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Explosive Behavior in the 1990s Nasdaq: When Did Exuberance Escalate Asset Values?*

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Abstract

A recursive test procedure is suggested that provides a mechanism for testing explosive behavior, date-stamping the origination and collapse of economic exuberance, and providing valid confidence intervals for explosive growth rates. The method involves the recursive implementation of a right-side unit root test and a sup test, both of which are easy to use in practical applications, and some new limit theory for mildly explosive processes. The test procedure is shown to have discriminatory power in

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detecting periodically collapsing bubbles, thereby overcoming a weakness in earlier applications of unit root tests for economic bubbles. Some asymptotic properties of the Evans (1991) model of periodically collapsing bubbles are analyzed and the paper develops a new model in which bubble duration depends on the strength of the cognitive bias underlying herd behavior in the market. The paper also explores alternative propagating mechanisms for explosive behavior based on economic fundamentals under time varying discount rates. An empirical application to the Nasdaq stock price index in the 1990s provides confirmation of explosiveness and date-stamps the origination of financial exuberance to June 1995, prior to the famous remark in December 1996 by Alan Greenspan about irrational exuberance in financial markets, thereby giving the remark empirical content.

Keywords : Explosive root, irrational exuberance, mildly explosive process, Nasdaq bubble, periodically collapsing bubble, sup test, time-varying discount rate, unit root test

JEL Classifications : G10, C22

How do we know when irrational exuberance has unduly escalated asset values? (Alan Greenspan, 1996)

Experience can be a powerful teacher. The rise and fall of internet stocks, which created and then destroyed $8 trillion of shareholder wealth, has led a new generation of economists to acknowledge that bubbles can occur. (Alan Krueger, 2005)

1. Introduction

During the 1990s, led by DotCom stocks and the internet sector, the U.S. stock market experienced a spectacular rise in all major indices, especially the Nasdaq index. Concomitant with this striking rise in stock market indices, there was much popular talk among economists about the effects of the internet and computing technology on productivity and the emergence of a "new economy" associated with these changes. What caused the unusual surge and fall in prices, whether there were bubbles, and whether the bubbles were rational or behavioral are among the most actively debated issues in macroeconomics and finance in recent years.

Many researchers attribute the episode to financial bubbles. Examples include Greenspan (1996), Thaler (1999), Shiller (2000), The Economist (2000), Cooper et al. (2001), Ritter and Welch (2002), Ofek and Richardson (2002), Lamont and Thaler (2003), and Cunado *et al*. (2005). More recently, economists have sought to rationalize the equity boom using a variety of economic variables, including uncertainty about firm profitability (Pastor and Veronesi, 2006), declining macroeconomic risk (Lettau et al., 2006), high and volatile revenue growth (Schwartz and Moon, 2000), learning (Pastor and Veronesi, 2007) and other fundamentals.

Among the many references, the remark by Greenspan (1996) on December 5, 1996, is the most celebrated, involving as it did the coining of the phrase "irrational exuberance" to characterize herd stock market behavior, a phrase which remains the most oft-quoted remark of the former chairman of the Federal Reserve Board. The remark has been influential in thinking about financial markets and herd behavior and it also had some short-term market effects. Indeed, after Greenspan coined the phrase in a dinner party speech, stock markets fell sharply worldwide the next day.[1] However, in spite of this correction, the Greenspan remark did not halt the general upward march of the U.S. market. On the contrary, over the full decade of the 1990s, the Nasdaq index rose to the historical high of 5,048.62 points on March 10, 2000 from 329.80 on October 31, 1990 (see Figure 1).

One purpose of the present article is to examine empirically the Nasdaq market performance in relation to the market perceptions of exuberance by Greenspan and other commentators. In particular, it is of interest to determine whether the Greenspan perception of herd behavior was supported by empirical

[1] For example, the stock markets in Frankfurt, Hong Kong, London, Toyko and the U.S. fell by 4, 3, 4, 3 and 2 percent, respectively.

evidence in the data or if Greenspan actually foresaw the outbreak of exuberance and its dangers when he made the remark. To achieve this goal, we first define financial exuberance in the time series context in terms of explosive autoregressive behavior and then introduce some new econometric methodology based on forward recursive regression tests and mildly explosive regression asymptotics to assess the empirical evidence of exuberant behavior in the Nasdaq stock market index. In this context, the approach is compatible with several different explanations of this period of market activity, including the rational bubble literature, herd behavior, and exuberant and rational responses to economic fundamentals. All these propagating mechanisms can lead to explosive characteristics in the data. Hence, the empirical issue becomes one of identifying the origination, termination and extent of the explosive behavior. While with traditional test procedures "there is little evidence of explosive behavior" (Campbell, Lo and MacKinlay, 1997, p. 260), with the recursive procedure, we successfully document explosive periods of price exuberance in the Nasdaq.

Among the potential explanations of explosive behavior in economic variables, the most prominent are models with rational bubbles, herd behavior, and propagating mechanisms based on economic fundamentals such as models with time varying discount rates. Accordingly, we first relate our analysis of explosive behavior to the rational bubble literature, where it is well known that standard econometric tests encounter difficulties in identifying rational asset bubbles (Flood and Garber, 1980; Flood and Hodrick, 1986; and Evans, 1991). The use of recursive tests enables us to locate exploding subsamples of data and detect periods of exuberance. The econometric approach utilizes some new machinery that permits the construction of valid asymptotic confidence intervals for explosive autoregressive processes and tests of explosive characteristics in time series data. This approach can detect the presence of bubbles in the data and date stamp for the origination and collapse of the bubble. Second, we consider an alternative source of market exuberance and explosive behavior in terms of time varying discount rates and show that time variation in discounting can produce explosive autoregressive behavior in market prices in which the growth rate is related to the dynamic path of the discount rate. By doing so we provide an alternative economic mechanism for exuberance without resorting to rational bubble models. In this sense we complement the recent literature on rationalizing the 1990s internet episode.

We apply our econometric approach to the Nasdaq index over the full sample period from 1973 to 2005 and some sub-periods. Using the forward recursive regression technique, we date stamp the origin and conclusion of the explosive behavior. To answer the question raised by Greenspan in the first header leading this article, we match the empirical time stamp of the origination against the dating of Greenspan's remark. The statistical evidence from these methods indicates that explosiveness started in June 1995, thereby predating and providing empirical content to the Greenspan remark in December 1996. The empirical evidence indicates that the explosive environment continued until August 2001.

If the discount rate is time invariant, the identification of explosive characteristics in the data is equivalent to the detection of a stock bubble, as argued in Diba and Grossman (1987, 1988). Using standard unit root tests applied to the real U.S. Standard and Poor's Composite Stock Price Index over the period 1871-1986, Diba and Grossman (1988) tested levels and differences of stock prices for nonstationarity, finding support in the data for nonstationarity in levels but stationarity in differences. Since differences of an explosive process still manifest explosive characteristics, these findings appear to reject the presence of a market bubble in the data. Although the results were less definitive, further tests by Diba and Grossman (1988) provided confirmation of cointegration between stock prices and dividends over the same period, supporting the conclusion that prices did not diverge from long-run fundamentals and thereby giving additional evidence against bubble behavior. Evans (1991) criticized this approach, showing that time series simulated from a nonlinear model that produces periodically collapsing bubbles manifests more complex bubble characteristics that are typically not uncoverable by standard unit root and cointegration tests. He concluded that standard unit root and cointegration tests are inappropriate tools for detecting bubble behavior because they cannot effectively distinguish between a stationary process and a periodically collapsing bubble model. Patterns of periodically collapsing bubbles in the data look more like data generated from a unit-root or stationary autoregression than a potentially explosive process. Recursive tests of the type undertaken in our paper are not subject to the same criticism and, as demonstrated in our analysis and simulations reported below, are capable of distinguishing periodically collapsing bubbles from pure unit root processes.

The remainder of the paper is organized as follows. Section 2 defines market exuberance, discusses model specification issues and relates exuberance to the earlier literature on rational bubbles as well as the recent literature on rationalizing episodes of escalation in asset prices in terms of economic fundamentals. Section 3 discusses some econometric issues, such as finite sample estimation bias and the construction of valid asymptotic confidence intervals for mildly explosive processes. Section 4 describes the data used in this study. The empirical results are reported in Section 5. Section 6 documents the finite sample properties of our tests. This section also develops some asymptotic properties of the Evans (1991) model of periodically collapsing bubbles and develops a new model in which the bubble duration depends on the strength of the cognitive bias underlying the herd behavior in the market. Simulations with these models are conducted and the finite sample properties of the tests are analyzed. Section 7 concludes.

2. Specification Issues

2.1 Exuberance, Explosiveness, and Bubbles

When Greenspan coined “irrational exuberance”, the phrase was not defined – see the primary header to this article. Instead, the appellation can be interpreted as a typically cryptic warning that the market might be overvalued and in risk of a financial bubble. In the event, as the second header leading this article indicates, the subsequent rise and fall of internet stocks to the extent of $8 trillion of shareholder wealth renewed a long-standing interest among economists in the possibility of financial bubbles. Theoretical studies on rational bubbles in the stock market include Blanchard (1979), Blanchard and Watson (1982), Shiller (1984), Tirole (1982, 1985), Evans (1989), and Evans and Honkapohja (1992), among many others; and empirical studies include Shiller (1981), West (1987, 1988), Campbell and Shiller (1987, 1989), Diba and Grossman (1988), Froot and Obstfeld (1991), and Wu (1997). Flood and Hodrick (1990) and Gurkaynak (2005) survey existing econometric methodologies and test results for financial bubbles.

It is well known in the rational bubble literature that bubbles, if they are present, should manifest explosive characteristics in prices. This statistical property motivates a definition of exuberance in terms of explosive autoregressive behavior propagated by a process of the form $x_t = \mu_x + \delta x_{t-1} + \varepsilon_{x,t}$ where for certain subperiods of the data $\delta > 1$. Figure 2 gives typical time series plots for stationary ($\delta = 0.9$), random walk ($\delta = 1.0$) and explosive processes ($\delta = 1.02$) with intercept $\mu_x = 0$ and inputs $\varepsilon_{x,t} \sim iid\ N(0, 1)$. The differences in the trajectories are quite apparent.

The concept of rational bubbles can be illustrated using the present value theory of finance whereby fundamental asset prices are determined by the sum of the present discounted values of expected future dividend sequence. Most tests begin with the standard no arbitrage condition below

$$P_t = \frac{1}{1+R} E_t(P_{t+1} + D_{t+1}) \tag{1}$$

where P_t is the real stock price (ex-dividend) at time t, D_t is the real dividend received from the asset for ownership between t –1 and t, and R is the discount rate ($R > 0$). This section assumes R to be time invariant. Section 2.2 examines the effects of time varying discount rates.

We follow Campbell and Shiller (1989) by taking a *log-linear* approximation[2] of (1), which yields the following solution through recursive substitution:

$$p_t = p_t^f + b_t \tag{2}$$

where

$$p_t^f = \frac{\kappa - \gamma}{1-\rho} + (1-\rho) \sum_{i=0}^{\infty} \rho^i E_t d_{t+1+i} \tag{3}$$

$$b_t = \lim_{i \to \infty} \rho^i E_t p_{t+i}$$

$$E_t(b_{t+1}) = \frac{1}{\rho} b_t = \left(1 + \exp(\overline{d-p})\right) b_t \tag{4}$$

with $p_t = \log(P_t),\ d_t = \log(D_t),\ \gamma = \log(1+R)\ \rho = 1/(1+\exp(\overline{d-p}))$, with $\overline{d-p}$ being the average log dividend-price ratio, and

$$\kappa = -\log(\rho) - (1-\rho)\log(\frac{1}{\rho} - 1)$$

Obviously, $0 < \rho < 1$. Following convention, we call p_t^f , which is exclusively determined by expected dividends, the fundamental component of the stock price, and b_t, which satisfies the difference equation

[2] While log linear approximations of this type about the sample mean are commonly employed in both theoretical and empirical work, we remark that they may be less satisfactory in nonstationary contexts where the sample means do not converge to population constants. We therefore used the series both in log levels and in levels in our empirical work and found very similar results for both cases.

(5) below, the rational bubble component. Both components are expressed in natural logarithms. As $\exp(\overline{d-p}) > 0$, the rational bubble b_t is a submartingale and is explosive in expectation. Equation (4) implies the following process

$$b_t = \frac{1}{\rho} b_{t-1} + \varepsilon_{b,t} \equiv (1+g) b_{t-1} + \varepsilon_{b,t},\ \mathrm{E}_{t-1}(\varepsilon_{b,t}) = 0 \tag{5}$$

where $g = \frac{1}{\rho} - 1 = \exp(\overline{d-p}) > 0$ is the growth rate of the natural logarithm of the bubble and $\varepsilon_{b,t}$ is a martingale difference.

As evident from (2), the stochastic properties of p_t are determined by those of p_t^f and b_t. In the absence of bubbles, *i.e.*, $b_t = 0, \forall t$, we will have $p_t = p_t^f$, and p_t is determined solely by p_t^f and hence by d_t. In this case, from (3), we obtain

$$d_t - p_t = -\frac{\kappa - \gamma}{1-\rho} - \sum_{i=0}^{\infty} \rho^j E_t(\Delta d_{t+1+i}) \tag{6}$$

If p_t and d_t are both integrated processes of order one, denoted by I(1), then (6) implies that p_t and d_t are cointegrated with the cointegrating vector [1,-1].

If bubbles are present, *i.e.*, $b_t \neq 0$, since (5) implies explosive behavior in b_t, p_t will also be explosive by equation (2), irrespective of whether d_t is an integrated process, I(1), or a stationary process, denoted by I(0). In this case, Δp_t is also explosive and therefore cannot be stationary. This implication motivated Diba and Grossman (1988) to look for the presence of bubble behavior by applying unit root tests to Δp_t. Finding an empirical rejection of the null of a unit root in Δp_t, Diba and Grossman (1988) concluded that p_t was not explosive and therefore there was no bubble in the stock market.

In the case where d_t is I(1) and hence Δd_t is I(0), equation (6) motivated Diba and Grossman (1988) to look for evidence of the absence of bubbles by testing for a cointegrating relation between p_t and d_t. In the presence of bubbles, p_t is always explosive and hence cannot co-move or be cointegrated with d_t if d_t is itself not explosive. Therefore, an empirical finding of cointegration between p_t and d_t may be taken as evidence against the presence of bubbles.

Evans (1991) questioned the validity of the empirical tests employed by Diba and Grossman (1988) by arguing that none of these tests have much power to detect periodically collapsing bubbles. He demonstrated by simulation that the low power of standard unit root and cointegration tests in this context is due to the fact that a periodically collapsing bubble process can behave much like an I(1) process or even like a stationary linear autoregressive process provided that the probability of collapse of the bubble is not negligible. As a result, Evans (1991, p.927) claimed that "periodically collapsing bubbles are not detectable by using standard tests. "

Equations (5) and (2) suggest that a direct way to test for bubbles is to examine evidence for explosive behavior in p_t and d_t when the discount rate is time invariant. Of course, explosive characteristics in p_t could in principle arise from d_t and the two processes would then be explosively cointegrated. However, if d_t is demonstrated to be nonexplosive, then the explosive behavior in p_t will provide sufficient evidence for the presence of bubbles because the observed behavior may only arise through the presence of b_t. Of course, it seems likely that in practice explosive behavior in p_t may only be temporary or short-lived, as in the case of stock market bubbles that collapse after a certain period of time. Thus, the actual generating mechanism for b_t can be much more complex than (5) and can reflect aspects of the market such as herd behavior and cognitive bias (see Section 6.2). Some of these possibilities can be taken into account empirically by looking at subsamples of the data.

Looking directly for explosive behavior in p_t and non-explosive behavior in d_t via right-tailed unit root tests is one aspect of the empirical methodology of the present paper. Although this approach is straightforward, it has received little attention in the literature. One possible explanation is the consensus view that "empirically there is little evidence of explosive behavior" in stock prices, as noted in Campbell, Lo and MacKinlay (1997, p.260) for instance. However, as Evans (1991) noted, explosive behavior is only temporary when economic bubbles periodically collapse and in such cases the observed trajectories may appear more like an I(1) or even stationary series than an explosive series, thereby confounding empirical evidence. He demonstrated by simulation that standard unit root tests had difficulties in detecting such periodically collapsing bubbles. In order for unit root test procedures to be powerful in detecting explosiveness, we propose the use of recursive regression techniques and show below by analytic methods and simulations that this approach is effective in detecting periodically collapsing bubbles. Using these methods, the present paper finds that when recursive tests are conducted and data from the 1990s are included in the sample, some strong evidence of explosive characteristics in p_t emerges.

Our tests are implemented as follows. For each time series x_t (log stock price or log dividend), we apply the augmented Dickey-Fuller (*ADF*) test for a unit root against the alternative of an explosive root (the right-tailed). That is, we estimate the following autoregressive specification by least squares[3]

$$x_t = \mu_x + \delta x_{t-1} + \sum_{j=1}^{J} \phi_j \Delta x_{t-j} + \varepsilon_{x,t}, \varepsilon_{x,t} \sim \ \mathrm{NID}(0, \sigma_x^2) \tag{7}$$

for some given value of the lag parameter J, where NID denotes independent and normal distribution. In our empirical application we use significance tests to determine the lag order J, as suggested in Campbell and Perron (1991). The unit root null hypothesis is $H_0 : \delta = 1$ and the right-tailed alternative hypothesis is $H_1 : \delta > 1$.

In forward recursive regressions, model (7) is estimated repeatedly, using subsets of the sample data incremented by one observation at each pass. If the first regression involves $n_{r_0} = [nr_0]$ observations, for some fraction r_0 of the total sample where [ ] signifies the integer part of its argument, subsequent

3 We also implemented the Phillips (1987) test and obtained very similar results to the ADF test.

regressions employ this originating data set supplemented by successive observations giving a sample of size $n_r = [nr]$ for $r_0 \leq r \leq 1$. Denote the corresponding *t*-statistic by ADF_r and hence ADF_1 corresponds to the full sample. Under the null we have

$$ADF_r \Rightarrow \frac{\int_0^r W dW}{\int_0^r W^2}$$

and

$$\sup_{r \in [r_0,1]} ADF_r \Rightarrow \sup_{r \in [r_0,1]} \frac{\int_0^r W dW}{\int_0^r W^2}$$

where W is the standard Brownian motion.

Comparison of $\sup_r ADF_r$ with the right tailed critical values from sup $_{r \in [r_0,1]} \frac{\int_0^r W dW}{\int_0^r W^2}$ makes it possible to test for a unit root against explosiveness. However, this testing procedure cannot stamp the emergence or the burst of bubbles. To locate the origin and the conclusion of bubbles, one can match the time series of ADF_r, with $r \in [r_0, 1]$, against the right tailed critical values from the asymptotic distribution of the standard Dickey-Fuller *t*-statistic.

If these tests lead to a rejection of H_0 in favor of H_1, then we may construct a valid asymptotic confidence interval for δ using some new econometric theory for the explosive case, as explained in Section 3.

2.2 The Effects of a Time Varying Discount Rate

The above discussion uses a fixed discount rate R. Allowing R to vary over time does not change the explosive behavior of the bubble component, but it can have important effects on the fundamental price. Recent work by Pastor and Veronesi (2006) has emphasized this possibility. These authors show that the combination of a time varying discount rate and high uncertainty about future dividend growth can substantially inflate market prices. Using a regime-switching model, Lettau *et al*. (2006) similarly found that a significant portion of the price run-up can be accounted for by declining macroeconomic risk. Neither of these papers discuss the potential for explosive dynamic effects in prices. The present section illustrates this possibility by developing a simple propagating mechanism for explosive behavior in the fundamental price under a time varying discount rate.

If dividends grow at a constant rate R_D with $R_D < R$ in (1), the fundamental value of the stock price[4]

$$P_t^f = \frac{D_t}{R - R_D} \tag{8}$$

This is the well-known Gordon growth model. It is evident that in this case the fundamental value can be very sensitive to changes in R when R is close to R_D. In fact, the fundamental value diverges as $R \searrow R_D$, so that a price run-up is evidently possible under certain time profiles for the discount rate. This simple

[4] This section develops the model in levels for analytic convenience.

Gordon model reveals the potential impact of a time varying discount rate, but it provides no price dynamics. The following argument provides an analytic formulation that shows how an explosive time path in fundamental values can be generated by time variation in the discount rate.

Consider a continuous time version of (8) with time varing discount rate R_t, viz.,

$$P_t^f = \int_0^\infty \exp(-sR_{t+s})E_t D_{t+s} ds \tag{9}$$

Suppose dividends have a constant expected growth rate R_D such that

$$E_t D_{t+s} = \exp(R_D s) D_t \tag{10}$$

and then D_t is a martingale when R_D = 0. Combining (9) and (10)

$$P_t^f = \int_0^\infty \exp(-s(R_{t+s} - R_D)) D_t ds \tag{11}$$

Given some fixed time point t_b, constants c_a> 0 and λ_1 >λ_2 > 0, let the time profile of the discount rate R_{t+s} for $t \in (0, t_b]$ be as follows:

$$R_{t+s} = \begin{cases} R_D + \frac{t_b - t - s}{s} c_a + \frac{\lambda_1}{s} & \text{for } 0 \leq s < t_b - t \\ R_D + c_a + \frac{\lambda_2}{s} & \text{for } s \geq t_b - t \end{cases} \tag{12}$$

Then, the discount rate decreases towards some level $R_D + \frac{\lambda_1}{t_b - t}$ as $t + s \nearrow t_b$ and jumps to the level $R_D + c_a + \frac{\lambda_2}{t_b - t}$ immediately thereafter, as shown in Figure 3. Thus, the time profile of the discount factor has a structural break at t_b in which a higher rate of discounting occurs at t_b. The break itself widens asymptotically as $t \nearrow t_b$.

We then have

$$\begin{aligned} P_t^f / D_t &= \int_0^\infty \exp(-s(R_{t+s} - R_D)) ds \\ &= \int_0^{t_b - t} \exp(-c_a(t_b - t - s) - \lambda_1) ds + \int_{t_b - t}^\infty \exp(-c_a s - \lambda_2) ds \\ &= e^{-\lambda_1} \left[\frac{e^{-c_a(t_b - t - s)}}{c_a} \right]_0^{t_b - t} + e^{-\lambda_2} \left[\frac{e^{-c_a s}}{-c_a} \right]_{t_b - t}^\infty \\ &= \frac{e^{-\lambda_1}}{c_a} \left[1 - e^{-c_a(t_b - t)} \right] + \frac{e^{-\lambda_2}}{c_a} e^{-c_a(t_b - t)} \\ &= \frac{e^{-\lambda_1}}{c_a} + \frac{\left(e^{-\lambda_2} - e^{-\lambda_1}\right)}{c_a} e^{-c_a(t_b - t)} := \sigma_t \end{aligned}$$

and the time path of P_t^f / D_t is explosive over $t \in (0, t_b]$. Over this interval, P_t^f evolves according to the differential equation

$$dP_t^f = \left(e^{-\lambda_2} - e^{-\lambda_1}\right) e^{-c_a(t_b - t)} D_t dt + \sigma_t dD_t$$

Since $c_a P_t^f / D_t = e^{-\lambda_1} + \left(e^{-\lambda_2} - e^{-\lambda_1}\right) e^{-c_a(t_b - t)}$, we have

$$dP_t^f = \frac{\left(e^{-\lambda_2} - e^{-\lambda_1}\right) e^{-c_a(t_b - t)}}{e^{-\lambda_1} + \left(e^{-\lambda_2} - e^{-\lambda_1}\right) e^{-c_a(t_b - t)}} c_a P_t^f dt + \sigma_t dD_t, \quad \text{for } t \in (0, t_b]$$

For t close to t_b the generating mechanism for P_t^f is approximately

$$\begin{aligned} dP_t^f &= \frac{\left(e^{-\lambda_2} - e^{-\lambda_1}\right)}{e^{-\lambda_1} + \left(e^{-\lambda_2} - e^{-\lambda_1}\right)} c_a P_t^f dt + \sigma_t dD_t \\ &= \left\{1 - e^{-(\lambda_1 - \lambda_2)}\right\} c_a P_t^f dt + \sigma_t dD_t \end{aligned}$$

which is an explosive diffusion because

$$c_b = \left\{1 - e^{-(\lambda_1 - \lambda_2)}\right\} c_a > 0$$

since $c_a > 0$ and $e^{-(\lambda_1 - \lambda_2)} < 1$. The discrete time path of P_t^f in this neighbourhood is therefore propagated by an explosive autoregressive process with coefficient $\rho = e^{c_b} > 1$.

The heuristic explanation of this behavior is as follows. As $t \nearrow t_b$ there is growing anticipation that the discount factor will soon increase. Under such conditions, investors anticipate the present to become more important in valuing assets. This anticipation in turn leads to an inflation of current valuations and price fundamentals P_t^f become explosive as this process continues.

On the other hand, for $t > t_b$ we have

$$R_{t+s} = R_D + c_a + \tfrac{\lambda_2}{s} \quad \text{for } s > 0$$

and then

$$\begin{aligned} P_t^f / D_t &= \int_0^\infty \exp(-s(R_{t+s} - R_D)) ds \\ &= \int_0^\infty \exp(-c_a s - \lambda_2) ds \\ &= e^{-\lambda_2} \left[\frac{e^{-c_a s}}{-c_a}\right]_0^\infty = \frac{e^{-\lambda_2}}{c_a} \end{aligned}$$

So, $P_t^f = \frac{e^{-\lambda_2}}{c_a} D_t$ for $t > t_b$ and price fundamentals are collinear with D_t. When D_t is a Brownian motion or an integrated process in discrete time, P_t^f and D_t are cointegrated. Thus, after time t_b, price fundamentals comove with D_t.

It follows that the time profile (12) for the discount rate R_t induces a subinterval of explosive behavior in P_t^f before t_b. In this deterministic setting, it is known as time t_b approaches that there will be an upwards shift in the discount rate that makes present valuations more important. A more realistic model might allow for uncertainty in this time profile and a stochastic trajectory for R_t that accommodated potential upwards shifts of this type. Recursive tests of the type such as those described in the last section may be used to assess evidence for subperiods of explosive price behavior that are induced by such time variation in the discount rate.

3. Econometric Issues

3.1 Econometric Analysis of Explosive Processes

Recent work by Phillips and Magdalinos (2007a, 2007b) has provided an asymptotic distribution theory for mildly explosive processes that can be used for confidence interval construction in the present context. These papers deal with an explosive model of the form

$$x_t = \delta_n x_{t-1} + \varepsilon_{x,t},\ t = 1, ..., n;\ \delta_n = 1 + \frac{c}{k_n},\ c > 0 \tag{13}$$

which is initialized at some $x_0 = o_p\left(\sqrt{k_n}\right)$ independent of $\left\{\varepsilon_{x,t}, t \geq 1\right\}$, and where $(k_n)_{n\geq 1}$ is a sequence increasing to ∞ such that $k_n = o(n)$ as $n \to \infty$. The error process $\varepsilon_{x,t}$ may comprise either independent and identically distributed random variables or a weakly dependent time series with $E\varepsilon_{x,t} = 0$ and uniform finite second moments so that $\sup_t E\varepsilon_{x,t}^2 < \infty$.

The sequence $\delta_n = 1 + \frac{c}{k_n} > 1$ is local to the origin in the sense that $\delta_n \to 1$ as $n \to \infty$, but for any finite n it involves moderate deviations from a unit root, *i.e.*, deviations that are greater than the conventional $O\left(n^{-1}\right)$ deviations for which unit root tests have nontrivial local power properties (see Phillips, 1987) and unit root type distributions apply. The corresponding time series (which is strictly speaking an array process) x_t in (13) is mildly explosive. Importantly, k_n may be within a slowly varying factor of n, for instance log n, so that we may have $k_n = n/\log n$.

Models of the form (13) seem well suited to capturing the essential features of economic and financial time series that undergo mildly explosive behavior. They also seem appropriate for capturing periodically collapsing bubble behavior where the bubble may appear over a subperiod of length $k_n < n$. These mildly explosive models have the very interesting and somewhat unexpected property, established in Phillips and Magdalinos (2007a, 2007b), that they are amenable to central limit theory. Moreover, the limit theory turns out to be invariant to the short memory properties of the innovations $\varepsilon_{x,t}$, so that inferential procedures based on this limit theory is robust to many different departures from simple $i.i.d.$ errors. This means that the models and the limit theory may be used as a basis for statistical inference with processes that manifest mildly explosive trajectories. For economic and financial data, this typically means values of δ_n that are in the region [1.005, 1.05]. In particular, if $k_n = n/\log n$ and n = 200, we have $\delta_n = 1 + \frac{c}{k_n} \in$ [1.002, 1.053] for $c \in$ [0.1, 2] .

Under some general regularity conditions, Phillips and Magdalinos show that the least squares regression estimator $\hat{\delta}_n = \sum_{t=1}^{n} x_{t-1}x_t / \sum_{t=1}^{n} x_{t-1}^2$ has the following limit theory for mildly explosive processes of the form (13):

$$\frac{k_n(\delta_n)^n}{2c}\left(\hat{\delta}_n - \delta_n\right), \frac{(\delta_n)^n}{(\delta_n)^2 - 1}\left(\hat{\delta}_n - \delta_n\right) \Longrightarrow C.$$

where C is a standard Cauchy random variable. It follows that a 100 (1 $-\alpha$)% confidence interval for δ_n is given by the region

$$\left(\hat{\delta}_n \pm \frac{(\hat{\delta}_n)^2 - 1}{(\hat{\delta}_n)^n} C_\alpha\right)$$

where C_α is the two tailed α percentile critical value of the standard Cauchy distribution. For 90, 95 and 99 percent confidence intervals, these critical values are as follows:

$C_{0.10}$ = 6.315, $C_{0.05}$ = 12.7, $C_{0.01}$ = 63.65674

These values can be compared with the corresponding Gaussian critical values of 1.645, 1.96, 2.576.

The confidence intervals and limit theory are also invariant to the initial condition x_0 being any fixed constant value or random process of smaller asymptotic order than $k_n^{1/2}$. This property provides some further robustness to the procedure.

3.2 Finite Sample Bias Correction via Indirect Inference Estimation

Least squares (LS) regression is well known to produce downward biased coefficient estimates in the first order autoregression (AR). This bias does not go to zero as the AR coefficient $\delta \to 0$ and the bias increases as δ gets larger. It is therefore helpful to take account of this bias in conducting inference on autoregressive coefficients such as δ in (13). Several statistical procedures are available for doing so, including the use of asymptotic expansion formulae (Kendall, 1954), jackknifing (Quenouille, 1956; and Efron, 1982), median unbiased estimation (Andrews, 1993) and indirect inference (Gouriéroux et al., 1993).

Indirect inference was originally suggested and has been found to be highly useful when the moments and the likelihood function of the true model are difficult to deal with, but the true model is amenable to data simulation. In fact, the procedure also produces improved small sample properties and has the capacity to reduce autoregressive bias, as shown by Gouriéroux et al. (2000) in the time series context and Gouriéroux et al. (2007) in the dynamic panel context. We shall use indirect inference in the present application because of its known good performance characteristics and convenience in autoregressive model estimation.

To illustrate, suppose we need to estimate the parameter δ in the simple AR(1) model (*i.e.* J = 0 in model (7)[5])

$$x_t = \mu_x + \delta x_{t-1} + \varepsilon_{x,t}, \ \varepsilon_{x,t} \sim \ \mathrm{NID}(0, \sigma_x^2) \tag{14}$$

from observations $\{x_t; t \leq n\}$, where the true value of δ is δ_0. Some autoregressive bias reduction methods, such as Kendall's (1954) procedure, require explicit knowledge of the first term of the asymptotic expansion of the bias in powers of n^{-1}. Such explicit knowledge of the bias is not needed in indirect inference. Instead, indirect inference calibrates the bias function by simulation. The idea is as follows. When applying LS to estimate the AR(1) model with the observed data, we obtain the estimate $\hat{\delta}_n^{LS}$ and can think of this estimate and its properties (including bias) as being dependent on δ through the data. Given a parameter choice δ, let $\{\tilde{x}_t^h(\delta); t \leq n\}$ be data simulated from the true model, for h = 1,··· , H with H being the total number of simulated paths. Note that these simulations rely on the distributional assumption made in (14). Let the LS estimator based on the h^{th} simulated path, given δ, be denoted by $\widetilde{\delta}_n^h(\delta)$.

The indirect inference estimator is defined as the extremum estimator

$$\hat{\delta}_{n,H}^{II} = \mathrm{argmin}_{\delta \in \Phi} \parallel \hat{\delta}_n^{LS} - \frac{1}{H} \sum_{h=1}^{H} \widetilde{\delta}_n^h(\delta) \parallel \tag{15}$$

where $\parallel \cdot \parallel$ is some finite dimensional distance metric, and Φ is the parameter space which is compact. In the case where H tends to infinity, the indirect inference estimator becomes

$$\hat{\delta}_n^{II} = \mathrm{argmin}_{\delta \in \Phi} \parallel \hat{\delta}_n^{LS} - q_n(\delta) \parallel \tag{16}$$

where $q_n(\delta) = E(\widetilde{\delta}_n^h(\delta))$ is the so-called binding function. In this case, assuming the function q_n to be invertible, the indirect inference estimator is given by

$$\hat{\delta}_n^{II} = q_n^{-1}(\hat{\delta}_n^{LS})$$

The procedure essentially builds in a small-sample bias correction to parameter estimation, with the bias being computed directly by simulation.

It can be shown that the asymptotic distribution of $\hat{\delta}^{II}$ is the same as that of $\hat{\delta}_n^{LS}$ as $n \to \infty$ and $H \to \infty$. So the asymptotic confidence interval derived in the previous section applies equally well to the indirect inference estimator and will be implemented in what follows.

[5] When J > 0, we need to augment model (14) accordingly.

4. Data

Our data are taken from *Datastream International*. We collect monthly observations on the Nasdaq composite price index (without dividends) and the Nasdaq composite dividend yields, and compute the Nasdaq composite dividend series from these two series. We use the Consumer Price Index (CPI), which is obtained from the Federal Reserve Bank of St. Louis, to convert nominal series to real series. Our sample covers the period from February 1973 to June 2005 and comprises 389 monthly observations.

Figure 1 plots the time series trajectories of the Nasdaq real price and real dividend indices. Both series are normalized to 100 at the beginning of the sample. As can be seen, both price and dividend grew steadily from the beginning of the sample until the early 1990s. The price series then began to surge and the steep upward movement in the series continued until the late 1990s as investment in DotCom stocks grew in popularity. Early in the year 2000 the price abruptly dropped and continued to fall to the mid 1990s level. The dividend series, on the other hand, remained steady throughout the sample period.

5. Testing and Dating Exuberance

Table 1 reports the ADF_1 and $\sup_{r\in[r_0,1]} ADF_r$ test statistics for both the log Nasdaq real price and log Nasdaq real dividend indices for the full sample from February 1973 to June 2005, where $r_0 = 0.10$. Also reported are the various critical values for each of the two tests. For the ADF_1 test, the asymptotic critical values are obtained from Fuller (1996, Table 10.A.2). For $\sup_{r\in[r_0,1]} ADF_r$, the critical values are obtained using Monte Carlo simulation based on 10,000 replications. Several conclusions are drawn from the table. First, if we were to follow the convention and apply the ADF test to the full sample (February 1973 to June 2005), the tests could not reject the null hypothesis $H_0 : \delta = 1$ in favor of the right-tailed alternative hypothesis $H_1 : \delta > 1$ at the 5 percent significance level for the price series, and therefore one would conclude that there was no significant evidence of exuberance in the price data. If one believes in a constant discount rate, the result is consistent with Diba and Grossman (1988) and is subject to the criticism leveled by Evans (1991) because standard unit root tests for the full sample naturally have difficulty in detecting periodically collapsing bubbles. Second, the $\sup_{r\in[r_0,1]} ADF_r$ test, on the other hand, provides significant evidence of explosiveness in the price data at the 1 percent level, suggesting the presence of price exuberance, but no evidence in the dividend data. However, $\sup_{r\in[r_0,1]} ADF_r$ cannot reveal the location of the exuberance.

To locate the origin and the conclusion of exuberance, Figure 4 plots the time series of the ADF_r statistics together with the 5 percent asymptotic critical value for the log real price and the log real dividend.[6] The optimal lag length is determined using the procedure suggested by Campbell and Perron

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6 The 5 percent asymptotic critical value is -0.08, which is very close to the 5 percent finite-sample critical value (-0.03) with 50 observations. Therefore, our conclusions from Figure 4 stay the same even if we apply the latter conservative critical value of -0.03 to the forward recursive regressions.

We have also conducted the tests using price and dividend series in *levels* rather than in *natural logarithms*. The results are similar and the conclusions remain qualitatively unchanged. They are not reported to conserve space and are available upon request.

The above remarks apply to Figure 5 discussed below as well.

(1991). Starting with 12 lags in the model, coefficients are sequentially tested for significance at the 5 percent level, leading to the selection of the model for which the coefficient of the last included lag is significant at the 5 percent level.[7] The initial start-up sample for the recursive regression covers the period from February 1973 to April 1976 (10 percent of the full sample).

The forward recursive regressions give some interesting new findings (see Figure 4). The dividend series is always nonexplosive. The stock price series is also tested to be nonexplosive for the initial sample, which suggests no evidence of exuberance in the initial data. This feature is maintained until May 1995. In June 1995, the test detects the presence of exuberance in the data and the evidence in support of price exuberance becomes stronger from this point on and peaks in February 2000. The exuberance is detected as continuing until July 2001, and by August 2001 there is no longer any evidence of exuberance in the data. Interestingly, the first occurrence date for price exuberance in the data is June 1995, which is 18 months before Greenspan's historic remark of "irrational exuberance" made in December 1996.

To highlight the explosive behavior in the Nasdaq during the 1990s, we carry out the analysis using two sub-samples. The first sub-sample is from January 1990 to December 1999, the 10-year period that recent researchers have focused on (e.g., Pastor and Veronesi, 2006; Ofek and Richardson, 2003; and Brunnermeier and Nagel, 2004). Panel A of Table 2 reports the test results. As above, we apply the ADF_1 and $\sup_r ADF_r$ tests for a unit root against the alternative of an explosive root to both the log real price and log real dividend series. We also obtain the least squares estimate $\hat{\delta}^{LS}$, the indirect inference estimate $\hat{\delta}^{II}$, the 95 percent asymptotic confidence interval of δ based on $\hat{\delta}^{II}$, and critical values for the unit root tests.

All the results give strong evidence of explosiveness in p_t . For example, for the log real Nasdaq price index, the ADF_1 statistic for the full sample is 2.309, far exceeding the 1 percent critical value of 0.60. Similar results occur with the $\sup_r ADF_r$ test. We therefore reject the null hypothesis of a unit root at the 1 percent significance level in favor of explosive behavior for the Nasdaq stock index. In contrast, there is no evidence that the log real dividend series exhibits explosive behavior.[8]

Figure 5 graphs the trajectory of the ADF_r statistics together with the 5 percent asymptotic critical value for sample observations from January 1990 to the end of the sample. As for the full sample, we choose r_0 = 0.10. Similar to Figure 4, we again date the start of price exuberance in June 1995, so the empirically determined date of origination of the exuberance appears robust to the choice of the initial sample. The recursive regressions detect the conclusion of exuberance in October 2000, somewhat earlier than that reported in Figure 4.

7 Ng and Perron (1995) demonstrate that a too parsimonious model can have large size distortions, while an over-parameterized model may result in reduction of test power. They show that methods based on sequential tests have an advantage over information-based rules because the former have less size distortions and have comparable power.

8 The 5 and 1 percent critical values for a unit root against the stationary alternative are -2.86 and -3.42, respectively. Based on these critical values, the ADF_1 test will indeed reject the null hypothesis of a unit root in favor of the alternative of stationarity for the dividend series.

The autoregression gives the AR coefficient estimate $\hat{\delta}^{LS}$ = 1.025 in stock price. Assuming that the error term in the regression follows an $i.i.d.$ normal distribution and J = 0, we obtain the indirect inference estimate $\hat{\delta}^{II}$ = 1.033 via simulation with 10,000 replications. The associated 95 percent asymptotic confidence interval for δ is [1.016, 1.050]. This implies that the log stock price p_t will grow at the explosive rate of 3.3 percent per month. Since the dividend series d_t is not explosive, with a constant discount rate the fundamental price p_t^f is also not explosive, being determined exclusively by dividends according to (3). Therefore, from (2), b_t (the log bubble) must also be explosive with a growth rate at least as high as the growth rate of stock price, g = 3.3 percent per month. With 95 percent confidence, the true growth rate g lies in the range between 1.6 and 5 percent per month. Under the assumption of constant discount rate, this provides sufficient conditions for the presence of a bubble.

To understand the implication of the estimated explosive rate for stock price, suppose that the Nasdaq index were over-valued by around 10 percent when Greenspan made his "irrational exuberance" comment in December 1996. Then the initial size of the log bubble would be $b_0 = \log(\frac{P_0}{P_0^f}) = \log(1.10) = 0.0953$ in December 1996. Using the indirect inference estimate of the growth rate g = 0.033, we may calculate that, by March 2000 when the Nasdaq index reached its historic high (39 months later), the expected log level of the price bubble would have risen to $b_t = (1 + 0.033)^{39} \times 0.0953 = 0.338$, and the ratio of the expected Nasdaq price to its fundamental value would have been $\frac{P_t}{P_t^f} = \exp(b_t) = \exp(0.338) = 1.40$. In other words, after 39 months, the expected Nasdaq index would have become around 40 percent over-valued relative to its fundamental.

Notice that $\hat{\delta}^{II}$ = 1.033 reported in Panel A of Table 2 gives an unbiased estimate of the explosive root for the stock price process p_t, which can be considered a lower bound of the explosive root of the unobservable bubble process b_t. The reason is as follows. From (2), we know that the actual stock price consists of the fundamental component and the bubble component. Under the assumption that the fundamental component is either I(1) or I(0) and the bubble component is explosive, if a bubble lasts for a sufficiently long period of time, the bubble component will dominate the fundamental component and the actual stock price will grow at around the same speed as the bubble component does. However, within a limited time period when a bubble is first developing, the magnitude of the bubble component may be small relative to the fundamental component even though the process is explosive, and therefore employing the stock price series for estimation will under-estimate the true growth rate of the bubble.

To provide a more realistic estimate of the growth rate of the bubble, since the Nasdaq index kept rising after December 1999, we implement the ADF_1 test by extending the first sub-sample to June 2000 when the test detects explosive price behavior with the most significant ADF test statistic. Panel B of Table 2 reports the least squares estimate for this sample, $\hat{\delta}^{LS}$ = 1.036, which yields the indirect inference estimate $\hat{\delta}^{II}$ = 1.040. This implies a growth rate g = 4 percent per month. While this is still a lower bound estimate of the growth rate of the bubble process, it is plausible to think of it as the closest to the true growth rate.

Suppose that the Nasdaq index were over-valued by 10 percent when our test first detected the bubble to start in June 1995, then the initial size of the bubble is $b_0 = \log(\frac{P_0}{P_0^f}) = \log(1.10) = 0.0953$. Using the above unbiased estimate of the speed of the bubble, by June 2000 (60 months later) when our test detected the bubble to be the strongest, the expected size of the bubble would have become b_t = (1 + 0.04)^{60}X 0.0953 =1.0025. This implies that the ratio of the expected Nasdaq price to its fundamental value would have been $\frac{P_t}{P_t^f}$ = exp(b_t) = exp(1.0025) = 2.73. In other words, the expected value of the Nasdaq index would have been 173 percent over-priced relative to its fundamental value after 60 months. The actual Nasdaq index peaked at 5,048.62 points on March 10, 2000, then dropped to 1,950.4 by December 31, 2001 and to 1,335.31 by December 31, 2002. If the year 2001 end value is considered close to the "fundamental" value, then the Nasdaq index would be 159 percent over-priced at the peak (5049/1950 = 2.59). On the other hand, if the year 2002 end value is considered the "fundamental" value, the peak value would be 278 percent over-priced (5049/1335 = 3.78). Therefore, the above estimate of the growth rate of the bubble matches the actual Nasdaq price dynamics reasonably well.

6. Finite Sample Properties

6.1 Unit Root Tests for an Explosive Bubble

While standard unit root tests have been applied to test for unit roots against explosiveness in the price series p_t in Diba and Grossman (1988) and Evans (1991), both papers only examined the finite sample performance of the standard unit root tests for the bubble b_t (see Section VI in Diba and Grossman and Section III in Evans). Naturally, however, it is more informative to verify the finite sample performance of the standard unit root tests in the price series itself p_t because in practice the price series is observed but the bubble series is not.

Consider the following data generating process, where the fundamental price follows a random walk with drift and the bubble process is a linear explosive process without collapsing:

$$p_t = p_t^f + b_t, \quad p_t^f = \mu_f + p_{t-1}^f + \varepsilon_{f,t} \tag{17}$$

$$b_t = (1+g)b_{t-1} + \varepsilon_{b,t} \tag{18}$$

where $\varepsilon_{f,t} \sim$ NID(0, σ_f^2) and $\varepsilon_{b,t} \sim$ NID(0, σ_b^2). We use Nasdaq price index data from February 1973 to December 1989 (*i.e.*, before the 1990s explosive price period started) to estimate the fundamental process, assuming that there was no bubble during this period so that $p_t = p_t^f$. This estimation yields the values μ_f = 0.00227 and σ_f = 0.05403. We then use these two parameter values along with g = 0.04 (based on the indirect inference estimate of δ in Panel B of Table 2) to obtain the estimate of the bubble innovation σ_b = 0.0324 by employing data for the explosive period January 1990 to June 2000 via the Kalman filter, as in Wu (1997). These parameters μ_f , σ_f , and σ_b are used to conduct simulations under different assumptions about the speed parameter g and the initial level of the bubble b_0 with 120 observations and 10,000 replications. The simulation results are reported in Panel A of Table 3. Panel B displays the results for different values for the bubble innovation standard deviation σ_b, while the speed parameter g is set to 0.04, which is the indirect inference estimate of δ – 1 reported in Panel B of Table 2.

It is known from Diba and Grossman (1988) that standard unit tests can detect explosive characteristics in b_t. Our simulation results suggest that the standard unit root tests can also detect the explosive characteristics in p_t when bubbles appear in the empirically realistic settings as long as the bubbles are not periodically collapsing. Panel A of Table 3 clearly demonstrates that the test power is higher, the larger is the growth rate g and/or the larger is the initialization b_0. When the growth rate g is larger than 0.01, the test has substantial power against the explosive alternative and when g = 0.04 (the indirect inference estimate using the Nasdaq stock index during the bubble period), the test has nearly perfect power against the explosiveness alternative regardless of the initial level of the bubble b_0. Panel B of Table 3 shows that smaller values of the standard deviation σ_b lead to greater test power provided the initial value b_0 is not too small (here b_0 > 0.03). Overall, the power is not very sensitive to the innovation standard deviation σ_b or to the initial value of t he bubble b_0 and is quite high with the growth rate g = 0.04.

6.2 Recursive Unit Root Tests and Periodically Collapsing Bubbles

The above simulation design does not allow for the possibility of periodically collapsing bubbles, an important class of bubbles that seem more relevant in practical economic and financial applications. Evans (1991) proposed a model to simulate such collapsing bubble and showed that standard unit root tests had little power to detect this type of bubble. In this section, we first design a simulation experiment to assess the capacity of our recursive regression tests to detect this type of periodically collapsing bubble. We show that although the tests are inconsistent in the context, in finite samples the tests have good power. We further validate the consistency of the tests by introducing an alternative model for periodically collapsing bubbles.

Evans's (1991) Periodically Collapsing Bubble Model

Evans (1991) suggested the following model for a bubble process B_t that collapses periodically:[9]

$$B_{t+1} = (1+g)B_t\varepsilon_{b,t+1}, \quad \text{if } B_t \leq \alpha \tag{19}$$

$$B_{t+1} = [\zeta + \pi^{-1}(1+g)\theta_{t+1}\left(B_t - (1+g)^{-1}\zeta\right)]\varepsilon_{b,t+1}, \quad \text{if } B_t > \alpha \tag{20}$$

where $g > 0$, $\varepsilon_{b,t}$ = exp ($y_t - \tau^2/2$) with $y_t \sim$ NID(0, τ^2), θ_t is an exogenous Bernoulli process which takes the value 1 with probability π and 0 with probability 1− π . Evans (1991) specifies his model in levels and so price, dividend and bubble are in levels and are expressed in upper-case letters. This model has the property that B_t+1 satisfies E_t(B_t+1) = (1 + g) B_t, analogous to (4). The model generates bubbles that survive as long as the initial bounding condition $B_t \leq \alpha$ applies (say $t \leq T_\alpha$) and thereafter only as long as the succession of identical realizations $\theta_{T_\alpha+k}$ = 1, k = 1, 2, . . . hold. The bubble bursts when θ_t = 0.

9 Blanchard (1979), Flood and Garber (1980), and Blanchard and Watson (1982) first propose stochastic bubbles that can burst with a fixed probability. Burmeister, Flood and Garber (1983) show the equivalence of a class of different-looking stochastic bubble processes.

To facilitate comparisons between our simulation results with those of Evans (1991), we use the same simulation design and parameter settings as his. In particular, a bubble process B_t of 100 observations is simulated from the model (19) and (20) with the parameter settings g = 0.05, α = 1, ζ= 0.5, B_0 = 0.5, and τ= 0.05, and θ_t is a Bernoulli process which takes the value 1 with probability π and 0 with probability 1− π. When θ_t = 0, the bubble bursts. We choose the value π =0.999, 0.99, 0.95, 0.85, 0.75, 0.5, 0.25. In addition, a dividend series (in levels) of 100 observations is simulated from the following random walk model with drift:

$$D_t = \mu_D + D_{t-1} + \varepsilon_{d,t},\ \varepsilon_{b,t} \sim \mathrm{NID}(0, \sigma_d^2)$$

where μ_D = 0.0373, σ_d^2 = 0.1574, D_0 = 1.3. Consequently, the fundamental price is generated from

$$P_t^f \ = \mu_D(1+g)g^{-2} + D_t/g$$

and the simulated price series follows as $P_t = P_t^f + B_t$. In the simulations reported, B_t is scaled upwards by a factor of 20, as suggested in Evans (1991).

Table 4 reports the empirical power of the ADF_1 and sup$_r$ ADF_r statistics for testing an explosive bubble based on the 5 percent critical value reported in Table 1 and 10,000 replications. We should emphasize that, unlike Evans (1991) who assumed that B_t is observed and tested the explosiveness in B_t, we apply the ADF_1 test to the price series itself P_t. Several interesting results emerge from the table. First, the power of the ADF_1 test depends critically on π. When π = 0.999 or 0.99, the ADF_1 test has considerably good power (0.914 and 0.460 respectively). When $\pi \leq$ 0.95, the ADF_1 test has essentially no power. These results are consistent with those reported in Evans (1991, Table 1). Second, the power of the sup$_r$ ADF_r statistic also depends on π, but in a much less drastic way. For example, when π= 0.25, it still has considerable power (0.340). For empirically more relevant cases, say when π = 0.95, the power of sup$_r$ ADF_r becomes much higher (0.714).

Clearly the performance of the tests is determined by the time span of a bubble. In Appendix A, we formally show that the maximum time span of a collapsing bubble in Evans's (1991) model is O_p (log n) , which is very short relative to the full sample size n, so that standard unit root tests cannot be expected to perform well. This Appendix A further shows that in a regression of B_t +1 on B_t with O_p (log n) observations from an explosive period, the signal in the regression has the maximum order of O_p ($n^{2\log\left(\frac{1+g}{\pi}\right)-\frac{\tau^2}{2}}$). When log ($\frac{1+g}{\pi}$) < 1+ $\frac{\tau^2}{4}$, this signal is smaller than that of an integrated process, whose signal is O_p (n^2), and significantly less than that of an explosive process. These findings explain the failure of conventional unit root tests to detect bubbles of this type, confirming the simulations in Evans (1991) and in our Table 4.

In recursive regressions, the signal will be comparatively stronger because the data set is shorter and it will be emphasized when the end point in the recursion occurs toward the end of a bubble. This argument suggests that there will be some statistical advantage to the use of recursive regression techniques and the use of a sup test in assessing the evidence for periodically collapsing bubbles, as confirmed in Table 4. However, in a recursive regression using samples of size n_r = [n_r] for r > 0, the maximum length of the bubble is still O_p (log n) and this is still not long enough relative to n_r for a recursive test to be consistent essentially because the signal is not strong enough. This limitation shows up in the simulations as the test performs worse when π gets smaller, although the power for the sup test is clearly non trivial and substantially better than that of conventional tests. We might expect some additional gain from the use of a rolling regression in conducting the test, where the sample size (*N*) used for the regression has smaller order than n, for instance, N = $[n^\eta]$ for some γ < 1, or even *N* = *O* (log n). When *N* = $[n^\eta]$, for instance, the signal from the explosive part of the data, which still has the time span of O_p (log n), will dominate provided that n < 2 log ($\frac{1+g}{\pi}$)– $\frac{\tau^2}{2}$. However, in the case of rolling regressions of this type, tests generally have different limit distributions from those studied already in the unit root and structural break literature, for example by Banerjee et al. (1993), where rolling regressions of length proportional to the sample size n are used.

An Alternative Model of Periodically Collapsing Bubbles

In Evans's (1991) periodically collapsing bubble model, whether a bubble collapses or not depends entirely on a Bernoulli process. In this simple setup, the bubble collapses with a constant probability π, regardless of the state of the economy, such as the strength of the cognitive bias. We next develop a model in which the duration of the periodically collapsing bubble depends on the strength of the cognitive bias underlying herd behavior in the market. While this quantity is obviously unobserved, there may be proxy or determining variables for it that could be used in empirical modeling, as in the approach taken by Hu and Phillips (2004) in modeling Federal funds rate target decisions by the Federal Open Market Committee, where certain relevant observable macroeconomic variables were used to capture the determining fundamentals used by the Fed in assessing prevailing economic conditions and the need for market intervention.

The model of Hu and Phillips was formulated as an ordered discrete choice model where the determining variables involved some integrated processes. In the present case, the mechanism may be assumed to be governed by an unobserved stochastic process representing an index of herd market thinking. Suppose that this index follows an integrated process $S_t = \sum_{l=1}^{t} \xi_l$, where ξ_t is a zero mean, stationary process with continuous spectral density $f_\xi(\nu)$, representing the latest contribution to cognitive bias in the market. When S_t > 0 there will be a positive cognitive bias that supports the bubble phenomenon, and when S_t < 0 the cognitive bias is negative. Assume that there is some level ν_n < 0 of this index that represents a degree of negative cognitive bias that is sufficient to end the bubble phenomena. Then, bubble conditions will be sustained while the index S_t remains above ν_n. If the bubble begins at *t* = 0 at level B_0 as in (19) and reaches the level $B_{T_\alpha} > \alpha$ at time T_α, bubble conditions will last as long as they are sustained by herd market thinking. So our model for a bubble process B_t that collapses periodically can be written as

$$B_{t+1} = (1+g)B_t\varepsilon_{b,t+1}, \quad \text{if } B_t \leq \alpha \tag{21}$$

$$B_{t+1} = [\zeta + \pi_{t+1}^{-1}(1+g)\theta_{t+1}\left(B_t - (1+g)^{-1}\zeta\right)]\varepsilon_{b,t+1}, \quad \text{if } B_t > \alpha \tag{22}$$

where g > 0, $\varepsilon_{b,t+1}$ = exp(y_t – τ^2/2) with $y_t \sim$ NID(0, τ^2), θ_{t+1} takes the value 1 when $S_{t+1} > \nu_n$ and 0 when $S_{t+1} \leq \nu_n$ and is independent of $\varepsilon_{b,t}$, and π_{t+1} = Pr($S_{t+1} > \nu_n \mid B_t > \alpha$). We again have E_t (B_{t+1}) = (1 + g) B_t, consistent with (4).

In Appendix B, we show that if ξ_t ~ NID(0, σ_ξ^2) and $\nu_n = \sqrt{n}\nu$ for some $\nu < 0$, the duration of the bubble is measured by $[nh_\nu]+ O_p(1)$ where $h_\nu = \inf_r \{r : \sigma_\xi W(r) \leq \nu\}$ is the first passage time of a standard Brownian motion $W(r)$ to the level ν/σ_ξ . For this model, we can expect recursive tests to be consistent because the length of the explosive component is of order $O_p(nh_\nu)$ and therefore long enough to ensure that the signal is exponentially large with positive probability at some point in the recursion.

To facilitate comparisons with the simulation results reported in the previous section, we use the same simulation design and parameter settings. As to the simulation of cognitive bias, we chose the value ν/σ_ξ =-0.3, -0.2, -0.15, -0.1. Table 5 reports the empirical power of the ADF_1 and $\sup_r ADF_r$ statistics for testing an explosive bubble based on the 5 percent critical value reported in Table 1 and 10,000 replications. Several interesting results emerge from the table. First, the ADF_1 test performs better in our model than in the Evans model. So the standard unit root tests are more likely to pick up bubbles in our model than in the Evans model. However, the power remains low. For example, when ν/σ_ξ =-0.3, the power of the ADF_1 test is 0.346; when ν/σ_ξ =-0.1, the power of the ADF_1 test becomes 0.350. Second, the power of the $\sup_r ADF_r$ test is considerably higher than that of ADF_1 in all cases and also higher than that of the $\sup_r ADF_r$ test under the Evans model. For example, when ν/σ_ξ =-0.1, the power of the $\sup_r ADF_r$ test under the new bubble model is 0.730.

To further investigate the consistency of the tests, we examine the power of the tests when the sample size increases to 400. Table 6 reports the empirical power of the ADF_1 and $\sup_r ADF_r$ statistics for testing an explosive bubble based on the 5 percent critical value. As with the case of the sample size of 100, the power of ADF_1 remains low. Indeed, when ν/σ_ξ =-0.1, the power of ADF_1 becomes even smaller, confirming our theoretical argument about the inconsistency of the test. On the other hand, the power of $\sup_r ADF_r$ increases in all cases. These results confirm our theoretical argument about the consistency of the tests.

7. Conclusion

This paper has proposed a new approach to testing for explosive behavior in stock prices that makes use of recursive regression, right-sided unit root tests and a new method of confidence interval construction for the growth parameter in stock market exuberance. Simulations reveal that the approach works well in finite samples and has discriminatory power to detect explosive processes and periodically collapsing bubbles when the discount rate is time invariant.

The empirical application of these methods to the Nasdaq experience in the 1990s confirms the existence of exuberance and date stamps its origination and collapse. As the second quotation that heads this article indicates, the existence of exuberance or “bubble” activity may be self evident to some economists in view of the sheer size of the wealth created and subsequently destroyed in the Nasdaq market. Of primary interest, therefore, are its particular characteristics such as the origination date, which we find to be June 1995, the peak in February 2000, and the conclusion in August 2001. Comparison of this statistical origination to the timing of the famous remark by Greenspan in December 1996 affirms that Greenspan’s perceptions were actually supported by empirical evidence of exuberance in the data at that time.

Greenspan's remarks are often taken to indicate foresight concerning the subsequent path of Nasdaq stocks. The present findings indicate that his remarks were also supported in some measure by the track record of empirical experience up to that time. Thus, Greenspan's perspective concerning irrational exuberance in stock prices and future profitability in December 1996 showed hindsight as well as foresight concerning the impending escalation in technology asset values.

This paper has not attempted to identify explicit sources of the 1990s exuberance in internet stocks. Several possibilities exist, including the presence of a rational bubble, herd behavior, or explosive effects on economic fundamentals arising from time variation in discount rates. Identification of the explicit economic source or sources will involve more explicit formulation of the alternative models and suitable model determination techniques to empirically distinguish between such models. The present econometric methodology shows how the data may be studied as a mildly explosive propagating mechanism. The results confirm strong empirical support for such activity in the Nasdaq data over the 1990s.

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Table 1. Testing for Explosive Behavior in the Nasdaq Index from February 1973 to June 2005

This table reports ADF_1 and $\sup_{r\in[r_0,1]} ADF_r$ tests of the null hypothesis of a unit root against the alternative of an explosive root, where r_0 = 0.10. The optimal lag length for the ADF test is selected according to top-down sequential significance testing, as suggested by Campbell and Perron (1991), with the maximum lag set to 12 and the significance level set to 5 percent. The series are the log real Nasdaq price index and log real Nasdaq dividend. The sample period is February 1973 to June 2005 with 389 monthly observations. The critical values for the ADF statistic are obtained from Fuller (1996, Table 10.A.2) and the critical values for $\sup_{r\in[r_0,1]} ADF_r$ are obtained by Monte-Carlo simulation with 10,000 replications.

	ADF_1	$\sup_{r\in[r_0,1]} ADF_r$
log price p_t	-0.826	2.894
log dividend d_t	-1.348	-1.018
	Critical Values for the Explosive Alternative	
1 percent	0.60	2.094
5 percent	-0.08	1.468
10 percent	-0.44	1.184

Table 2. Testing for Explosive Behavior in the Nasdaq Index in the 1990s

This table reports ADF_1 and $\sup_{r\in[r_0,1]} ADF_r$ tests of the null hypothesis of a unit root against the alternative of an explosive root, where r_0 = 0.10. The optimal lag length for the ADF test is selected according to top-down sequential significance testing, as suggested by Campbell and Perron (1991), with the maximum lag set to 12 and the significance level set to 5 percent. The series are the log real Nasdaq price index and log real Nasdaq dividend. Panel A reports the results for the period January 1990 to December 1999; Panel B reports the results for the period January 1990 to June 2000 when explosive behavior is detected to be the strongest. The critical values for the ADF statistic are obtained from Fuller (1996, Table 10.A.2) and the critical values for $\sup_{r\in[r_0,1]} ADF_r$ are obtained by Monte-Carlo simulation with 10,000 replications.

	ADF_1	$\sup_{r\in[r_0,1]} ADF_r$	$\hat{\delta}^{LS}$	$\hat{\delta}^{II}$	95% Confidence Interval
	Panel A. Sample Period: January 1990 to December 1999				
log price p_t	2.309	2.894	1.025	1.033	[1.016,1.050]
log dividend d_t	-8.140	-1.626	0.258		
	Panel B. Sample Period: January 1990 to June 2000				
log price p_t	2.975	2.975	1.036	1.040	[1.033,1.047]
log dividend d_t	-8.600	-1.626	0.204		
	Critical Values for the Explosive Alternative				
1 percent	0.60	2.094			
5 percent	-0.08	1.468			
10 percent	-0.44	1.184			

Table 3. Power of the ADF_1 Test

This table reports the empirical power of the ADF test for an explosive stock market bubble at the 5 percent nominal size level with 120 observations and 10,000 Monte-Carlo replications. The model used for the experiment is $p_t = p_t^f + b_t$, $p_t^f = \mu + p_{t-1}^f + \varepsilon_{f,t}$, $b_t = (1+g)b_{t-1} + \varepsilon_{b,t}$, with parameter values $\mu = 0.00227$, $\sigma_f = 0.05403$, $\sigma_b = 0.0324$, estimated based on the Nasdaq price index data as described in the text. These parameter values are used to conduct simulations under different assumptions about the speed parameter g and the initial level of the bubble process b_0. Results are reported in Panel A. Panel B displays results with different values assigned to b_0 and the bubble innovation standard deviation σ_b when the speed parameter g is set to its empirically fitted value of 0.04.

Panel A

Initial Value b_0	g = 0.00 (size)	g = 0.01	g = 0.02	g = 0.03	g =0.04
0.00	0.049	0.107	0.458	0.806	0.934
0.02	0.049	0.111	0.464	0.810	0.937
0.04	0.049	0.115	0.476	0.818	0.935
0.06	0.049	0.119	0.495	0.828	0.951
0.08	0.049	0.125	0.522	0.848	0.954
0.10	0.049	0.134	0.550	0.866	0.961

Panel B

Initial Value b_0	σ_b = 0.005	σ_b = 0.01	σ_b = 0.02	σ_b = 0.03	σ_b = 0.04
0.00	0.652	0.817	0.901	0.930	0.942
0.02	0.822	0.851	0.905	0.933	0.944
0.04	0.972	0.911	0.924	0.934	0.948
0.06	0.999	0.962	0.936	0.945	0.950
0.08	1.000	0.988	0.953	0.952	0.955
0.10	1.000	0.998	0.968	0.961	0.961

Table 4. Power of the ADF_1 and $\sup_{r\in[r_0,1]} ADF_r$ Tests under the Evans (1991) Model

This table reports the empirical power of the ADF test for an explosive bubble at the 5 percent nominal size level with 100 observations and 10,000 Monte-Carlo replications. The model used for the experiment is $P_t = P_t^f + 20B_t$ where $P_t^f = \mu(1+g)g^{-2} + D_t/g$ with $D_t = \mu + D_{t-1} + \varepsilon_{d,t}$, $\varepsilon_{b,t} \sim \mathrm{NID}(0, \sigma_d^2)$ and B_t collapses periodically according to

$$B_{t+1} = (1+g)B_t\varepsilon_{b,t+1}, \quad \text{if } B_t \leq \alpha$$

$$B_{t+1} = [\zeta + \pi^{-1}(1+g)\theta_{t+1}\left(B_t - (1+g)^{-1}\zeta\right)]\varepsilon_{b,t+1}, \quad \text{if } B_t > \alpha$$

with $g > 0$, $\varepsilon_{b,t} = \exp(y_t - \tau^2/2)$, $y_t \sim$ NID $(0, \tau^2)$, θ_t being a Bernoulli process which takes the value 1 with probability π and 0 with probability $1 - \pi$. We set $g = 0.05$, $\alpha = 1$, $\zeta = 0.5$, $B_0 = 0.5$, $\tau = 0.05$, $\mu = 0.0373$, $\sigma_d^2 = 0.1574$, $D_0 = 1.3$. We choose different values for π.

π	0.999	0.99	0.95	0.85	0.75	0.50	0.25
ADF_1	0.914	0.460	0.069	0.022	0.016	0.026	0.044
$\sup_{r\in[r_0,1]} ADF_r$	0.992	0.927	0.714	0.432	0.351	0.342	0.340

Table 5. Power of the ADF_1 and $\sup_{r\in[r_0,1]} ADF_r$ Tests under the Alternative Bubble Model

This table reports the empirical power of the ADF test for an explosive bubble at the 5 percent nominal size level with 100 observations and 10,000 Monte-Carlo replications. The model used for the experiment is $P_t = P_t^f + 20B_t$ where $P_t^f = \mu(1+g)g^{-2} + D_t/g$ with $D_t = \mu + D_{t-1} + \varepsilon_{d,t}$, $\varepsilon_{b,t} \sim \mathrm{NID}(0, \sigma_d^2)$ and B_t collapses periodically according to

$$B_{t+1} = (1+g)B_t\varepsilon_{b,t+1}, \quad \text{if } B_t \leq \alpha$$

$$B_{t+1} = [\zeta + \pi_{t+1}^{-1}(1+g)\theta_{t+1}\left(B_t - (1+g)^{-1}\zeta\right)]\varepsilon_{b,t+1}, \quad \text{if } B_t > \alpha$$

with $g > 0$, $\varepsilon_{b,t} = \exp(y_t - \tau^2/2)$, $y_t \sim$ NID $(0, \tau^2)$, θ_t taking the value 1 when $S_{t+1} > \sqrt{n}\nu$ and 0 when $S_{t+1} \leq \sqrt{n}\nu$, $S_t = \sum_{k=1}^{t} \xi_k$, $\xi_t \sim \mathrm{NID}(0, \sigma_\xi^2)$. We set $g = 0.05$, $\alpha = 1$, $\zeta = 0.5$, $B_0 = 0.5$, $\tau = 0.05$, $\mu = 0.0373$, $\sigma_d^2 = 0.1574$, $D_0 = 1.3$. We choose different values for ν/σ_ξ .

ν/σ_ξ	-0.3	-0.2	-0.15	-0.1
ADF_1	0.346	0.305	0.339	0.350
$\sup_{r\in[r_0,1]} ADF_r$	0.672	0.595	0.638	0.730

Table 6. Power of the ADF_1 and $\sup_{r\in[r_0,1]} ADF_r$ Tests under the Alternative Bubble Model

This table reports the empirical power of the ADF test for an explosive bubble at the 5 percent nominal size level with 400 observations and 10,000 Monte-Carlo replications. The model used for the experiment is $P_t = P_t^f + 20B_t$ where $P_t^f = \mu(1+g)g^{-2} + D_t/g$ with $D_t = \mu + D_{t-1} + \varepsilon_{d,t}, \varepsilon_{b,t} \sim \mathrm{NID}(0, \sigma_d^2)$ and B_t collapses periodically according to

$$B_{t+1} = (1+g)B_t\varepsilon_{b,t+1}, \quad \text{if } B_t \leq \alpha$$

$$B_{t+1} = [\zeta + \pi_{t+1}^{-1}(1+g)\theta_{t+1}\left(B_t - (1+g)^{-1}\zeta\right)]\varepsilon_{b,t+1}, \quad \text{if } B_t > \alpha$$

with $g > 0$, $\varepsilon_{b,t} = \exp(y_t - \tau^2/2)$, $y_t \sim$ NID $(0, \tau^2)$, θ_t taking the value 1 when $S_{t+1} > \sqrt{n}\nu$ and 0 when $S_{t+1} \leq \sqrt{n}\nu$, $S_t = \sum_{k=1}^{t} \xi_k$, $\xi_t \sim \mathrm{NID}(0, \sigma_\xi^2)$. We set g = 0.05, α = 1, ζ = 0.5, B_0 = 0.5, τ = 0.05, μ = 0.0373, σ_d^2 = 0.1574, D_0 = 1.3. We choose different values for ν/σ_ξ .

ν/σ_ξ	-0.3	-0.2	-0.15	-0.1
ADF_1	0.416	0.390	0.376	0.322
$\sup_{r\in[r_0,1]} ADF_r$	0.847	0.794	0.807	0.826

Figure 1. Time Series Plots of Real Nasdaq Price and Real Nasdaq Dividend from February 1973 to June 2005. Both series are normalized to 100 at the beginning of the sample.

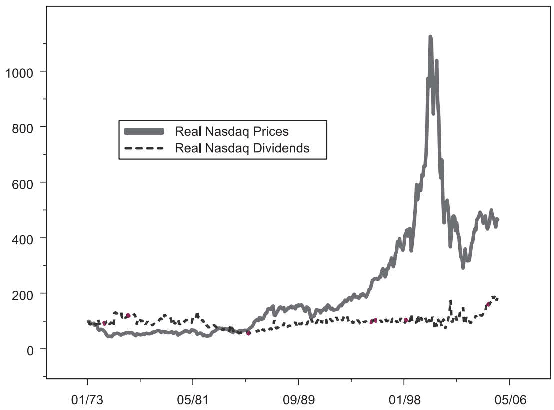


Figure 2. Typical Stationary, Random Walk and Explosive Autoregressive Trajectories

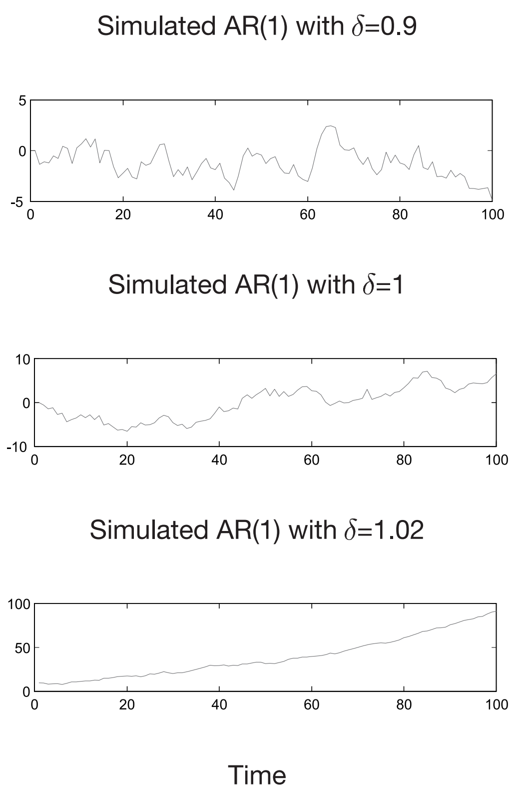


Figure 3. Time Path of the Discount Rate R_{t+s} in (12)

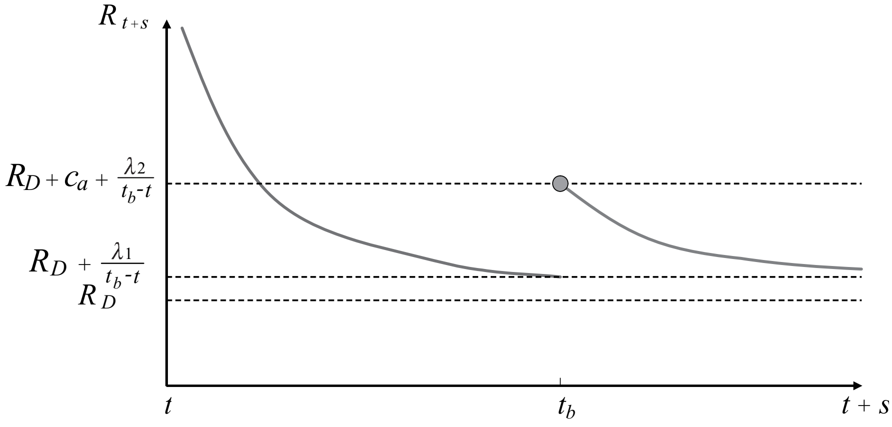


Figure 4. Time Series of ADF_r *t*-statistic for the Logarithmic Real Nasdaq Price and the Logarithmic Real Nasdaq Dividend (r_0 = 0.1) from May 1976 to June 2005. The ADF_r *t*-statistic is obtained from the forward recursive regression with the first observation in February 1973.

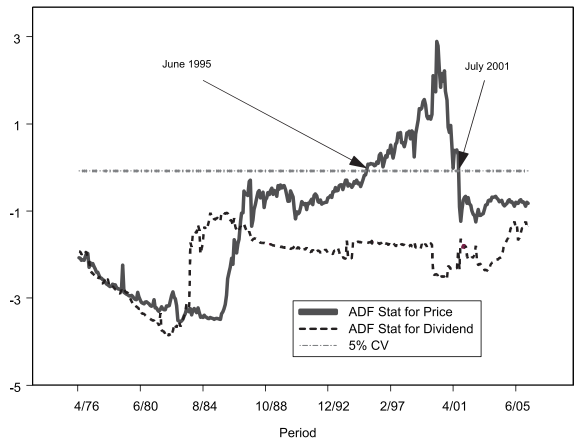
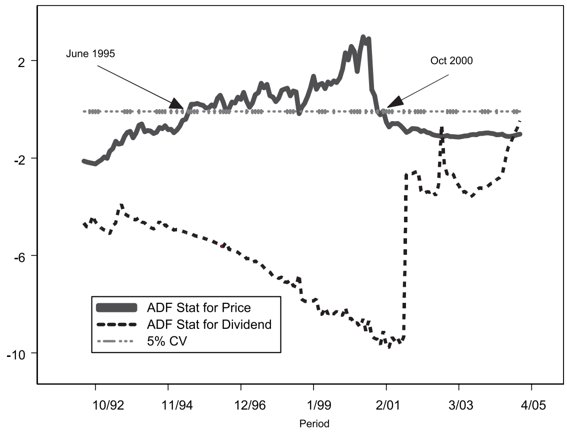


Figure 5. Time Series of ADF_r *t* -statistic for the Logarithmic Real Nasdaq Price and the Logarithmic Real Nasdaq Dividend (r_0 = 0.1) from June 1991 to June 2005. The ADF_r *t* -statistic is obtained from the forward recursive regression with the first observation in January 1990.



Appendix A. Properties of Evans's (1991) Model

We may write the initial stopping time T_α for which the boundary value α is attained as

$$T_\alpha = \inf_t \{t : B_t \geq \alpha\}$$

Subsequent stopping times are determined in the same way after the initial bubble collapses. The duration of each of the bubbles depends on these stopping times plus the number of repeated subsequent draws of $\theta_{T_\alpha+1}$ = 1. It is known (e.g, Schilling, 1990) that the maximum run time, R_n, for a sequence of identical Bernoulli draws in a sample of size n has mean *E* (R_n) = $O\left(\log_{1/\pi}\{n(1-\pi)\}\right) = O\left(\frac{\log\{n(1-\pi)\}}{\log\frac{1}{\pi}}\right)$ and variance Var (R_n) = $\frac{\pi^2}{6\log^2\left(\frac{1}{\pi}\right)}$. It follows that R_n= O_p (log n) . Hence, the maximum time span of a collapsing bubble over the full sample will be T_α + R_n = T_α + O_p (log n). To determine the length of the stopping time T_α, observe that the condition in (19) requires

$$B_{T_\alpha} = (1+g)^{T_\alpha} B_0 \prod_{s=1}^{T_\alpha} u_s = (1+g)^{T_\alpha} B_0 \prod_{s=1}^{T_\alpha} e^{y_s - \frac{1}{2}\tau^2} \leq \alpha$$

which holds if

$$T_\alpha \log(1+g) + \log B_0 + \sum_{s=1}^{T_\alpha}\left(y_s - \frac{1}{2}\tau^2\right) \leq \log\alpha$$

or

$$T_\alpha \left\{\log(1+g) - \frac{\tau^2}{2}\right\} + \sum_{s=1}^{T_\alpha} y_s \leq \log\alpha - \log B_0$$

Writing $\sum_{s=1}^{T_\alpha} y_s = \tau W(T_\alpha)$ where W is a standard Brownian motion, this condition can be rewritten as

$$W(T_\alpha) + \mu T_\alpha \leq A \tag{23}$$

where

$$\mu = \frac{1}{\tau}\log(1+g) - \frac{\tau}{2}, \quad A = \frac{1}{\tau}\{\log\alpha - \log B_0\}$$

The time span T_α of the first component in the bubble (19) is therefore the passage time until a standard Brownian motion $W(t)$ with drift μ hits the boundary value A. That is

$$T_\alpha = \inf_s \{W(s) + \mu s \geq A\} \tag{24}$$

It is well known (e.g. Borodin and Salminen, 1996, p.223) that this passage time satisfies

$$P(T_\alpha = \infty) = 1 - e^{\mu A - |\mu A|}$$

and, since for small values of τ and with $B_0 < 1$ we have μ, $A > 0$, it follows that $P(T_\alpha = \infty) = 0$. Also, T_α has moment generating function (Borodin and Salminen, 1996, p.223)

$$E\left(e^{-gT_\alpha}\right) = e^{\mu A - |A|\left\{2g+\mu^2\right\}^{1/2}}$$

so that the expected hitting time

$$E(T_\alpha) = |A\mu|\, e^{\mu A - |A\mu|} = A\mu = A\left\{\frac{1}{\tau}\log(1+g) - \frac{\tau}{2}\right\}$$

is finite, as is the variance. It follows that the maximum time span of a collapsing bubble generated by (19) and (20) over the full sample is $T_\alpha + R_n = O_p(\log n)$ and, in general, the time span will be shorter than $T_\alpha + R_n$ because the maximum run time R_n will not usually be attained.

This finding explains the failure of conventional unit root tests to detect bubbles of this type, confirming the simulations in Evans (1991). In effect, even the maximum time span of $O_p(\log n)$ for these collapsing bubbles is so short relative to the full sample size n that full sample tests for explosive behavior are inconsistent. Heuristically, this is because the signal from the explosive part of the trajectory is generally not strong enough to dominate the regression before the bubble collapses. In particular, if data $\{B_t\}_{t=1}^n$ were available, the signal from an explosive period initialized at T_0 and of duration $T_\alpha + R_n$ in the regression of B_{t+1} on B_t has order

$$
\begin{aligned}
& O_p\left(\sum_{t=T_0}^{T_\alpha+R_n+T_0} B_t^2\right) \\
= \; & O_p\left(T_\alpha \alpha^2 + \sum_{t=T_0+T_\alpha+1}^{R_n+T_0} B_t^2\right) = O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\prod_{s=T_0+T_\alpha+1}^{T_0+T_\alpha+R_n} u_s\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\prod_{s=T_0+T_\alpha+1}^{T_0+T_\alpha+R_n} u_s\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\exp\left[\sum_{s=T_0+T_\alpha+1}^{T_0+T_\alpha+R_n} y_s - R_n\frac{\tau^2}{2}\right]\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\exp\left[\sum_{s=T_0+T_\alpha+1}^{T_0+T_\alpha+R_n} y_s - R_n\frac{\tau^2}{2}\right]\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\exp\left[\tau\left\{W\left(T_0+T_\alpha+R_n\right)-W\left(T_0+T_\alpha\right)\right\} - R_n\frac{\tau^2}{2}\right]\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\exp\left[O_p\left(\sqrt{R_n}\right) - R_n\frac{\tau^2}{2}\right]\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left\{\left(\frac{1+g}{\pi}\right)^{2t}\alpha^2\exp\left[-\frac{\tau^2}{2}\log n\right]\right\}\right) \\
= \; & O_p\left(\sum_{t=T_0+T_\alpha+1}^{R_n+T_0+T_\alpha}\left(\frac{1+g}{\pi}\right)^{2t}\right)\times O_p\left(e^{-\frac{\tau^2}{2}\log n}\right) = O_p\left(\left(\frac{1+g}{\pi}\right)^{2R_n}\right)\times O_p\left(e^{-\frac{\tau^2}{2}\log n}\right) \\
= \; & O_p\left(n^{2\log\left(\frac{1+g}{\pi}\right)-\frac{\tau^2}{2}}\right) \qquad (25)
\end{aligned}
$$

since $R_n = O_p(\log n)$. The signal from a stationary autoregression is $O_p(n)$ and from a unit root autoregression is $O_p\left(n^2\right)$ so that the signal from the explosive component above will be of maximal order $O_p\left(n^{2\log\left(\frac{1+g}{\pi}\right)-\frac{\tau^2}{2}}\right)$, which is still a power law in n and no greater than that of an integrated process, whose signal is $O_p\left(n^2\right)$, when

$$
\log\left(\frac{1+g}{\pi}\right) < \frac{1}{2} + \frac{\tau^2}{4}
$$

and no greater than that of a polynomial in an integrated process in general, thereby excluding explosive behavior.

Appendix B. Properties of the Alternative Bubble Model

Define the initial stopping time T_α for which the boundary value α is attained as

$$T_\alpha = \inf_t \{t : B_t \geq \alpha\}$$

and the stopping time $H_n^{T_\alpha}$ of the bubble that follows time T_α as

$$H_n^{T_\alpha} = \min_s \{s : S_{T_\alpha+s} \leq \nu_n\} \tag{26}$$

Suppose $\nu_n = \sqrt{n}\nu$ for some ν < 0, and let $s = [nr]$ for some r > 0. Then the condition $S_{T_\alpha+s} \leq \nu_n$ is asymptotically equivalent to $n^{-1/2} S_{T_\alpha+[nr]} \leq \nu$, or $\sigma_\xi V(r) \leq \nu$ in the limit where the Brownian motion $V(r)$ is the weak limit of $n^{-1/2} S_{T_\alpha+[nr]}$ given the initialization at T_α. We can then write $H_n^{T_\alpha} = [nh_\nu]$, where $h_\nu = \inf_r \{r : \sigma_\xi W(r) \leq \nu\}$ is the first passage time of a standard Brownian motion to the level ν/σ_ξ. The duration of the bubble is then measured by $T_\alpha + H_n^{T_\alpha} = [nh_\nu] + O_p(1)$ and the probability density of h_ν is the passage time distribution (e.g. Borodin and Salminen, 1996, p.223)

$$f(h) = \frac{|\nu/\sigma_\xi|}{\sqrt{2\pi}} h^{-3/2} \exp\left\{-\nu^2/2h\sigma_\xi^2\right\}$$

Hence,

$$\begin{aligned}
\pi_{t+1} &= P(S_{t+1} > \nu_n | B_t > \alpha) = P(H_n^{T_\alpha} > t) \\
&= P([nh_\nu] > t) = P(nh_\nu \geq t+1) \\
&= 1 - \int_0^{(t+1)/n} \frac{|\nu/\sigma_\xi|}{\sqrt{2\pi}} h^{-3/2} \exp\left\{-\nu^2/2h\sigma_\xi^2\right\} dh
\end{aligned}$$

In this framework, B_t is formally a triangular array process and generates bubbles whose duration depends on the distribution of h_ν.

An alternative possibility for duration determination that is convenient in practice is to use an arc sine law for the probability distribution of h_ν. This distribution is motivated by the fact that it represents the distribution of the amount of time that a Brownian motion spends above (or below) the origin. If $n^{-1/2}\nu_n \to 0$ in (26) then $S_{T_\alpha+s} \leq \nu_n$ is asymptotically equivalent to $\sigma_\xi V(r) \leq 0$ and the proportion of time h that this condition holds in the unit interval has an arc sine distribution with density

$$f(h) = \frac{1}{\pi\sqrt{h(1-h)}}, \quad \text{with } h \in [0,1]$$

as in Park and Phillips (2000). Again, the duration of the bubble is measured by $T_\alpha + H_n^{T_\alpha} = [nh_\nu] + O_p(1)$.