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Abstract

Empirical findings and theoretical studies suggest that firms adjust towards time-varying target leverage ratios. This paper studies the performances of the default probabilities generated from two stationary-leverage models with time-dependent and constant target ratios respectively. The time-dependent model consistently performs better in terms of discriminatory power of differentiating firms' default risk and capability for predicting default rates over the period 1996 to 2006. The model provides appropriate measures of credit risk of firms and evidence to support the existence of a time-varying target leverage ratio.

Keywords: Leverage, Default probabilities, Credit risk
JEL Classification: C60, G13, G32

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1. Introduction

While the empirical literature provides conflicting assessments about how firms choose their capital structures, Flannery and Rangan (2006) find strong evidence that non-financial firms identified and pursued long-run target leverage ratios during the l966-2001 period.[1] Their empirical model accounts for the potentially dynamic nature of a firm's capital structure. The evidence is equally strong across size classes and time periods, and indicates that a partial adjustment model with firm fixed effects fits the data very well. Firms that are under- or over-leveraged by this measure soon adjust their leverage ratios to offset the observed gap between their actual and target leverage ratios. Survey evidence by Graham and Harvey (2001) shows that 81% of firms consider a target leverage ratio or range when making their debt decisions.

Flannery and Rangan (2006) also find that firms adjust towards time-varying target leverage ratios, which depend on plausible firm features. Their base specification indicates that the typical firm's target debt ratio varies quite a lot. The cross-sectional mean target debt ratio starts at 32.1% in 1966, rises to a maximum of 64.0% in 1974, and ends the period at 27.0% in 2001. Over the entire sample, the estimated target has an average of 30.7% and a standard deviation of 25.1%. Consistent with this finding, Robert (2002) examines the dynamic properties of the capital structure of firms during the 1980-1998 period in a state-space framework and finds that firms gradually adjust their capital structure to a time-varying target, as opposed to a fixed level. In addition, mean reversion in leverage may be due to firms' credit considerations. Hovakimian et al. (2001) empirically find that the target ratio may change over time as the firm's profitability and stock price change. Korajczyka and Levy (2003) demonstrate that macroeconomic conditions affect firms' target leverages which could thus be cyclical and time varying.

Theoretical studies also suggest that the target leverage ratio is time varying. Hennessy and Whited (2005) use a dynamic programming approach to model the dynamic capital structure decision. They model debt as a one period security with no issuing cost and the adjustment therefore has to be made every period. They find that the target leverage ratio is time varying depending on firms' current state variables. Titman and Tsyplakov (2007) and Childs et al. (2005), which incorporate the interaction of firms' investment decisions and financing decisions into their models, also arrive at similar conclusions.

The empirical findings of a firm adjusting its outstanding debts in response to changes in its firm value in order to achieve a target level of leverage call for the stationary-leverage model for pricing corporate bonds, which has been studied by Collin-Dufresne and Goldstein (2001). The Collin-Dufresne and Goldstein model (hereafter referred to as the CG model) is based on the structural approach in Merton (1974) and considers a mean-reverting liability that is the default barrier.[2] The leverage ratio is defined

[1] Other empirical findings on the existence of a target leverage ratio can be found in Marsh (1982), Jalilvand and Harris (1984) and Auerbach (1985). It should be noted that the existence of target leverage ratios does not presume that the "trade-off" theory of capital structure dominates the packing order theory (see Shyam-Sunder and Myers, 1999).

[2] Black and Scholes (1973) and Merton (1974) have been the pioneers in the development of the structural approach to model credit risk of corporates using a contingent-claims framework. Under this approach, default could happen if the value of a firm's assets falls below the firm's liability (i.e. the value of promised payments). Credit risk therefore changes with variations in the value of the firm value and liability.

as a ratio of the liability to the asset value of a firm. The model parameters in the CG model are all constant. These assumptions make the leverage ratio approach a constant target liability-to-asset (leverage) ratio over time. Collin-Dufresne and Goldstein observe empirically that the long-term target ratio is close to the average leverage ratio of BBB-rated firms. They conclude that accounting for a bond issuer's ability to control its level of outstanding debt in the model has a significant impact on credit spread predictions. It helps reconcile some predictions of credit spreads with empirical observations. These include credit spreads that are larger for low-leverage firms and less sensitive to changes in firm value, and upward sloping term structures of credit spreads of speculative-grade bonds.

As there is evidence to support the existence of time-varying target leverage ratios and mean reversion in leverage may be due to firms' credit considerations, the mean-reverting dynamics of a leverage ratio should be a critical factor in modelling credit risk. Hui et al. (2006) show that the incorporation of time-varying target leverage ratios into a structural credit risk model is capable of producing term structures of probabilities of default (PDs) that are consistent with the default rates reported by Standard & Poor's (S&P's), in particular for ratings of BBB and below.

The objective of this paper is to study the capability of two stationary-leverage models to predict credit risk of firms in terms of their PDs. The two models are the CG model in which the target ratio is constant and the time-dependent stationary-leverage model (hereafter referred to as the TDSL model) in which the target ratio is time-varying. The study compares the performance of these two models and determines the extent to which the time-dependent target ratio in the model has an effect on the predictions of default rates. The results would have implications on the development in credit risk models under the structural approach. In particular, the Basel New Capital Accord (Basel II) allows sophisticated banks to use their internal rating systems and credit risk models to determine their capital requirements to cover the credit risk of exposures to corporates (i.e. the internal ratings-based approach).[3] Because of such development, the structural credit risk models have been increasingly studied or even employed by the industry. On the other hand, Basel II allows less sophisticated banks to use external credit ratings to classify their exposures into different risk classes for capital purposes (i.e. the standardised approach). Bank supervisors may therefore find that it is important to have an understanding of the dynamics of leverage ratios and its effect on predicting default rates as compared to those predicted by external credit ratings.

For comparative purposes, the KMV model based on the structural approach is used as a reference to assess the performance of the two stationary-leverage models. The KMV model with an extensive database is developed by Moody's KMV specialised in credit risk analysis. Based on historical information on a large sample of firms, a measures called distance-to-default can be mapped to the corresponding implied PD for a given time horizon. This implied PD is the expected default frequency (EDF) of the firm, which is a common market-based credit risk measure adopted by market practitioners. The calculations of PDs from the KMV model are presented in the Appendix.

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[3] The Basel Committee on Banking Supervision is responsible for proposing capital requirements for internationally active banks. The Committee first proposed Basel II, in 1999, with the final version (Basel, 2004) in June 2004. Basel II is expected to replace the original Basel Accord, which was implemented in 1988.

In the TDSL model a firm's liability is assumed to be governed by a mean-reverting stochastic process whilst the firm value follows a simple lognormal process. The two stochastic variables of the firm value and the liability are correlated. By incorporating time-dependent model parameters in the model, the target leverage ratio is thus time dependent. The time-dependent target leverage ratio can therefore reflect the movement of a firm's initial target leverage ratio towards a long-term target ratio over time which is the average leverage ratio of BBB-rated firms. When the volatility of the liability is set equal to zero and the model parameters are constant, the time-dependent model converges to the CG model.

The predictive capability of the models is assessed by comparing model-implied 1-year PDs of sample companies with their actual default rates in a one-year horizon during 1996-2006. The assessment is based on discriminatory power and calibration accuracy of the models.[4] To avoid the problem of downward-biased PDs at short maturities (e.g. one year) produced directly by the models, that is common to contingent-claims credit risk models including the stationary-leverage models which assume continuous dynamics, a simple mapping process is used. The idea is to map the term structure of PDs of a company generated by a model to the "closest" term structure of default rates of a credit rating (say BBB) reported by S&P's (2005). After mapping the two term structures, the corresponding 1-year default rate of the (BBB) rating is assigned as the model-implied 1-year PD of the company. Such model-implied PD of the company can be considered as an aggregate measure of the entire term structure of PDs of the company.

The remainder of the paper is organised as follows. In the following section we present the TDSL model. Section 3 presents the associated mapping process for model-implied 1-year PDs. The data used for the study and the empirical results of assessing the CG and TDSL models in terms of discriminatory power and capability for predicting default rates are presented in sections 4 and 5 respectively. The final section summarises and discusses the findings.

2. Time-dependent Stationary-leverage (TDSL) Model

A continuous-time framework is used to calculate PDs of a firm in the TDSL model as in Hui et al. (2006). The firm value V is assumed to follow a lognormal diffusion process. The firm liability Q is governed by a mean-reverting lognormal diffusion process. The risk-free interest rate r is assumed to be constant. Their continuous stochastic movements are modelled by the following stochastic differential equations:

$$\frac{dV}{V} = \mu_V(t)dt + \sigma_V(t)dz_V$$

$$\frac{dQ}{Q} = \left[\mu_Q(t) + \kappa_Q(t)(\ln V - \ln Q)\right]dt + \sigma_Q(t)dz_Q \qquad (1)$$

[4] These methodologies are set out in Basel (2005) for validating credit rating systems under Basel II.

where $\sigma_Q(t)$ and $\sigma_V(t)$ are the respective volatility values, $\mu_Q(t)$ and $\mu_Q(t)$ are the respective drift rates, and the firm liability Q is mean-reverting at speed $\kappa_Q(t)$. When InQ is less than InV, the firm acts to increase InQ, and vice-versa. This means that the firm tends to issue debt when its leverage ratio falls below a target and reduces its liability when its leverage ratio is above the target. Future changes in the liability structures of the firm give uncertainty to the value of the liability. There is no other explicit relationship assumed between the firm value and the way it will impact the value of the liability. All model parameters are explicitly time dependent. The Wiener processes dz_V and dz_Q are correlated with ρ_{VQ}.

We define $R \equiv Q/V$ to be the leverage ratio and apply Ito's lemma to derive the partial differential equation governing a corporate discount bond value $P(R,t)$ with constant risk-free interest rate r based on the model as follows:[5]

$$\frac{\partial P(R,t)}{\partial t} = \frac{1}{2}\sigma_R^2(t)R^2\frac{\partial^2 P}{\partial R^2} + \kappa_Q(t)[\ln\theta_R(t) - \ln R]R\frac{\partial P}{\partial R} - rP \tag{2}$$

where t is the time-to-maturity,

$$\sigma_R(t) \equiv \sqrt{\sigma_V^2(t) - 2\rho_{VQ}(t)\sigma_V(t)\sigma_Q(t) + \sigma_Q^2(t)}$$

$$\theta_R(t) \equiv \exp\left\{\left[\sigma_V^2(t) - \rho_{VQ}(t)\sigma_V(t)\sigma_Q(t) - \xi(t)\right]/\kappa_Q(t)\right\}$$

$$\xi(t) \equiv \lambda_Q(t)\sigma_Q(t) - \lambda_V(t)\sigma_V(t) \tag{3}$$

Here $\theta_R(t)$ is the time-dependent target leverage ratio, and the terms $-\lambda_Q(t)\sigma_Q(t)$ and $-\lambda_V(t)\sigma_V(t)$ are included to account for the risk premiums of the firm's value and liability. Under the risk-neutral measure, ξ is equal to zero. To obtain reasonable parameter values for the target leverage ratio, the dynamics of Q and V are under the actual measure such that ξ is non-zero. If the leverage ratio R and its associated model parameters in Eq. (3) could be directly observed in the market, Eq. (2) is still applicable to the model.

When the firm's leverage ratio is above a predefined level R_0, bankruptcy occurs before maturity. This is consistent with the event of bankruptcy being associated with abnormally high levels of debt relative to the market value of the firm's assets. As shown in Hui et al. (2006), the corresponding PD, $P_{\text{def}}(x,t)$, of a corporate discount bond over a period of time t based on Eq. (2) can be approximated by

$$\begin{aligned} P_{\text{def}}(x,t) = 1 - N&\left\{-\frac{1}{\sqrt{2c_1(t)}}\left[x\exp(\alpha(t)) + c_2(t)\right]\right\} \\ &+ N\left\{\frac{1}{\sqrt{2c_1(t)}}\left[x\exp(\alpha(t)) + c_2(t) + 8\beta c_1(t)\right]\right\} \\ &\times \exp\left\{4\beta\left[x\exp(\alpha(t)) + c_2(t) + 16\beta^2 c_1(t)\right]\right\} \end{aligned} \tag{4}$$

[5] As it is not a bond pricing analysis in this paper, the use of a constant risk-free interest rate can serve the purpose of default analysis. The corresponding bond pricing solution with the stochastic interest rate can be found in Hui et al. (2006).

where $x = \ln(R / R_0)$, $N(.)$ is the cumulative normal distribution function, β is a real number parameter, and

$$c_1(t) = \frac{1}{2}\int_0^t \sigma_R^2(t')\exp[2\alpha(t')]dt'$$

$$c_2(t) = \int_0^t F(t')\exp[\alpha(t')]dt'$$

$$\alpha(t) = -\int_0^t \kappa_Q(t')dt'$$

$$F(t) = \kappa_Q(t)[\ln\theta_R(t) - \ln R_0] - \frac{1}{2}\sigma_R^2(t)$$

The parameter β is adjusted such that the approximate solution in Eq. (4) provides the best approximation to the exact result. A simple and easy-to-use method that has been developed by Lo et al. (2003) for solving barrier option values with time-dependent model parameters is provided for computing accurate PD estimates based on Eq. (4).

To investigate whether the time-varying target ratio of a firm is an important factor for modelling credit risk, the time dependence of different firms' target leverage ratios is assumed to be uniform across firms.[6] This means that each firm at a different point in time has the same profile of a target ratio $\theta_R(t)$ over time.[7] This assumption puts uncertainty ε in each firm's target ratio at a different point in time as its actual target ratio should be $\theta_R(t) + \varepsilon$. ε is a random shift of the target ratio where $\varepsilon \sim n(0, \sigma_\varepsilon^2)$ and n is the normal distribution function. As shown in Madec and Japhet (2004), for a drifted Ornstein-Uhlenbeck process in Eq. (2) the effect of the random shift of the target ratio on the PD, $P_{\text{def}}(x,t)$, is an increase in the volatility of the leverage ratio of each firm by σ_ε^2. As target leverage ratios are not directly and timely observable in the market, we assume that such an increase in the volatility may probably be reflected in the observable volatility of the firm's equity price. Therefore, the use of the same profile of a target ratio $\theta_R(t)$ over time will not have a material impact on the volatility of the leverage ratios of the firms.

As noted in Titman and Tsyplakov (2005), a value-maximising firm which wishes to maximise the combined value of its debt and equity will initially choose a high leverage ratio, and will tend to make financing choices that move it towards a moving target leverage ratio. In addition, firms with lower costs of issuing equity initially have higher leverage ratios, since it is less expensive for them to issue equity to raise funds to pay down their debt if the performances of the firms are unsatisfactory. These factors may encourage firms to have higher initial target leverage ratios. Therefore, the profile of the time-varying target ratio is assumed to be a high initial value of 0.732 (i.e. the average leverage ratio of a corporate of 'CCC' rating) towards the long-term value of 0.315 (i.e. the average leverage ratio of a corporate of 'BBB' rating) observed empirically by Collin-Dufresne and Goldstein (2001) over time. The high initial value is in line with the maximum cross-sectional target ratio of 0.640 estimated by Flannery and Rangan (2006).

6 The leverage ratios of the BBB and CCC ratings are based on the industry medians reported by S&P's (2001).

7 It is beyond the scope of this paper to investigate the target leverage ratios at different time horizons of individual firms. No study we know of has reported the actual behaviour of time-varying target ratios of individual firms.

We use the simplest, yet non-trivial, scenario about the time-dependence of the target leverage ratio $\theta_R(t)$ to illustrate the effect of the time-dependence of a target leverage ratio on model PDs, which is a linear function of time with no adjustable parameter as follows:

$$\theta_R(t) = \theta_{R0}(1 - \eta t) \tag{5}$$

and an exponential form for the time-dependent target leverage ratio $\theta_R(t)$ as follows:

$$\theta_R(t) = \theta_{R0}[1 + \eta \exp(-\gamma t)] \tag{6}$$

where θ_{R0} and η are constants such that $\theta_R \approx 0.732$ in the first year and $\theta_R \approx 0.315$ in the fifteenth year. The conditions of $\theta_R(t = 1) \approx 0.732$ and $\theta_R(t = 15) \approx 0.315$ enable us to express θ_{R0} and η as functions of γ. Using the average of the optimal values of γ for different ratings found by Hui et al. (2006) such that model PDs fit the empirical default rates, γ is set to be -0.176 for the exponential TDSL model.[8] As γ is negative, the decrease in the target leverage ratio with time is faster than that based on the linear function.

3. Model-implied 1-year PD

The predictive capability of the models is assessed by comparing model-implied 1-year PDs of sample companies with their actual default rates. A simple mapping process of assigning a model-implied 1-year PD to a company follows that used in Hui et at. (2005).[9] The model-implied PD of a company based on each model is obtained by mapping the company's model term structure of PDs generated by the model to the "closest" term structure of default rates of an external credit rating reported in S&P's (2005).[10] The corresponding 1-year default rate of the rating is assigned as the model-implied 1-year PD of the company.

The mapping process is consistent with Leland's (2004) finding that PDs generated from the structural models fit actual default rates provided by Moody's (1998) for longer time horizons quite well for reasonable parameters with proper calibrations. The process can be thought of as characterising each company as if it were part of the pool of companies which have been covered by S&P's calculations of the actual default rates of different ratings. Mapping to the actual 1-year default rate can avoid the problem of downward-biased 1-year PDs produced directly by the stationary-leverage models which assume continuous dynamics.[11] The characterising factor used in this process is the term structures of PDs determined by the models and the term structures of cumulative default rates of different S&P's ratings. The model-implied 1-year PD of the company is thus an aggregate measure and contains the information of the entire term structure of PDs of the company.

8 The optimal values of γ for individual ratings, CCC, B, BB, BBB and A, are -0.23, -0.1, -0.15, -0.31 and -0.09 respectively.

9 Hui et at. (2005) propose a benchmarking model for validation of PDs of listed companies for Basel II purposes, in which the mapping process is used to assign benchmark ratings to the companies.

10 S&P's (2005) reports the cumulative average default rates by rating modifier 1981 to 2004.

11 The problem of downward-biased PDs at short maturities is common to all structural credit risk models which assume continuous dynamics.

Figure 1 illustrates the mapping process. For the stationary-leverage models, a company's leverage ratio (i.e. the ratio of liability to asset value), and asset volatility are the input parameters used to generate the model term structure of PDs of the company. Using the least square method, the model term structure of PDs is mapped to the "closest" term structure of default rates of an S&P's rating to determine the company's model-implied 1-year PD.

4. Data and Model Parameters

The full dataset for the analysis consists of 126,759 monthly observations from 2,294 publicly listed non-financial companies[12] in 46 economies covering the period March 1996 to November 2006.[13] The dataset is from Credit Monitor of Moody's KMV.[14] Only those companies with S&P's credit ratings are included in the analysis. Table 1 presents the distribution of S&P's ratings (AAA to CCC+ or below) of the samples, together with a 13-point scale of the ordinal ratings which will be used for the assignments of the model-implied 1-year PDs discussed in the previous section. Three groups of credit ratings are assigned the same ordinal numbers (e.g. both A+, A and A- are assigned a number of 3). This is consistent with that the term structures of default rates of A+, A, and A- are very close, with those of A+ and A crossing each other (see S&P's, 2005). The term structure of default rates of each of these three groups is the simple average of the default rates of its constituent ratings. The distribution of the samples across industries and economies are presented in Tables 2 and 3 respectively. The 1-year PD of S&P's credit rating of a company is referred to as the 1-year long-run default rate of its S&P's long-term issuer rating. The 1-year EDF from the KMV model and the input parameters of the models for generating companies' term structures of PDs including the asset volatility, the default point (firm liability) and the market asset value of the companies are extracted from the dataset.[15]

The mean-reverting parameter λ for the leverage ratio in the CG and TDSL models is 0.1 which is estimated by Fama and French (2002) who investigate the universe of firms. The target leverage ratio in the CG model is assumed to be 0.315 which is the average leverage ratio of BBB-rated firms reported by S&P's (2001). The predefined default-triggering level is set at $R_0 = 1$ which is used in Collin-Dufresne and Goldstein (2001). As σ_Q of individual ratings is not well studied empirically in the literature, the use of $\sigma_Q = 0$ for the calculations is the same as the model specification in the CG model and σ_L falls close to its asset volatility. The risk-free interest rate r is set to be 5%.

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12 We also exclude all utility companies.

13 Among the 46 economies, 30 are developed economies (i.e., high-income economies defined by the World Bank) which share about 96% of the samples. In terms of geographical regions of the economies, 68% of the sample is from North America, 15% is from the Asia-Pacific region, and 13% is from Western Europe.

14 According to Moody's KMV, Credit Monitor is a software tool that helps monitor and manage the credit risk of corporate obligors. Credit Monitor reports and calculates the default probabilities for corporate obligors for a term of one to five years. While private firm models are available in Credit Monitor, all data used in this study are extracted from the public firm model, namely the EDF™ Public Firm Model.

15 According to Moody's KMV, the default point is the point to which a firm's asset value must fall before the firm defaults. It is approximately equal to the total amount of short-term liabilities, plus half of the long-term liabilities. However, the exact definition varies across industries. Asset values refer to the underlying economic assets of firms instead of the book value reported on their balance sheets. For a public firm, its asset value is estimated from its equity market value, equity volatility and liability structure (see Appendix).

5. Empirical Results

5.1 Discriminatory Power

In this subsection, discriminatory power of the models is assessed. Discriminatory power refers to the capability of a credit risk model to discriminate *ex ante* between high and low credit risk companies. The receiver operating characteristic (ROC) curve, which is a visual tool, is adopted to evaluate discriminatory power of the models. To construct the ROC curves, the whole dataset described in the previous section is split into 117 monthly samples.[16] For each observation month, those companies which have already defaulted by that month are excluded. The 1-year PDs are generated from the stationary-leverage models based on the most updated information available in the observation month. Each company is observed for a one-year period after the end of the observation month to check if it subsequently defaults within the period. The company is classified as a *defaulter* if a default of the company is observed and a *non-defaulter* otherwise. Finally, all monthly samples are combined together to form an aggregate sample that will be used to produce the ROC curve for the models.

The construction of the ROC curve requires two important statistics, namely the hit rate and the false alarm rate. The calculation of these two statistics requires a specification of a PD threshold (i.e. PD^*) such that a company is predicted as a defaulter if its 1-year PD is higher than PD^* and is predicted as a non-defaulter otherwise.

The hit rate of PD^* is defined as

$$HR(PD^*) = \frac{H(PD^*)}{N_B} \tag{7}$$

where $H(PD^*)$ is the number of actual defaulters having their 1-year PD estimates higher than the threshold PD^* (i.e. the number of defaulters predicted correctly) and N_B is the total number of actual defaulters in the sample.

The false alarm rate of PD^* is defined as

$$FAR(PD^*) = \frac{F(PD^*)}{N_G} \tag{8}$$

where $F(PD^*)$ is the number of false alarms, that is, the number of actual non-defaulters having their 1-year PD estimates higher than the threshold PD^* (i.e. the number of actual non-defaulters that were wrongly predicted as defaulters). N_G is the total number of non-defaulters in the aggregate sample.

We set every possible value of the 1-year PD as PD^* and calculate the corresponding $HR(PD^*)$ and $FAR(PD^*)$. ROC curve is obtained by plotting $FAR(PD^*)$ in the x-axis against $HR(PD^*)$ in the y-axis, with $FAR(PD^*)$ being sorted by PD^* in descending order.

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16 It covers the period March 1996 to November 2005.

A model's performance is better the steeper the ROC curve is at the left end and the closer the ROC curve's position is to the point (0,1). This means that the model is better, the larger the area under the ROC curve (AUROC) is. The discriminatory power of a credit risk model can be evaluated by either the AUROC or accuracy ratio (AR) which is defined as

$$AR = 2\int_0^1 HR(FAR)d(FAR) - 1 \qquad (9)$$

where $\int_0^1 HR(FAR)d(FAR)$ is the AUROC. Engelmann et al. (2003) prove that there is a one-to-one relation between the AR and the AUROC and either one implies another. Therefore, these two performance statistics contain the same information. The AR is 0 for a random model without discriminatory power and it is 1.0 for a perfect model. The statistical significance of the differences in the discriminatory power of the models are evaluated by the non-parametric method proposed by Delong et al. (1998) using the AUROC.

In addition to the ROC curve, AUROC, and AR, the Kolmogorov-Smirnov (KS) statistic, which is defined as

$$\underset{PD}{Max} \mid HR(PD) - FAR(PD) \mid \qquad (10)$$

is also used to measure the discriminatory power of the models. The KS statistic, ranging from zero to one, measures the distance between the two distribution functions of PDs of the non-default and the default companies. A higher value of the KS statistic indicates that a model has higher discriminatory power to differentiate between defaulters and non-defaulters.

Regarding the default definition adopted in this study, a company is classified as a default if at least one of the following events is triggered: (a) the company receives "SD" or "D" for its long-term issuer credit rating from S&P's; (b) Moody's reports that the company defaults; and (c) the company is delisted from the stock exchange because of bankruptcy. The default date is defined as the earliest date of the company triggering any one of these three events. Among the 126,759 observations in the aggregate sample (as described in Tables 1 to 3), 2,157 are classified as defaults from 229 defaulted companies.[17]

Figure 2 presents the ROC curves, ARs and KS statistics of the CG, TDSL, KMV models and S&P's ratings. The results show that the models in general perform considerably better than a random model and have adequate discriminatory power of ranking the credit risk of the companies, as their ROC curves are far away from the diagonal (i.e. the random curve, which implies AR=0). The CG model is outperformed by the linear and exponential TDSL models, in terms of the ROC curves, AR and KS statistics. Consistent with this finding, Table 4 shows that the AUROC of the CG model is the lowest among the models and credit ratings, where the differences in AUROC are statistically significant at the 1% level based on the method proposed by Delong et al. (1998). The KMV model has the highest discriminatory power. This may be due to the mapping of the model based on historical information on a large sample of firms.

[17] For every default company, at a maximum, 12 monthly samples could be included as a default observation. Therefore, the number of default observations for discriminatory analyses is significantly higher than the number of defaulted companies.

It is generally suggested that discriminatory power analyses are sensitive to the number of defaults, especially when it is small. Given this, a robustness test is performed to test the sensitivity of the above empirical results to the number of defaults. The definition of defaults is thus relaxed such that the number of defaults in the sample increases. Specifically, the default definition specified in (a) above is modified as a company being rated CCC or below by S&P's.[18] Using the new default definition, the number of default observations increases to 2,964 from 2,157, while the total number of observations slightly decreases to 125,006 from 126,759. Figure 3 and Table 5 present the discriminatory analysis based on the new aggregate sample. The results show that the relative discriminatory power of the models is similar to that based on the original default definition. This indicates that the discriminatory power analysis is robust and the number of default observations does not alter the results.

5.2 Calibration Accuracy

The accuracy of calibration is another measure to assess credit risk models by examining how accurate the models' PDs are at predicting the actual default rates. The accuracy of calibration is measured as the Brier Score (BS) developed by Brier (1950), which is based on the difference between the PD estimated from a model and actual default rate. The BS statistic is defined as:

$$BS = \frac{1}{N}\sum_{i=1}^{N}(PD_i - \pi_i)^2 \tag{11}$$

where PD_i is the 1-year PD of the company i in a portfolio which contains N companies. π_i is a binary variable for the company i which is defined as 1 if the company defaults, and 0 otherwise. By definition, the BS statistic is the mean square error of default forecasts in which greater discrepancies between realised outcomes and forecasts are penalised using a quadratic function (see Rauhmeier and Schuele, 2005). The higher is the accuracy of calibration, the smaller the BS statistic is. The BS statistic ranges from zero to one and it equals zero when default forecasts are perfect.

Figure 4 shows the BS statistics for the CG, exponential TDSL and KMV models, and S&P's credit ratings. The figure also presents their monthly average PD estimates and the average default rates of the sample. Note that data in the period March-1996 to December-2007 are excluded from the analysis due to the extremely small numbers of defaults during this period.[19] According to the BS statistics, the CG model is outperformed by the TDSL model,[20] S&P's ratings and the KMV model. The KMV model

[18] The definition of default used in this robustness test is more conservative than a default event defined by S&P's. According to S&P's, CCC+ to CC- rated companies refer to those companies which are currently highly vulnerable to non-payment. However, unlike those SD or D rated companies which are defaulted companies under S&P's rating definitions, CCC+ to CC- rated companies have not triggered any default thresholds such as bankruptcy petition, loan restructuring and fail to pay its financial obligations as they come due. The definition of default here includes those "almost insolvent" companies.

[19] An observation month is regarded as having too small a number of defaults if the BS statistic of a "naïve" rating system which always predicts no default (i.e. $PD_i = 0$ for all i = 1, ..., N) is lower than the BS statistics of the credit risk models considered (including the KMV model and S&P's credit ratings). Among these 22 months, 16 found that the BS statistic of the "naïve" rating system outperforms all the credit risk models. This indicates that the data in this 22-month period may not be appropriate for calibration analyses.

[20] The CG model is also outperformed by the linear TDSL model. The BS statistic for the linear TDSL model is 0.01600 (not shown in the graph), which is lower than that of the CG model.

achieves the highest accuracy of calibration among the models considered. The time series of the PD estimates demonstrate that the KMV and exponential TDSL models could track the actual default rates significantly better than the CG model which in general tends to underestimate the 1-year PDs over time. In contrast, S&P's credit ratings exhibit a typical feature of a through-the-cycle rating system in which the credit ratings and corresponding PD estimates tend to be stable over time.

To compare the capability of the models' 1-year PD estimates to predict the actual 1-year default rates over time, the BS statistics for each pair of the models (including the S&P's ratings) in every observation month (in the period January 1998 to November 2005) are used for the comparisons. For a pair of models i and j, the difference between their BS statistics in observation month t, defined as $\Delta BS_{i,j;t} = BS_{i,t} - BS_{j,t}$ (where $BS_{i,t}$ denotes the BS statistic for model i in observation month t) is calculated. A positive (negative) value of $\Delta BS_{i,j;t}$ implies that the accuracy of calibration of model j is higher (lower) than that of model i in observation month t.

Distributional properties of $\Delta BS_{i,j;t}$ are derived from a nonparametric bootstrapping method to evaluate whether $\Delta BS_{i,j;t}$ is statistically significantly different from zero.[21] We randomly draw N companies with replacement from the original sample in time t, where N is the number of companies in the original sample in t. The process is repeated by B times such that B bootstrap samples of size N each are created. $\Delta BS_{i,j;t}$ is calculated for each bootstrap sample b (where $b = 1,..., B$) and the resultant statistic is denoted by $\Delta BS^{b}_{i,j;t}$. B is set to be 5,000 to give a reliable estimate.[22] The distributional properties of $\Delta BS_{i,j;t}$ are revealed from the vector $\{\Delta BS^{1}_{i,j;t},..., \Delta BS^{b}_{i,j;t},..., \Delta BS^{B}_{i,j;t}\}$. The 95% confidence interval is defined as the values covered by the 2.5 and 97.5 percentiles of the vector. The null hypothesis that the difference in the BS statistics between models i and j is zero (i.e. no difference in accuracy of calibration between models i and j) can be rejected at the 5% level if zero is out of the range of the bootstrapped 95% confidence interval. Model i is said to outperform (underperform) model j in t in terms of accuracy of calibration if the 97.5 (2.5) bootstrap percentile of the vector is a negative (positive) number.

For each pair of the models (including the KMV model and S&P's credit ratings), the bootstrapping test is performed for every observation month. There are 95 monthly comparisons of $\Delta BS_{i,j;t}$. The number of months that the model outperforms the other model is calculated to evaluate the relative performance of the two models. The performance statistics (i.e. point estimates of $\Delta BS_{i,j;t}$) and the corresponding 95% confidence intervals are shown in Figures 5 to 8.

Figure 5 shows the comparison between the exponential TDSL and CG models. The number of months that the exponential TDSL model outperforms the CG model is 15 (out of 95), while the CG model only outperforms the exponential TDSL model in 9 months (i.e. 15 versus 9). This suggests that in a relatively long time horizon, the exponential TDSL model could achieve a higher accuracy for predicting the actual default rates compared to the CG model. Comparing the exponential TDSL model to the two

[21] This method is adopted by Güttler (2005).

[22] For estimations of confidence intervals, Efron and Tibshirani (1993) suggested using at least 1000 replications (i.e. B = 1,000).

popular credit risk measures, it is found that the KMV model outperforms the exponential TDSL model[23] (13 versus 7, see Panel A in Figure 6), which in turn outperforms S&P's credit ratings quite significantly (20 versus 10, see Panel B in Figure 6). The CG model is outperformed by the KMV model considerably (1 versus 14, see Panel C in Figure 7) and marginally by S&P's credit ratings (10 versus 12, see Panel D in Figure 7). The linear TDSL model is outperformed by the CG model marginally (9 versus 11, see Panel E in Figure 8), and by KMV (1 versus 18, see panel F in Figure 8), but ties with S&P's credit ratings (15 versus. 15, see Panel G in Figure 8).

6. Conclusion

The empirical results show that default measures generated from the structural credit risk model being incorporated with a time-dependent target leverage ratio (i.e. the TDSL model) is more capable of predicting the credit risk of companies than those generated from the model with a constant target leverage ratio (i.e. the CG model) in terms of discriminatory power and calibration accuracy. The material difference between the predictive capability of the two models shows that the mean-reverting dynamics of a leverage ratio is a critical factor in modelling credit risk.

The less satisfactory results of the CG model compared with the TDSL model is caused by the assumption of a constant target leverage ratio value for companies that may lead to the PDs derived from the model less variable across the companies, and hence affects the predictive power of the model. A similar observation about the model feature was noted by Collin-Dufresne and Goldstein (2001). In the CG model, there is a driving force to drift down the leverage ratio to the target leverage ratio for those highly-leveraged companies. In contrast, the leverage ratios of those low-leveraged companies tend to increase and converge to the same target ratio. Given this model characteristic, the difference of cumulative PDs between a typical high leverage company and a low leverage company will tend to decrease with respect to time.

The analysis of the capability of the PDs generated from the TDSL model for predicting defaults of listed companies shows that the model provides appropriate measures of credit risk of listed companies. As a company's leverage ratio is a determinant factor of assessing its credit risk, further work will be needed to determine the detailed dynamics of the leverage ratio in the existence of a time-varying target leverage ratio. From a perspective of the risk-based capital standard (i.e. Basel II), the results indicate that the use of the agency-based and structural model-based measures for capital requirements of exposures to corporates could raise an issue of systematic differences between the two measures. The relative performance of the CG and TDSL models also provides evidence to support the existence of a time-varying target leverage ratio. This is consistent with the recent empirical findings.

[23] This may be due to the use of an empirical historical default database to calibrate PD estimates adopted by the KMV model.

References

Auerbach, A. J. (1985), *Real Determinants of Corporate Leverages, Corporate Capital Structures in the United States*, Chicago: University of Chicago Press.

Basel Committee on Banking Supervision (2004), "International Convergence of Capital Measurement and Capital Standards: a Revised Framework".

Basel Committee on Banking Supervision (2005), "Studies on the Validation of Internal Rating Systems," Working Paper No.14.

Black, F. and M. Scholes (1973), "The Pricing of Options and Corporate Liability," *Journal of Political Economics*, 81: 637-54.

Brier, G. (1950), "Verification of Forecast Expressed in Terms of Probability," *Monthly Weather Review*, 78: 1-3.

Childs, P. D., D. C. Mauer and S. H. Ott (2005), "Interactions of Corporate Financing and Investment Decisions: the Effects of Agency Conflicts," *Journal of Financial Economics*, 76: 667-90.

Collin-Dufresene, P. and R. S. Goldstein (2001), "Do Credit Spreads Reflect Stationary Leverage Ratio?" *Journal of Finance*, 56: 1929-57.

Delong, E. R., D. M. Delong and D. L. Clarke-Pearson (1988), "Comparing the Areas Under Two or More Correlated Receiver Operating Characteristic Curves: A Nonparametric Approach," *Biometrics*, 44: 837-45.

Efron, B. and R. J. Tibshirani (1993), *An Introduction to the Bootstrap*, Chapman & Hall, New York, NY.

Engelmann, B., E. Hayden and D. Tasche (2003), "Testing for Rating Accuracy," *Risk*, 16 (January): 82-6.

Fama, E. and K. R. French (2002), "Testing Tradeoff and Pecking Order Predications About Dividends and Debt," *Review of Financial Studies*, 15: 1-33.

Flannery, J. F., K. P. Rangan (2006), "Partial Adjustment Toward Target Capital Structures," *Journal of Financial Economics*, 79: 469-506.

Güttler, A. (2005), "Using a Bootstrap Approach to Rate the Raters," *Financial Markets and Portfolio Management*, 19: 277-95.

Graham, M. and C. Harvey (2001), "The Theory and Practice of Corporate Finance: Evidence from the Field," *Journal of Financial Economics*, 60: 187-243.

Hennessy, C. A. and T. M. Whited (2005), "Debt Dynamics," *Journal of Finance*, 60: 1129-65.

Hovakimian, A., T. Opler and S. Titman (2001), "The Debt-equity Choice," *Journal of Financial and Quantitative Analysis*, 36: 1-24.

Hui, C. H, T. C. Wong, C. F. Lo and M. X. Huang (2005), "Benchmarking Model of Default Probabilities of Listed Companies," *Journal of Fixed Income*, 15(2): 76-86.

Hui, C. H., C. F. Lo and M. X. Huang (2006), "Are Corporates' Target Leverage Ratios Time-dependent?" *International Review of Financial Analysis*, 15: 220-36.

Jalivand, A. and R. S. Harris (1984), "Corporate Behaviour in Adjusting to Capital Structure and Dividend Targets: An Econometric Investigation," *Journal of Finance*, 39: 611-25.

Kealhofer, S. (1998), "Portfolio Management of Default Risk," *Net Exposure*, 1(2).

Kealhofer, S. (2003), "Quantifying Credit Risk I: Default Prediction," *Financial Analysts Journal*, 59(1): 30-44.

Korajczyka, R. A. and A. Levy (2003), "Capital Structure Choice: Macroeconomic Conditions and Financial Constraints," *Journal of Financial Economics*, 68: 75-109.

Leland, H. E. (2004), "Predictions of Expected Default Frequencies in Structural Models of Debt," *Journal of Investment Management*, 2(2): 5-21.

Lo, C. F., H. C. Lee and C. H. Hui (2003), "A Simple Approach for Pricing Black-Scholes Barrier Options with Time-Dependent Parameters," *Quantitative Finance*, 3: 98-107.

Madec, Y. and C. Japhet (2004), "First Passage Time Problem for a Drifted Ornstein-Uhlenbeck Process," *Mathematical Biosciences*, 18: 131-40.

Marsh, P. (1982), "The Choice Between Equity and Debt: An Empirical Study," *Journal of Finance*, 37: 121-44.

Merton, R. C. (1974), "On the Pricing of Corporate Debt: the Risk Structure of Interest Rates," *Journal of Finance*, 29: 449-70.

Moody's Investors Services (1998), "Historical Default Rates of Corporate Bond Issuers, 1920-1997".

Rauhmeier, R. and R. Scheule (2005), "Rating Properties and Their Implications for Basel II Capital," *Risk*, (March): 78-81.

Robert, M. (2002), "The Dynamics of Capital Structure: an Empirical Analysis of a Partially Observable System," Working paper, Duke University.

Shyam-Sunder, L. and S. C. Myers (1999), "Testing Static Tradeoff Against Pecking Order Models of Capital Structure," *Journal of Financial Economics*, 51: 219-44.

Standard & Poor's (2001), "Adjusted Key U.S. Industrial Financial Ratios".

Standard & Poor's (2005), "Annual Global Corporate Default Study: Corporate Defaults Poised to Rise in 2005".

Titman, S. and S. Tsyplakov (2007), "A Dynamic Model of Optimal Capital Structure," Review of Finance, 11: 401-51.

Vasicek, O. A. (1997), "Credit Valuation," *Net Exposure*, 1(1).

Table 1. Assignment of Ordinal Ratings to S&P's Ratings and Distribution of S&P's Ratings of the Samples

S&P's ratings	Ordinal ratings	Numbers of sample companies	Percentage share (%)
AAA	1	716	0.6%
AA+	1	583	0.5%
AA	2	1,757	1.4%
AA-	2	3,496	2.8%
A+	3	5,435	4.3%
A	3	9,985	7.9%
A-	3	9,370	7.4%
BBB+	4	12,035	9.5%
BBB	5	15,141	11.9%
BBB-	6	12,192	9.6%
BB+	7	8,832	7.0%
BB	8	11,070	8.7%
BB-	9	13,525	10.7%
B+	10	12,009	9.5%
B	11	5,867	4.6%
B-	12	2,993	2.4%
CCC+ or below	13	1,753	1.4%
Total	-	126,759	100.0

Table 2. Distribution of Industries of the Samples

Industry	Numbers of sample companies	Percentage share (%)
Basic Materials	14,861	11.7%
Communications	17,219	13.6%
Consumer, Cyclical	26,327	20.8%
Consumer, Non-cyclical	23,736	18.7%
Energy	11,416	9.0%
Industrial	25,742	20.3%
Technology	7,458	5.9%

Table 3. Distribution of Economies of the Samples

Economy	Number of samples	% share	Economy	Number of samples	% share
Argentina	571	0.45%	Italy	447	0.35%
Australia	3,256	2.57%	Japan	13,115	10.35%
Austria	90	0.07%	South Korea	437	0.34%
Bahamas	155	0.12%	Luxembourg	202	0.16%
Belgium	74	0.06%	Mexico	1,233	0.97%
Bermuda	520	0.41%	Netherlands	1,357	1.07%
Brazil	314	0.25%	Netherlands Antilles	33	0.03%
Canada	5,500	4.34%	New Zealand	631	0.50%
Cayman Islands	126	0.10%	Norway	452	0.36%
Chile	539	0.43%	Philippines	305	0.24%
China	93	0.07%	Poland	34	0.03%
Denmark	97	0.08%	Portugal	122	0.10%
Dominican Republic	61	0.05%	Russia	375	0.30%
Finland	582	0.46%	Singapore	356	0.28%
France	2,626	2.07%	South Africa	149	0.12%
Germany	2,141	1.69%	Spain	432	0.34%
Greece	210	0.17%	Sweden	200	0.16%
Hong Kong, China	412	0.33%	Switzerland	781	0.62%
Hungary	17	0.01%	Taiwan, province of China	354	0.28%
India	12	0.01%	Thailand	210	0.17%
Indonesia	352	0.28%	Turkey	63	0.05%
Ireland	207	0.16%	UK	6,337	5.00%
Israel	328	0.26%	USA	80,851	63.78%
			Total	**126,759**	**100.0%**

Table 4. Statistical Significance of Discriminatory Power of the Models for Differentiating Default Companies: Pair-wise Comparisons Between the Areas Under the ROC Curves (AUROC) of the Models

Model ($AUROC_A$)	Model ($AUROC_B$)	$AUROC_A$ - $AUROC_B$	Chi-square statistics
CG model (0.8568)	Exponential TDSL model (0.9073)	-0.0505***	429.0447
	Linear TDSL model (0.9099)	-0.0531***	427.9750
	KMV (0.9346)	-0.0778***	485.4997
	S&P's ratings (0.9026)	-0.0458***	99.4715
Exponential TDSL model (0.9073)	Linear TDSL model (0.9099)	-0.0026***	13.2892
	KMV (0.9346)	-0.0273***	254.5143
	S&P's ratings (0.9026)	0.0047	1.9059
Linear TDSL model (0.9099)	KMV (0.9346)	-0.0247***	238.9063
	S&P's ratings (0.9026)	0.0073 **	4.4720
KMV (0.9346)	S&P's ratings (0.9026)	0.0320***	118.2239

Notes:

(1) The chi-square statistics are calculated based on Delong et al. (1998). For each pair of the models, a chi-square distribution with degree freedom of one is calculated based on the null hypothesis that AUROCs of the two selected models are the same.

(2) ***, **, and * denote statistical significance at the 1%, 5%, and 10% level.

Table 5. Statistical Significance of Discriminatory Power of the Models for Differentiating Companies with S&P Ratings CCC or Below: Pair-wise Comparisons Between the Areas under the ROC Curves (AUROC) of the Models

Model ($AUROC_A$)	Model ($AUROC_B$)	$AUROC_A$ - $AUROC_B$	Chi-square statistics
CG model (0.8381)	Exponential TDSL model (0.8865)	-0.0484***	498.8947
	Linear TDSL model (0.8898)	-0.0517***	523.3028
	KMV (0.9100)	-0.0719***	527.1459
	S&P's ratings (0.8914)	-0.0533***	168.9725
Exponential TDSL model (0.8865)	Linear TDSL model (0.8898)	-0.0033***	20.3243
	KMV (0.9100)	-0.0235***	217.1974
	S&P's ratings (0.8914)	-0.0049	2.1167
Linear TDSL model (0.8898)	KMV (0.9100)	-0.0202***	193.3229
	S&P's ratings (0.8914)	-0.0016	0.2486
KMV (0.9100)	S&P's ratings (0.8914)	0.0186***	35.8214

Notes:

(3) The chi-square statistics are calculated based on Delong et al. (1998). For each pair of the models, a chi-square distribution with degree freedom of one is calculated based on the null hypothesis that AUROCs of the two selected models are the same.

(4) ***, **, and * denote statistical significance at the 1%, 5%, and 10% level.

Figure 1. Process of Assigning Model-implied 1-year PD

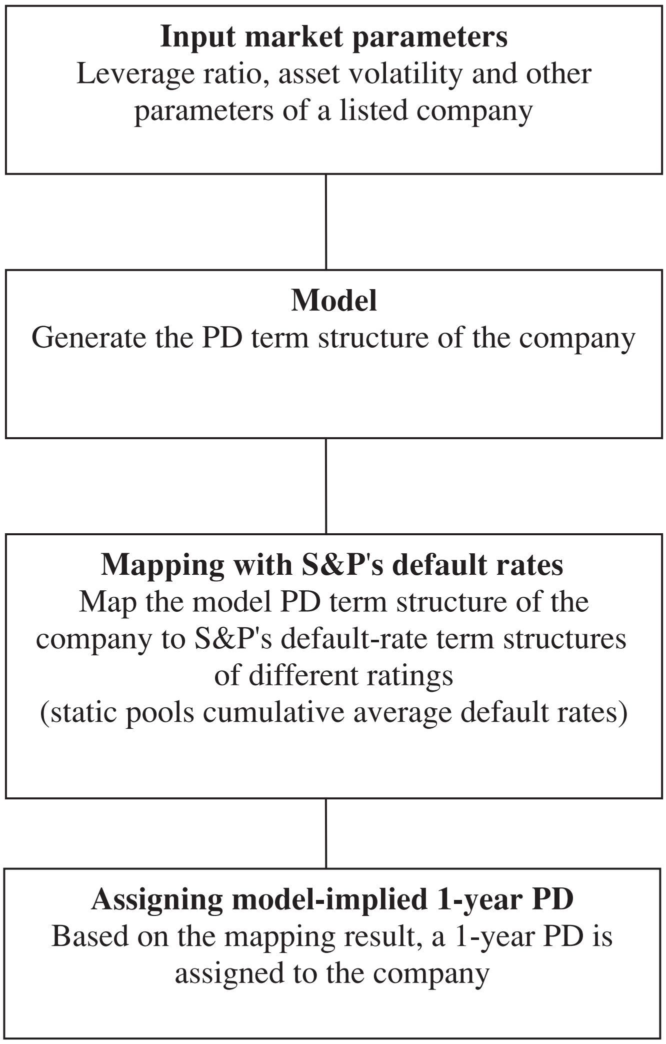


Figure 2. Discriminatory Power for Differentiating Defaulted Companies: Receiver Operating Characteristic Curves, Accuracy Ratios and Kolmogorov-Smirnov Statistics of the Models

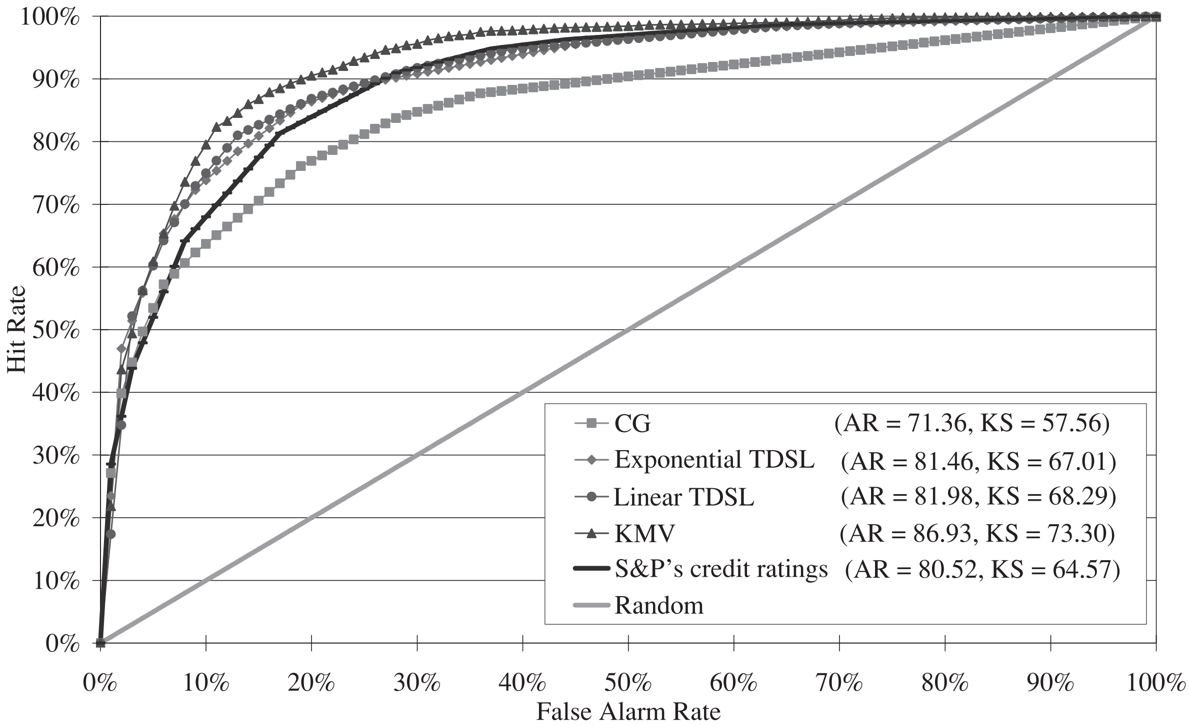


Figure 3. Discriminatory Power for Differentiating Companies with S&P's Credit Ratings of CCC or Below: Receiver Operating Characteristic Curves, Accuracy Ratios and Kolmogorov-Smirnov Statistics of the Models

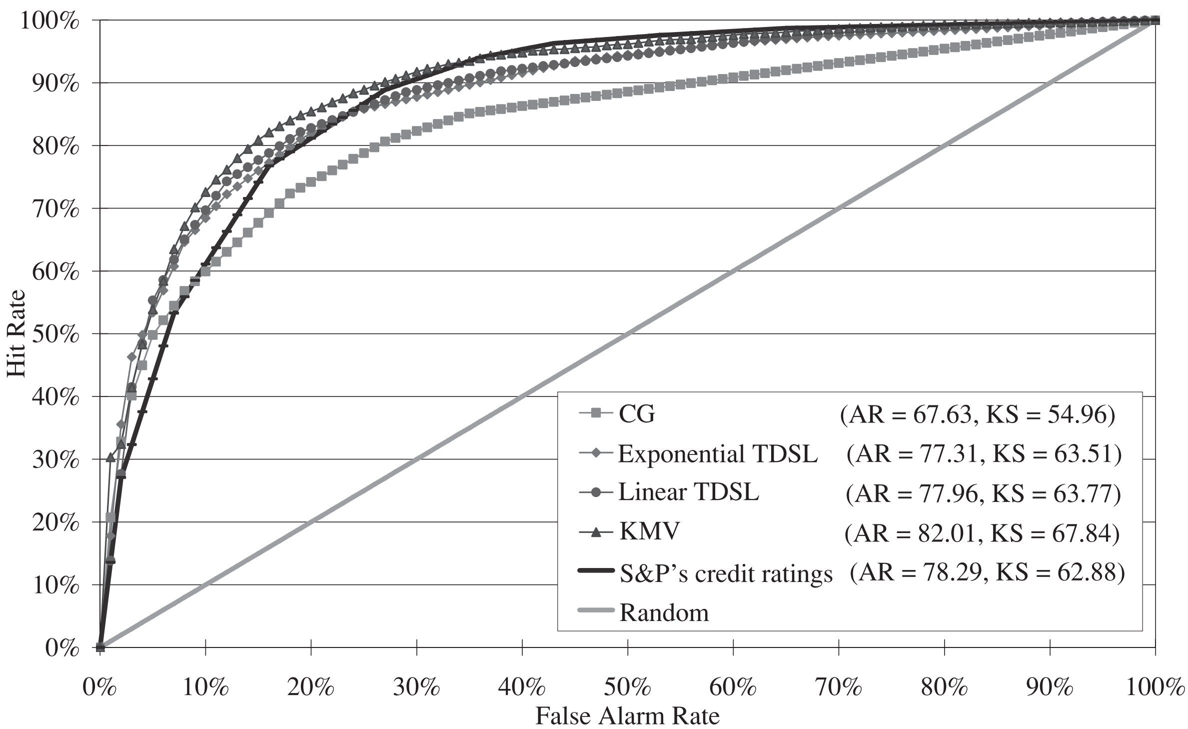
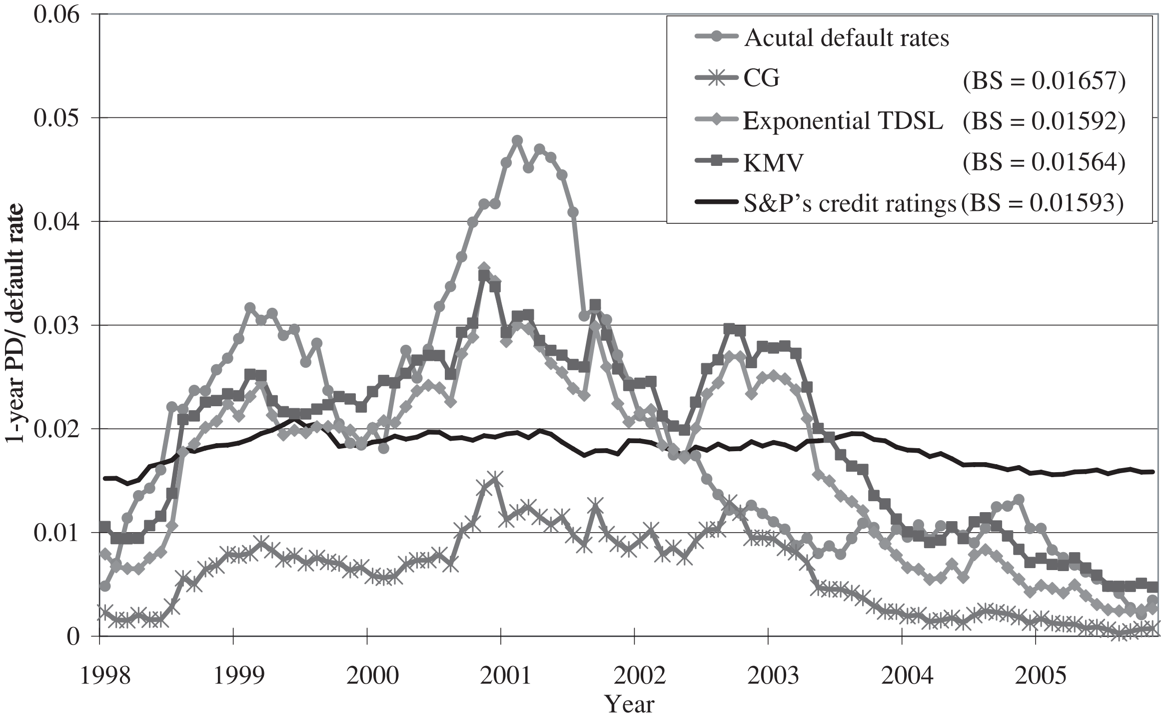


Figure 4. Time Series of Average Default Rates and Average PDs of the Models



Notes:

(1) BS (Brier score) statistics are calculated by using the aggregate sample covering the period January 1998 to November 2005.

(2) A 1-year rolling window is adopted to calculate the 1-year actual default rates. For each observation month, companies that have already defaulted by that month are excluded from the dataset. Each company in that monthly sample is observed for the 1-year period after the end of the observation month to check if it subsequently defaults within this period. The 1-year default rate for the observation month is defined as the ratio of the resulting number of defaults to the total number of sample in that observation month (excluding companies that have already defaulted by the observation month).

Figure 5. Calibration Accuracy of Exponential TDSL Model Relative to CG Model Based on Bootstrap Percentiles of Differences Between Brier Scores (BS)

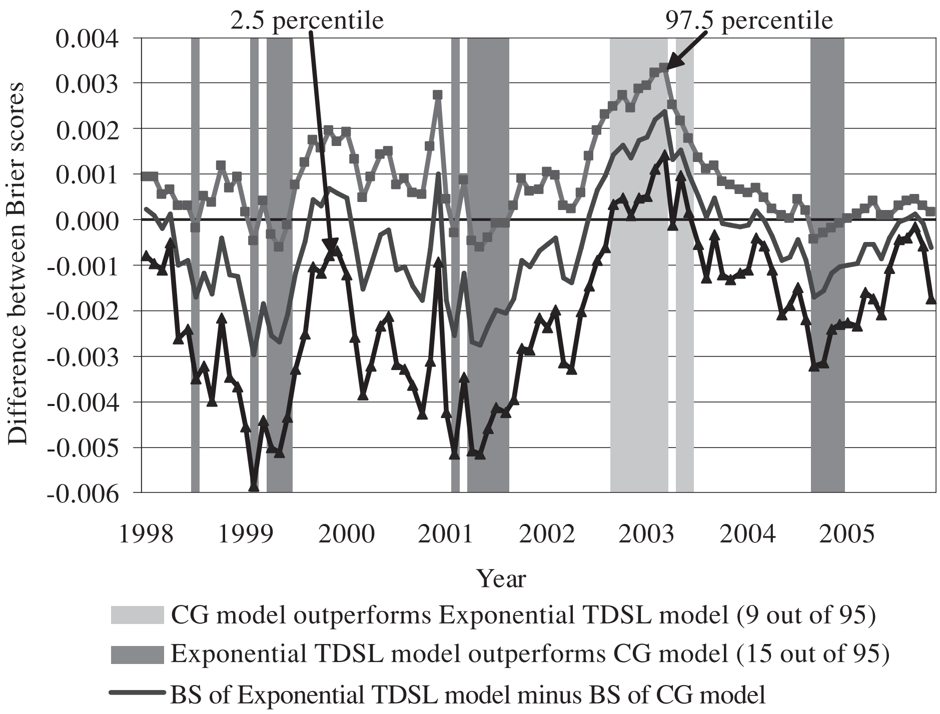
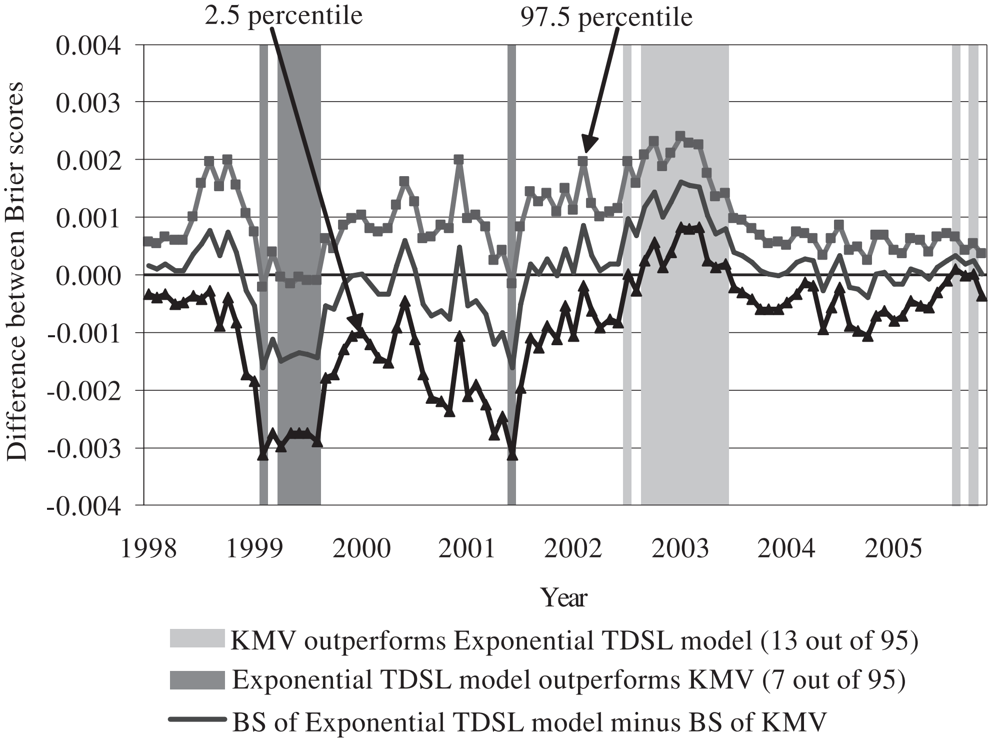


Figure 6. Calibration Accuracy of Exponential TDSL Model Relative to KMV Model and S&P's Credit Ratings Based on Bootstrap Percentiles of Differences Between Brier Scores (BS)

Panel A: Exponential TDSL model versus KMV



Panel B: Exponential TDSL model versus S&P's credit ratings

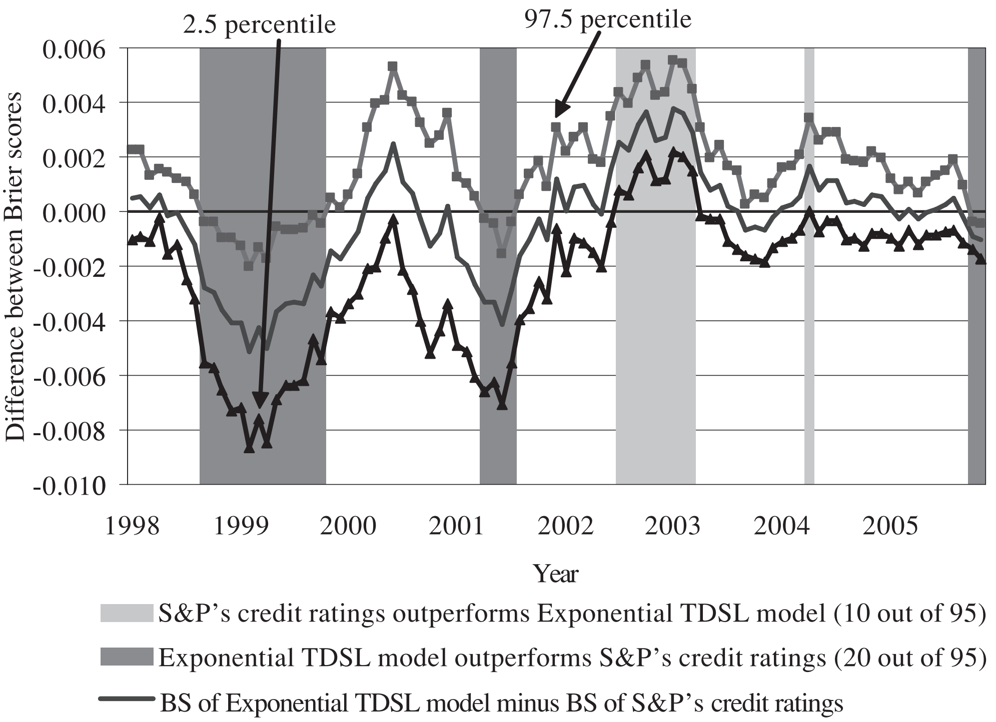
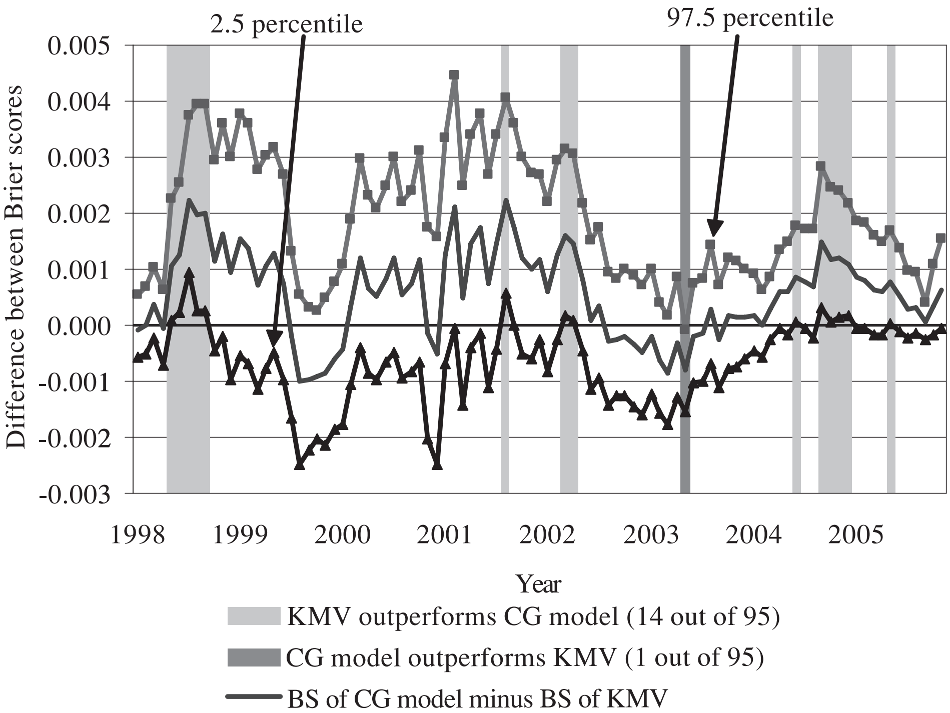


Figure 7. Calibration Accuracy of CG Model Relative to KMV Model and S&P's Credit Ratings Based on Bootstrap Percentiles of Differences Between Brier Scores (BS)

Panel C: CG model versus KMV



Panel D: CG model versus S&P's credit ratings

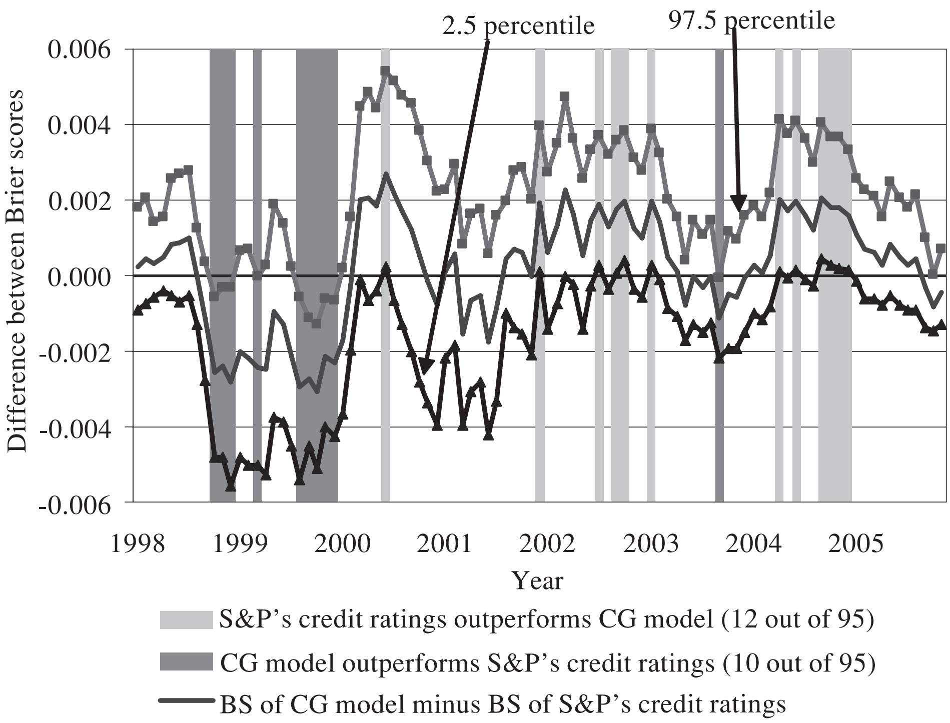
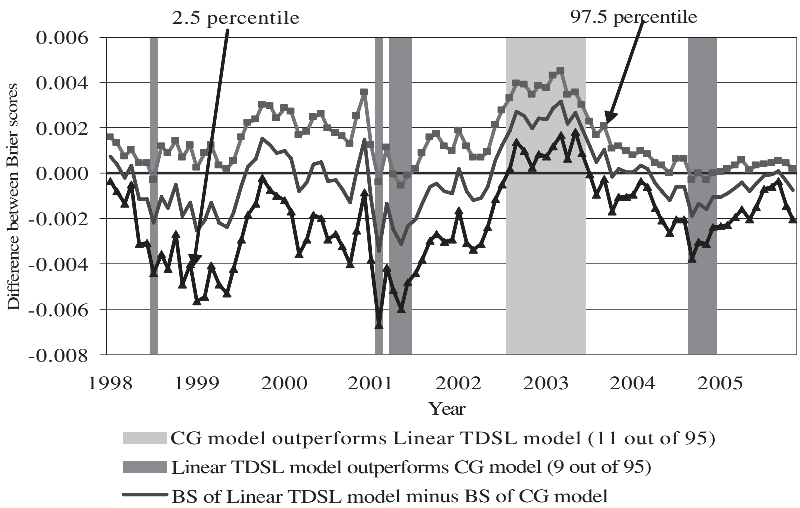
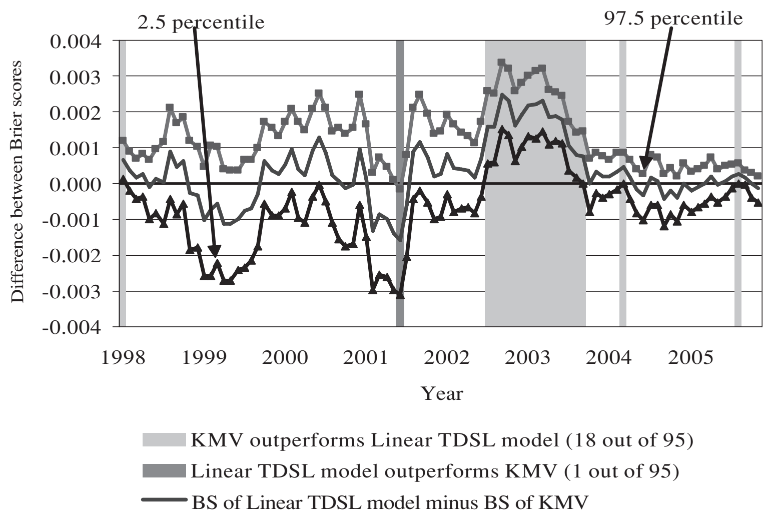


Figure 8. Calibration Accuracy of Linear TDSL Model Relative to CG Model, KMV Model, and S&P's Credit Ratings Based on Bootstrap Percentiles of Differences Between Brier Scores (BS)

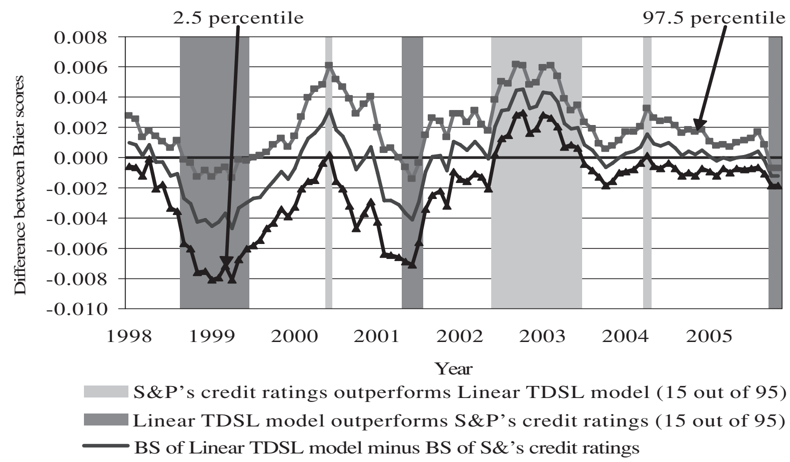
Panel E: Linear TDSL model versus the CG model



Panel F: Linear TDSL model versus KMV



Panel G: Linear TDSL model versus S&P's credit ratings



Appendix

The KMV model produces a PD for each firm at any given point in time. To calculate the PD, the model consists of the following procedures: estimation of the market value and volatility of the firm's asset; calculation of the distance-to-default; and scaling of the distance-to-default to actual PD using a proprietary default database.

The KMV model estimates the market value of a firm's asset by applying the Merton model.[24] The KMV model makes two assumptions. The first is that the total value of a firm is assumed to follow geometric Brownian motion,

$$dV = \mu V dt + \sigma_V V dz_V \tag{A1}$$

where V is the market value of the firm's assets, μ is the expected continuously compounded return on V, σ_V is the volatility of firm's asset value and dz_V is a standard Weiner process. The second assumption of the KMV model is that the capital structure of the firm is only composed of equity, short-term debt which is considered equivalent to cash, long-term debt and convertible preferred shares. With these simplifying assumptions it is then possible to derive analytical solutions for the value of equity E, and its volatility σ_E:

$$E = f(V, \sigma_V, K, c, r) \tag{A2}$$

$$\sigma_E = g(V, \sigma_V, K, c, r) \tag{A3}$$

where K denotes the leverage ratio in the capital structure, c is the average coupon paid on the long-term debt and r the risk-free interest rate.[25, 26]

The KMV model estimates σ_E from market data (i.e. from either historical stock returns data or from option implied volatility data). An iterative technique is used to simultaneously solve equations (A2) and (A3) numerically for values of V and σ_V.[27]

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24 See Vasicek (1997) and Kealhofer (1998, 2003).

25 In the simple Merton's framework, where the firm is financed only by equity and a zero coupon debt, equity is a call option on the assets of the firm with striking price the face value of the debt and maturity the redemption date of the bond. The equity value of a firm satisfies

$$E = VN(d_1) - e^{-rT} KN(d_2)$$

where, d_1 is given by

$$d_1 = \frac{\ln(V/K) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}$$

$d_2 = d_1 - \sigma_V \sqrt{T}$ and T is the time-to-maturity of the debt.

26 It can be shown that $\sigma_E = \eta_{E,V}\, \sigma_V$ where $\eta_{E,V}$ denotes the elasticity of equity to asset value, i.e. $\eta_{E,V} = (V/E)(\partial E / \partial V)$.

27 Vasicek (1997) notes that the numerical technique is complex due to the complexity of the boundary conditions attached to the various liabilities

Using the values of V and σ_V, the KMV model computes an index called "distance-to-default" (DD). DD is the number of standard deviations between the mean of the distribution of the asset value, and a critical threshold, the "default point", set at the par value of current liabilities including short-term debt to be serviced over the time horizon, plus half the long-term debt. The default point F is based on KMV's observations from a sample of several hundred companies that firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt. DD can be calculated as:

$$DD = \frac{\ln(V/F) + (\mu - \sigma_V^2/2)T}{\sigma_V \sqrt{T}} \tag{A4}$$

where μ is an estimate of the expected annual return of the firm's assets, and T is a forecasting horizon.

Based on historical information on a large sample of firms, the DD can be mapped to the corresponding implied PD for a given time horizon.[28] This implied PD is the expected default frequency (EDF) of the firm.[29]

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[28] In addition to the empirical approach of the DD-to-EDF mapping, there are some other notable differences between Merton's model and the current version of the KMV model in practice. According to Moody's KMV's information, these include: (1) the KMV model adopts a combined approach to estimate the asset volatility of a firm in which the estimate is a weighted average of empirical asset volatility and modelled asset volatility. The former is estimated from historical time series of asset returns, while the latter estimates volatility based on firms' size, income, profitability, industry and geographical region; (2) the model also allows default to occur at or before maturity, while the original Merton model allows default to occur only at maturity; and (3) the model is capable of dealing with a broader class of financial liabilities including preferred stocks and convertible bonds.

[29] The probability below the default point is $N(-DD)$ which is the EDF in the simple Merton's framework (see footnote 25 above).