 1. Monetary Base of HKD and HKD-USD Exchange Rate in the Convertibility Zone with Strong-Side Limit at *S* = 7.75 and Weak-Side Limit at *S* = 7.85.
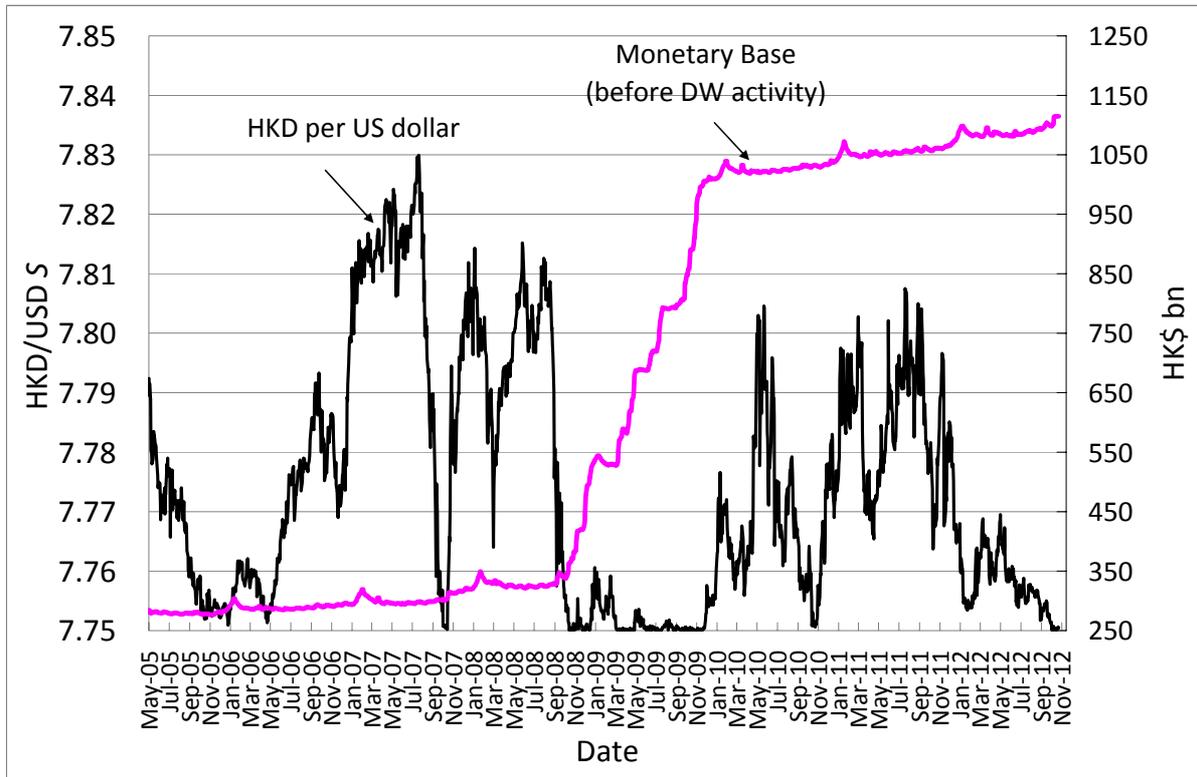

Figure 2. HKD-USD Exchange Rate in *X*-Scale with Strong-Side Limit at $X = \infty$ and Weak-Side Limit at *X* = 0.
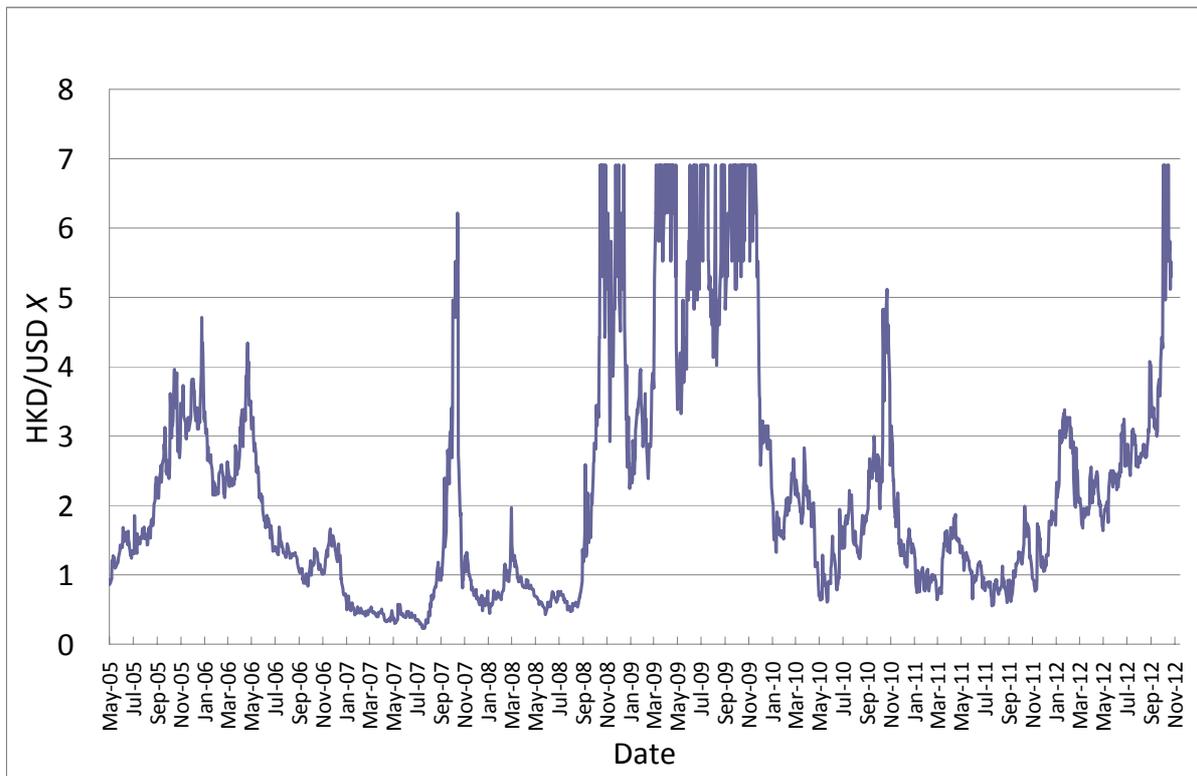
Figure 3. Probability Density Function $K(\xi, t; 5\theta, 0)$ Versus ξ at Time t = 0.1, 0.25, 0.75, 1.25, 2 and 5. Other Input Parameters are: $\theta = -\ln(0.5)$, $\xi' = 5\theta$, $\sigma = 0.5$ and $\kappa = 0.45$.
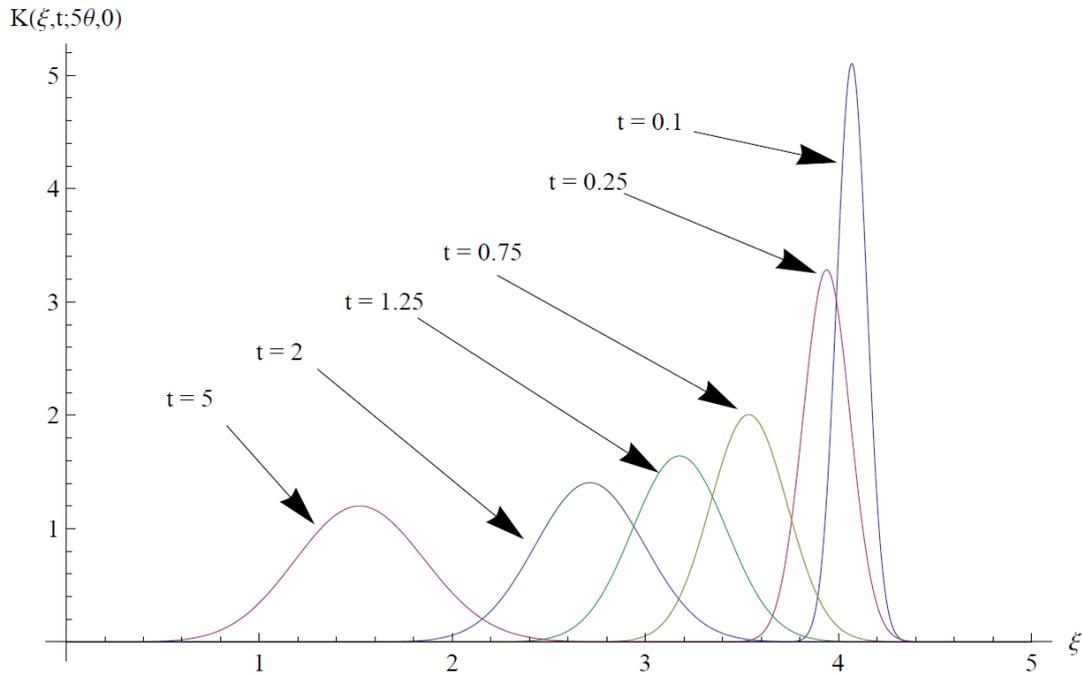

Figure 4. DF Test Statistics and VR Test Statistics Using a Moving Window Approach
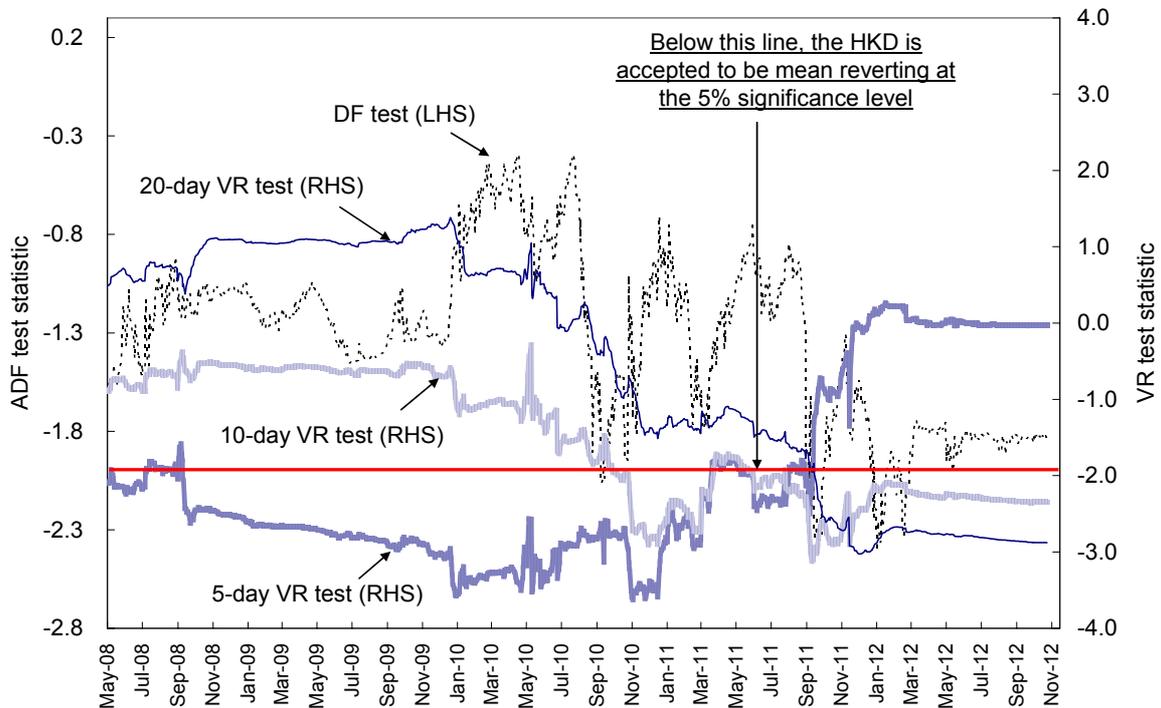
Figure 5. Estimated σ **(panel A) and Corresponding *Z*-Statistic (panel B) using Three-Year Rolling Window, and** σ **(Panel C) and Corresponding *Z*-Statistic (Panel D) using Expanding Window with (i) the Mean** θ **Fixed at the Middle of the Strong-Side Convertibility Zone, i.e.,** $S = 7.775$, $X = -\ln(0.25)$; **(ii) the Mean Fixed at the Central Parity, i.e.,** $S = 7.8$, $X = -\ln(0.5)$; **and (iii) the Mean is an Estimated Parameter.**
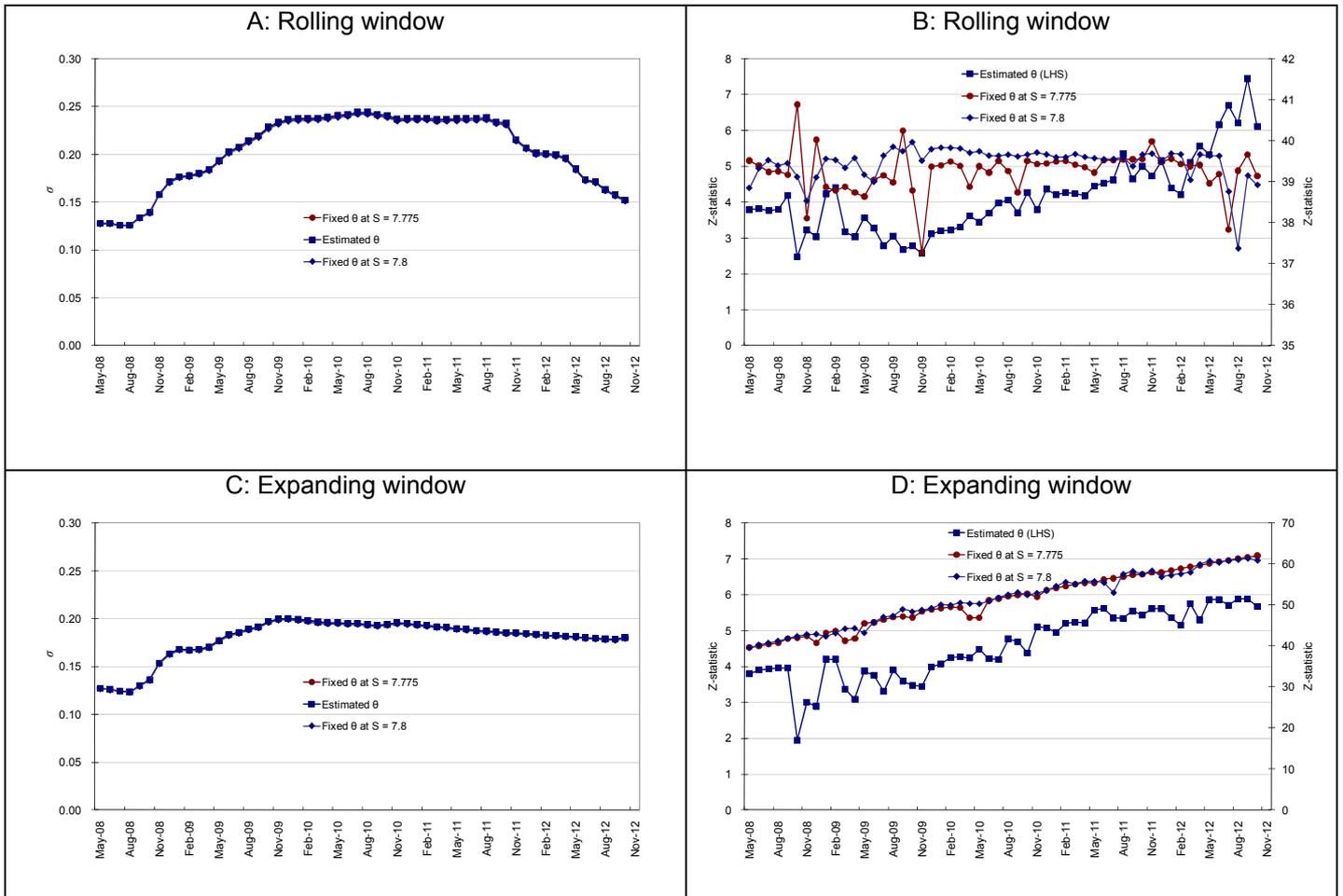
Figure 6. Estimated κ (Panel A) and Corresponding Z-Statistic (Panel B) using Three-Year Rolling Window, and κ (Panel C) and Corresponding Z-Statistic (Panel D) using Expanding Window with (i) the Mean θ Fixed at the Middle of the Strong-Side Convertibility Zone, i.e., $S = 7.775$; (ii) the Mean Fixed at the Central Parity, i.e., $S = 7.8$; and (iii) the Mean is an Estimated Parameters.
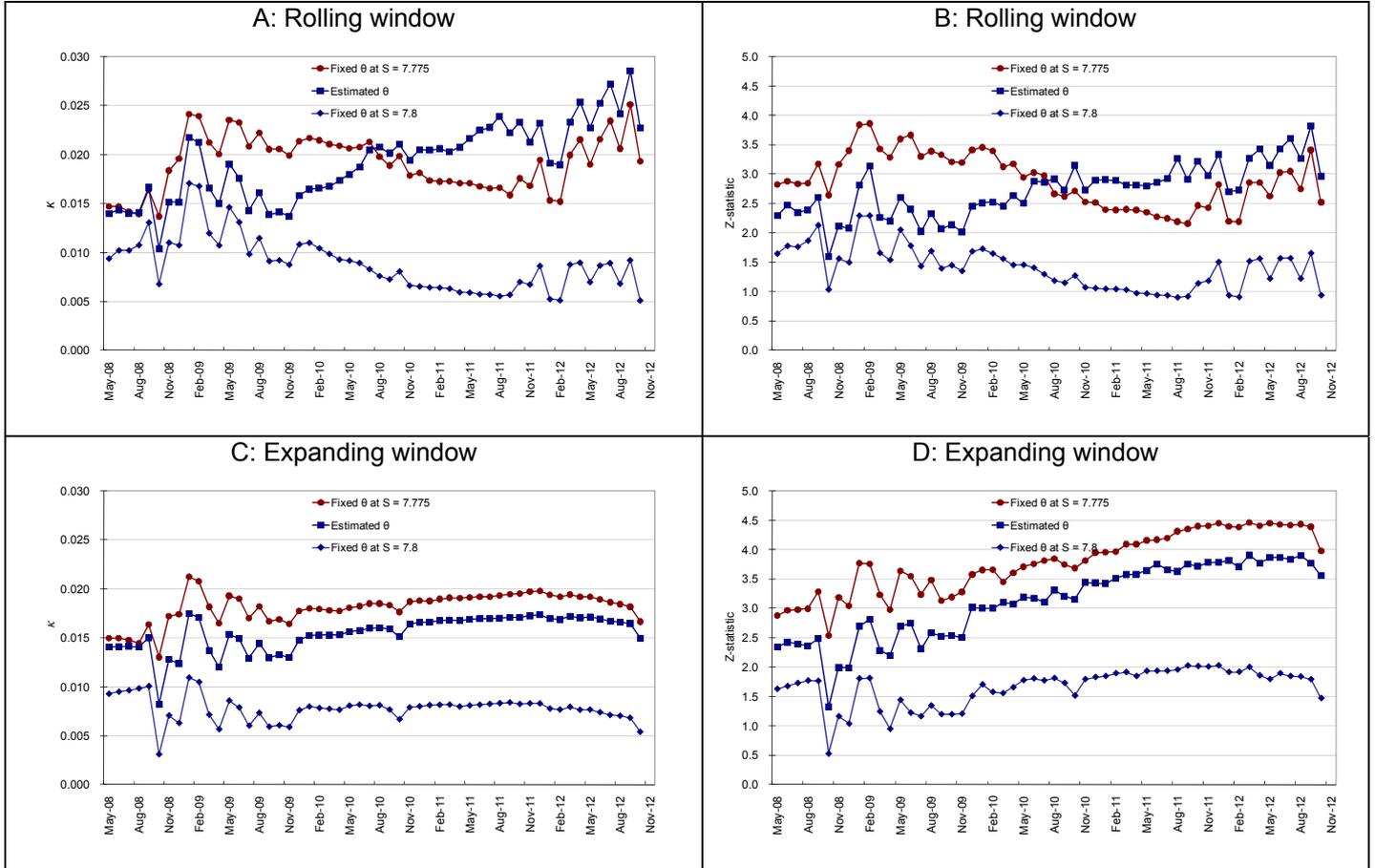
Figure 7. Estimated θ **and Corresponding** *Z*-**Statistic using Three-Year Rolling Window (Panel A) and Expanding Window (Panel B).**
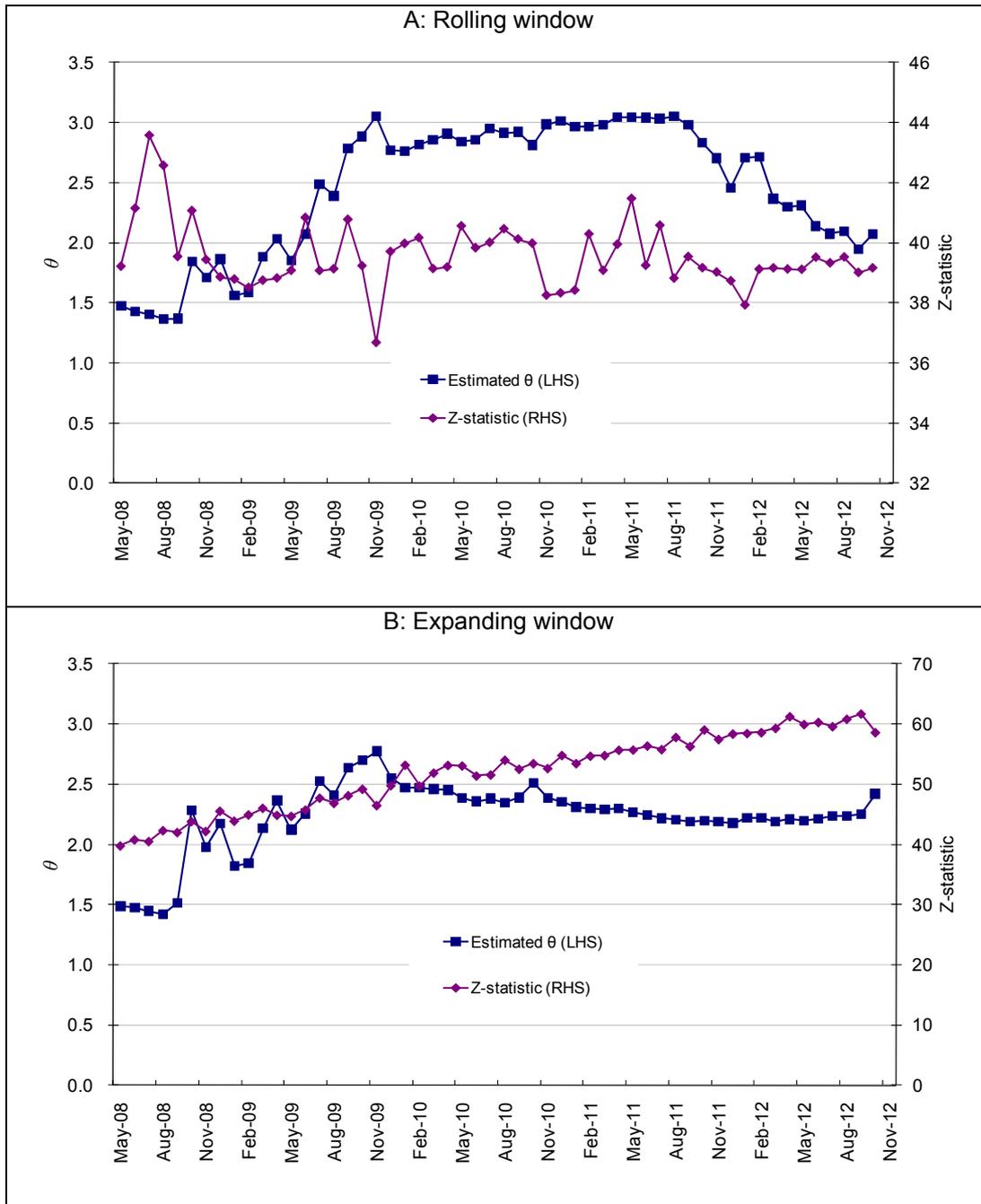
Figure 8. Estimated $\ln \kappa_{t+1} = \ln(\overline{\kappa}) + \beta \ln(MB_t / MB_{t-1})$ **using Three-Year Rolling Window (Panel A) and Expanding Window (Panel B).**
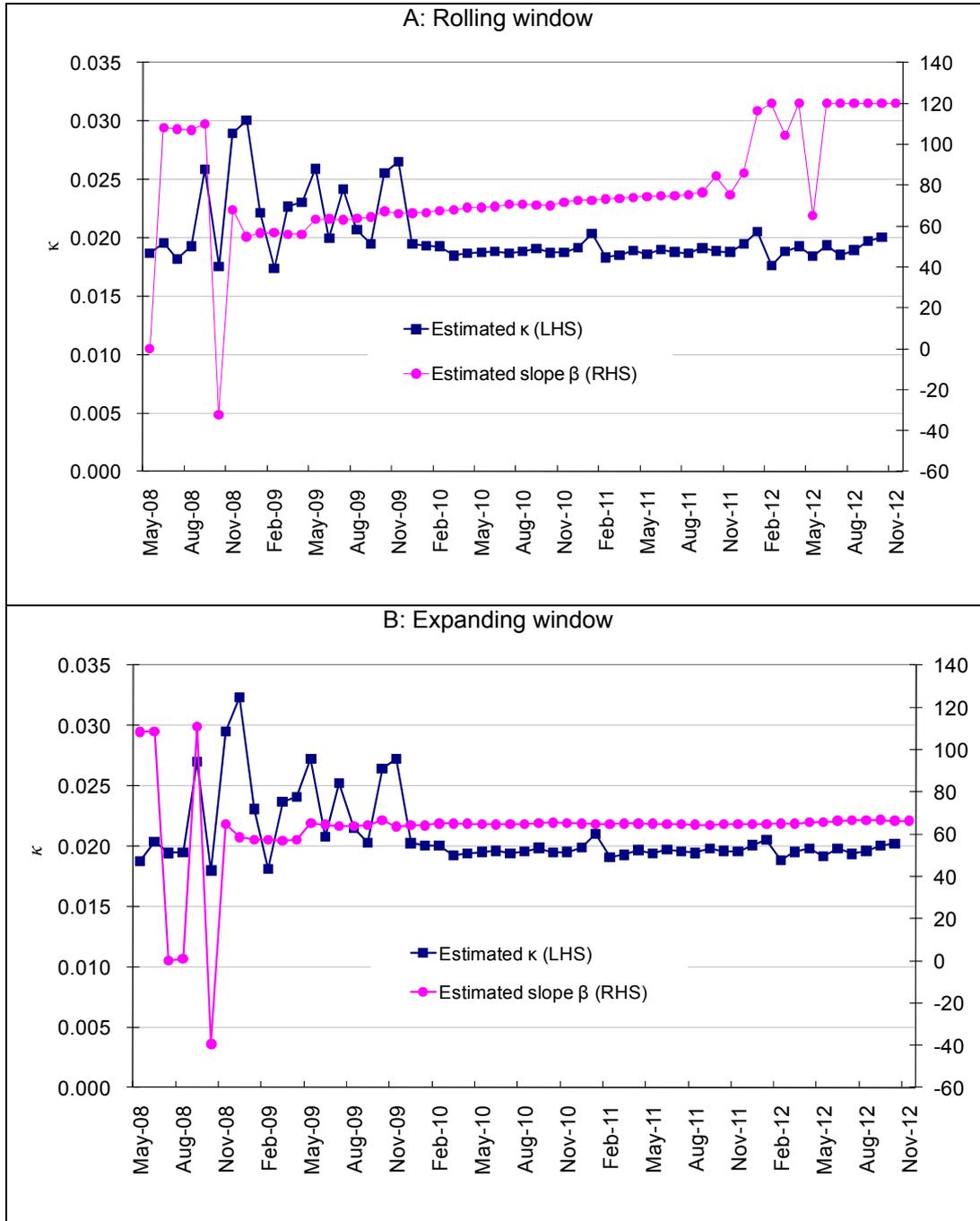
Figure 9. Estimated $\sigma^2 X$ using Expanding Window with the Estimated Mean.
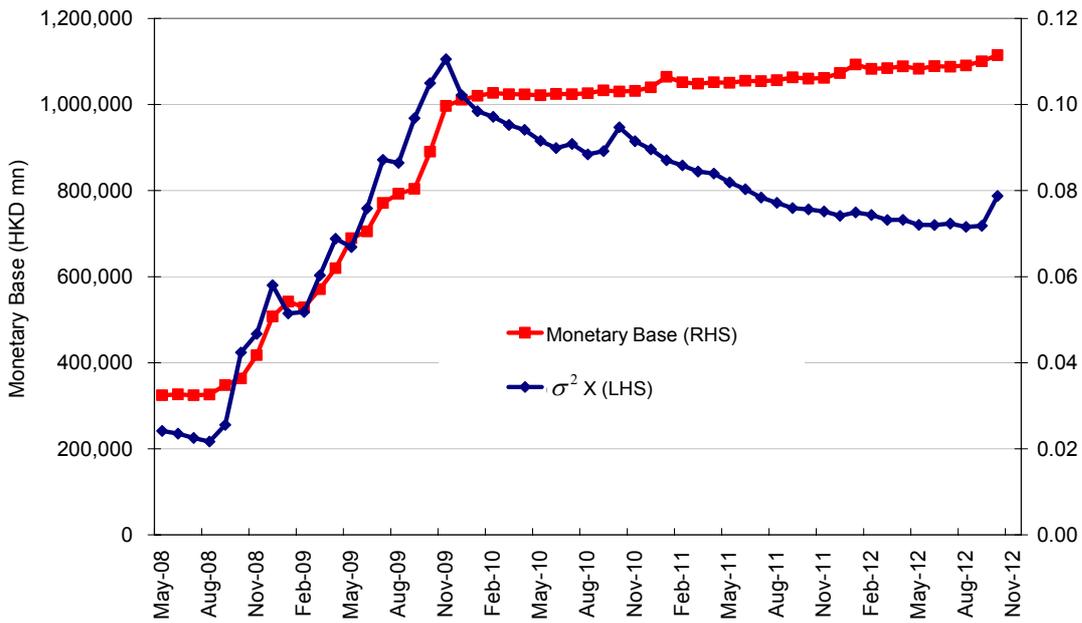

Figure 10. Probability Density Function $K(\xi, t \rightarrow \infty; \xi', t')$ and Corresponding Cumulative Distribution Function $Cf(\xi, \sigma, \kappa, \theta)$ Versus ξ using Estimated Parameters by Three-Year Rolling Window at September 2009 and September 2011.
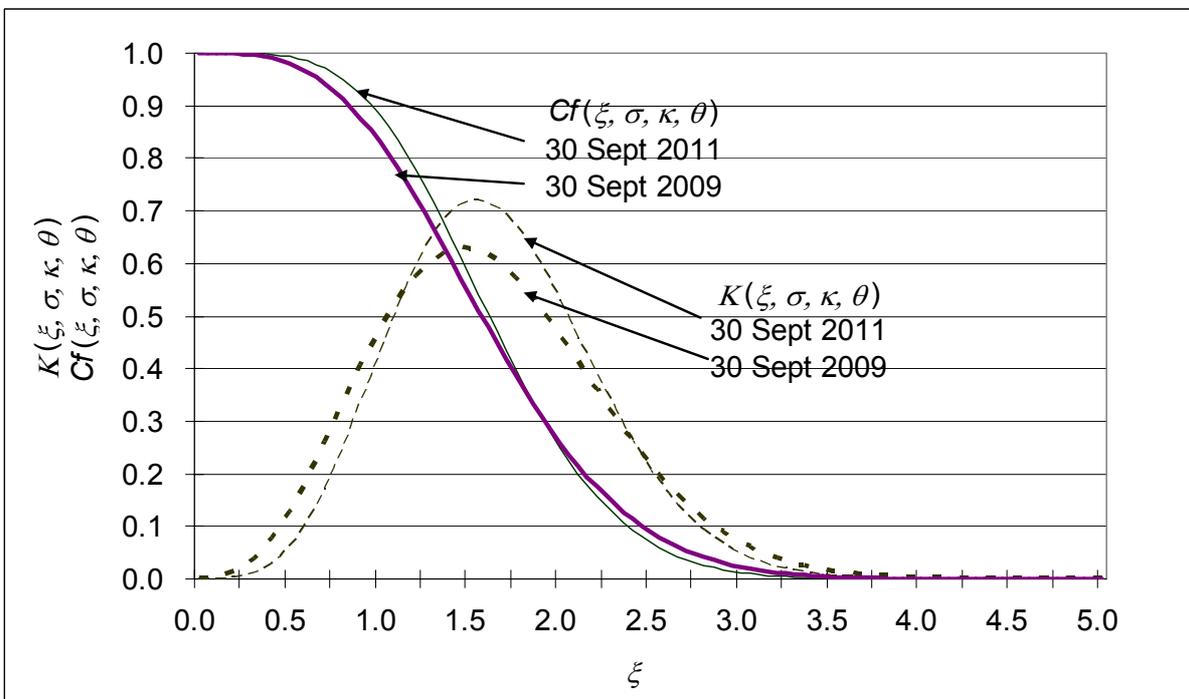
Figure 11. Estimated Values of the Critical Condition $\sigma^2/4\kappa\theta$ at the Weak-Side Limit using Three-Year Rolling Window with (i) the Mean θ Fixed at the Middle of the Strong-Side Convertibility Zone, i.e., $S = 7.775$, $X = -\ln(0.25)$; (ii) the Mean Fixed at the Central Parity, i.e., $S = 7.8$, $X = -\ln(0.5)$; and (iii) the Mean is an Estimated Parameter.