

# **Subordinated Binomial Tree**

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## **Abstract**

We develop a unique tree structure with random move time by subordinating asset price changes to random trade arrival. The asset price change is determined by two independent Bernoulli trials on trade arrival/non-arrival and up/down price movement. Convoluting the two leads to a trinomial tree and a corresponding option pricing problem with stochastic volatility. A time change from calendar-time to trading-time, however, restores the binomial tree and leads to an isomorphic option-pricing problem with constant volatility but random maturity. Utility-dependent valuation results are derived in both time dimensions and isomorphism is demonstrated. The binomial tree now grows with arrival of trades with a speed dictated by the intensity of arrival, irrespective of the passage of calendar-time. At the continuous-time limit, it contains the information-time option-pricing model of Chang, Chang and Lim (1998) as a risk-neutral special case. The trinomial tree is further extended to allow for correlated price revision and trade arrival to address the leverage effect and test for incremental improvements in pricing and hedging against the Black and Scholes and the CEV models.

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Tree-based discrete-time option pricing models start with the seminal binomial model of Cox, Ross and Rubinstein (1979) (CRR hereafter) who introduced an intuitive, parsimonious, and robust approach to derivatives valuation. A preference-free valuation result emerges from the simple set-up of a dynamically complete market with two states and two traded assets. The CRR model is widely used by practitioners for embedding the Black and Scholes (1973) model in the continuous-time limit as well as for the efficient pricing of American and path-dependent options.

There have been various extensions to the original binomial model. Some aim at improving convergence efficiency while others attempt to incorporate more complex stochastic processes. A completely different set of tree-based models has also been developed for the evolution of interest rates.<sup>1</sup> More recently, much attention has been paid to the so-called implied tree models.<sup>2</sup> These models are the result of concerted efforts to deal with volatility smiles and/or sneers commonly observed in prices of traded options. One attractive feature of implied tree models is that they price all traded options consistently thus eliminate the pricing biases present in parametric models, however the tree parameters are likely to be unstable over time and need to be recalibrated as market conditions change.

Throughout these developments however, the original CRR assumption that asset price changes occur uniformly over calendar-time continues to exist. The maturity of the option is divided into equal time intervals over which the asset price moves are of fixed magnitude, implying that a trade should always arrive at the end of each fixed calendar-time period. Taking it to the continuous-time limit, this assumption implies continuous trading. Although convenient, this simplification rules out actual trade arrival dynamics that a trade may not arrive at the end of each calendar-time period, an asset may be thinly traded during certain time periods with lower volatility but heavily traded during other time periods with higher volatility, and that asset price changes may be correlated to the pattern of trade arrival. Although it has been recognized in Amin (1993) and Rubinstein (1994) and some tree-based term structure models that faster convergence can be achieved by introducing non-equally spaced steps, none has randomized trade arrivals.

In this vein, the purpose of this research is to investigate tree-based option pricing with random move time by randomizing trade arrival. This idea originates with the celebrated empirical evidence in the subordinated process<sup>3,4</sup> literature that subordinating asset price changes to random trade arrivals significantly reduces asset return leptokurtosis. It is also consistent with the recent advancements in the Autoregressive Conditional Duration (ACD hereafter) and Autoregressive Conditional Multinomial (ACM hereafter) models literature (e.g., Defour and Engel (2000)) that price and trade co-move with random trade inter-arrival time.

Specifically, we first let asset price evolution in each calendar-time interval be governed by two independent i.i.d. (independently and identically distributed) Bernoulli trials. The first trial resolves the trade arrival/non-arrival uncertainty. When no trade arrives, the price of the underlying asset remains unchanged at the end of the time interval. When a trade does arrive however, the asset price evolves in a standard binomial tree with its up/down movement determined by the second Bernoulli trial. Because of this trade inter-arrival

time uncertainty, the binomial tree now comprises variable moves - it will only grow when a trade arrives, with faster growth when more trades arrive, and vice versa. Convoluting the two trials over time leads to a unique and simple trinomial tree, where in each calendar-time period asset price jumps either up or down when there is an arrival and stays constant when there is no arrival.

This trinomial tree structure embeds stochastic volatility in the context of a subordinated process model with the total volatility in a fixed time period, for example a day, determined by the product of the constant per trade volatility and the “random” number of trade arrival in the period. On less eventful days, trading is slow and prices evolve slowly, while on more eventful days, prices evolve faster to reflect the increasing speed of information arrival. The phrase “stochastic volatility” here is thus different from the usual sense of stochastic volatility in the current option pricing literature. In a standard stochastic volatility model, trade arrival is deterministic but up and down rates at each node of the tree is stochastic. In here however, stochastic volatility is induced by random trade arrival with the per-arrival up and down rates assumed to be constant. These two approaches are not necessarily at odds because it is conceivable that both trade arrival and up/down rates can be stochastic. The advantage of the tree structure here is parsimony because it embeds stochastic volatility within a state space of only three states and requires only one additional parameter than CRR in option pricing and hedging. Conversely, in attaining parsimony power is sacrificed in that its simple i.i.d. Bernoulli trade arrival specification assumes away certain stylized trade arrival dynamics like correlation between trade arrivals and price changes and seasonality and clustering in the trade arrival intensity process.

Compared to CRR, the above trinomial tree provides a more precise description of the actual speed of price evolution in the financial market, explains better that information arrivals generate trading volume, and reconciles why heavier unexpected trading volume is often empirically found to be associated with higher volatility. It may also answer one of Rubinstein’s (1994) suggested future research agenda of developing a unique trinomial tree that incorporates stochastic volatility.

It is also interesting to note that making a time change from calendar-time to trading-time can restore the original binomial tree. Mandelbrot and Taylor (1967), Granger and Morgenstern (1970), and Clark (1973) first introduced the concept of trading-time to refine subordinated process models by recognizing that asset price changes over trade but not just over the passage of calendar-time. Trading-time refers to a time scale where price changes are measured from trade to trade rather than from time to time as in the calendar-time norm. Trading-time subordinated process models utilize the trade arrival process as the randomizing subordinator and predict that we can significantly reduce asset return leptokurtosis by measuring time from trade to trade, as has been verified by Ané and Geman (2000) and others. This application of the concept of “operational time dimension” has been widely applied in the field of systems science because when the choice of the time scale is dictated by the nature of things, a simple change of the time scale can often reduce a complex process to a stationary one. In our application, when trade arrival is utilized as the subordinator, the trinomial calendar-time process can be reduced to a stationary binomial one in trading-time. This time change establishes an alternative

linkage between trinomial and binomial trees to Rubinstein's (2000) suggestion of skipping all the odd time steps in a binomial tree. In trading-time now however, the isomorphic option has a random maturity, as the number of trade arrival during an option's maturity is random.

Our application of a stochastic time change to valuation is by no means novel. Prior to our study, Chang, Chang, and Lim (1998, CCL hereafter) have applied this time change concept to option pricing. In an information-time setting, CCL derive an option pricing formula with trade arrival driven by a Poisson process. There is a problem however associated with the CCL model's mathematical structure - in obtaining the underlying asset price process in information-time, CCL have defined a continuous Brownian motion over the discrete Poisson trade arrival filtration. They also require the existence of a traded information-time riskless bond to derive risk-neutral option pricing formulas. This assumption requires that in calendar-time there is a bond that has stochastic interest rates perfectly correlated with information arrival. In contrast, we subordinate discrete binomial price changes to a discrete Bernoulli trade arrival filtration and do not require the information-time bond to be traded. As a result, we can show that at the continuous-time limit our model contains CCL as a special case under risk-neutrality.

The above stochastic time change implies that option pricing with stochastic volatility in calendar-time be isomorphic to option pricing with constant volatility but random maturity in trading-time. To verify this hypothesis, we will derive and compare option-pricing results in both time dimensions with isomorphism demonstrated. The above discussion also implies that given the non-traded nature of the random trade arrival uncertainty, complete-market preference-free pricing is infeasible. This leads us to first investigate the properties of a shadow security that could span and hedge the trade arrival uncertainty - a default-free bond issued to mature upon the random arrival of the next trade. Equilibrium arguments are then necessarily invoked in the context of a representative agent economy with power utility<sup>5</sup> to determine its value and required return. We show that its return comprises of a riskless component as well as a random maturity risk premium. In line with Rubinstein's (1994) idea that the interest return is the "clock" running a binomial tree, the return on this one-trade bond here runs the clock of our binomial tree with variable move time.

Given the shadow price, next we value options using the discrete-time martingale pricing methodology of Huang and Litzenberger (1988),<sup>6</sup> with three distinct securities and three states. We first extract two distinct equivalent martingale probability measures for the trade arrival and asset price uncertainties from the underlying prices, then we price options as martingales. Although the two measures exist under no-arbitrage, they are not unique precisely because the one-trade bond is not traded, which leads to utility-dependent option pricing formulas. In calendar-time, the resulting trinomial option formula with stochastic volatility degenerates into the original CRR formula when a trade always arrives in each consecutive time period and has the advantage of being applicable to American and path-dependent options. In trading-time, the resulting binomial formula with random maturity naturally contains the CCL information-time formula as a continuous-time limit when risk-neutrality is imposed.

To demonstrate the robustness of our basic framework, finally we extend and empirically test the trinomial model by relaxing the assumption of i.i.d. Bernoulli trade arrivals. Although Ané and Geman (2000) and others have empirically shown that transaction clock alone generates virtually perfect normality in stock returns, in microstructure research however, Hasbrouck (1991) and others have shown that the change in prices depends on the sign and size of trades as well as the bid-ask spread. In the more recent ACD and ACM work, Dufour and Engel (2000) have shown that the higher the trade arrival intensity the larger is the price revision. Bakshi, Cao, and Chen (2000) further report that when volatility and asset price changes are uncorrelated with each other, the alternative stochastic volatility option pricing models to the Black and Scholes (1973, and BS hereafter) that ignore the leverage effect are unlikely to generate the levels of return skewness and kurtosis necessary to reconcile the BS implied-volatility smiles.<sup>7</sup> To this end, we extend our trinomial tree by assuming that price revision and trade arrival intensity are correlated according to a deterministic specification similar to that of CEV (Cox and Ross, 1976). This deterministic intensity arrival structure is chosen over a stochastic structure for the reason of parsimony for requiring only one more parameter to control for the speed and direction of intensity change. Incorporating stochastic intensity arrival will certainly free out the structure, but at the expense of a more complex state space with at least four states in each period and several more parameters needed in option pricing and hedging. Since with a tree model computing time grows exponentially with the number of parameters required, a more general space structure as such will be computationally unbearable when brought to extensive empirical testing. Therefore we focus on a deterministic intensity arrival structure in this study.

The rest of the paper is organized as follows. Section I contains the main result with the development of the calendar-time trinomial tree and the corresponding option-pricing model. The underlying asset price, trade arrival, and pricing kernel processes are first introduced to facilitate the valuation of the shadow security using the fundamental martingale Euler valuation equation. Equivalent martingale probability measures associated with asset price and trade arrival risks are then derived using the discrete-time martingale pricing methodology of Huang and Litzenberger (1988). Section II develops the trading-time binomial counterpart and demonstrates isomorphism to the calendar-time trinomial model and convergence to CCL. Section III contains the extended calendar-time trinomial model with correlated trade arrival and asset price change and the corresponding empirical tests vis-à-vis the Black and Scholes and the CEV models for the purpose of evaluating incremental improvements in pricing and hedging. Intra-day data on S&P 500 index options are used for our empirical tests. Finally, section VI concludes the paper and discusses future research directions.

## **I. Calendar-Time Trinomial Tree and Option Pricing**

We apply the discrete-time martingale pricing methodology of Huang and Litzenberger (1988) to derive the trinomial option-pricing model when trade arrival is random, referred to as the “TRI” model. Asset price

evolution and the issue of market incompleteness are first discussed, which leads to the investigation of a non-traded hedge security - the one-trade bond. Pricing result and properties of this bond in the context of a representative agent economy with power utility are then investigated. Finally, we derive the equivalent martingale probability measures with respect to both asset price changes and trade arrival uncertainties for pricing options as martingales.

### A. Price Evolution

The original binomial option pricing model is developed based on the assumption that the asset price evolves in a binomial random walk, where a Bernoulli trial is made at the end of each of the  $n$  successive time periods to determine the up/down movement of the asset price. To randomize trade arrivals such that the asset price may not change at each turn of the game, let us consider a more general setup where two independent and successive Bernoulli trials are played at the end of each time period. The first trial is made to determine if a trade will arrive. When a trade does arrive, a second trial is made to determine the up/down movement of the asset price change. When a trade does not arrive, the second trial is not made and the asset price remains unchanged. This construction essentially defines a subordinated binomial process, where trade arrivals serves as the directing process and price changes from trade to trade the parent process. In other words, we subordinate binomial price changes to random trade arrivals such that the asset price will only change when a trade arrives, irrespective of the passage of calendar-time. Convoluting these two trials results in an i.i.d. trinomial representation of the asset price movement as depicted in Figure 1. We also let  $R$  denote one plus the riskless rate over one period, with the usual regularity condition that  $u > R > d$  to prevent riskless arbitrage.

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Insert Figure 1 About Here

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The directing process, a Bernoulli trade arrival structure, implies a geometric inter-arrival time distribution such that the probability that the next trade will arrive in the  $k$ -th period is  $(1-g)^{k-1}g$ . It is well known that this discrete process converges to a homogeneous Poisson-jump process in the limit as the length of the time period shrinks to zero, with the inter-arrival time exponentially distributed. The parent process, or the price change from trade to trade, is a stationary recombining binomial lattice that will be used later for developing the trading-time option-pricing model in section II.A.

Despite the seemingly identical appearance of the above tree to the standard trinomial models (e.g., Boyle 1986) that all possible multiplicative combinations of asset price movement at the end of each successive turn are likely outcomes, they are fundamentally different. The standard trinomial model is a computational method for evaluating integrals and thus focuses on efficient approximations of the continuous-time counterpart. Our trinomial specification is obtained by generalizing the original binomial model to incorporate random trade

arrivals. Additionally, the standard trinomial model is stationary while our trinomial representation embeds stochastic volatility as explained before.

Compared to the standard binomial model, our trinomial model has two distinct features. First, it contains the standard model as a special case. When a trade always arrives in each consecutive calendar-time period, our model degenerates into the standard binomial one. Secondly, it randomizes trade arrival. The market thus becomes incomplete with three states but only two traded assets. One way to hedge this additional uncertainty is to price the option based on the prices of the underlying asset, the riskless bond, and a second option. However, this is a recursive argument since we first need to price the second option. Ideally what we need is a simple hedge security that spans the trade arrival uncertainty. For example, it would solve the problem should there exist a one-trade riskless numeraire bond with price  $\beta$  that matures to its face value of \$1 upon the arrival of the next trade. When there is no trade arrival, the bond price remains unchanged. As such, this bond evolves in an i.i.d. trinomial representation over calendar-time as illustrated in Figure 2. Under this construction, this one-trade bond has random maturity since the next trade may arrive in any of the future calendar-time periods. The role of this bond, which spans the state space of trade arrivals on the Bernoulli filtration, is therefore to capture the time value in trading-time.

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Insert Figure 2 About Here

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### *B. Pricing the One-Trade Riskless Bond*

Should the one-trade bond exist and be traded, the market would be complete in the triplet consisting of the underlying asset, the matching bond that matures with the option, and the one-trade bond that matures upon arrival of the next trade. Unfortunately this is not the case. Thus to price the option, we have to make use of the shadow price of this one-trade bond. We determine the shadow price and properties of the one-trade bond using the fundamental martingale Euler valuation equation (Euler equation thereafter) in the context of a representative agent economy with power utility.

Let  $q_i$  be the one-period pricing kernel (the normalized state price or the Radon-Nikodym derivative) for payoffs at the end of the period defined as  $s(i)/\pi(i)$ , where  $s(i)$  is the state price for a sure payoff of \$1 in state  $i$ , and  $\pi(i)$  is the probability that state  $i$  will occur. Note that at the beginning of the period, the pricing kernel equals 1 and  $E(q_i) = B$ , where  $B$  is the price of a one-period riskless bond. Given the trinomial asset price filtration that defines the state space, the evolution of the pricing kernel over a generic calendar-time period is as shown in Figure 3 where  $q_u$ ,  $q_n$ , and  $q_d$  denote the pricing kernel when the asset price jumps up, does not jump, and jumps down, respectively. Therefore, over two successive periods for example, the pricing

kernel in the state where the asset price does not change in the first period but jumps up in the second period should be  $q_n q_u$ , with the corresponding probability  $(1-g)gh$ .

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Insert Figure 3 About Here

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The Euler equation dictates that under no arbitrage, the pricing kernel must exist such that a pricing-kernel-scaled asset price process should follow a martingale. Since our market is incomplete, the pricing kernel is neither unique nor attainable from observable market prices. Instead, it has to be determined by resorting to equilibrium arguments that we now turn to.

It is well known that in an equilibrium representative investor economy, the pricing kernel is the investor's time-value-discounted marginal rate of substitution. Therefore for an average investor whose utility exhibits constant relative risk aversion (CRRA thereafter),

$$(1) \quad q_i = \frac{U'(A_i c, t)}{U'(c, t)} e^{-\theta} = A_i^{\delta-1} e^{-\theta},$$

where  $U$  is the investor's utility function,  $A_i$  is the consumption jump size in state  $i$ ,  $c$  is aggregate consumption,  $\theta$  is the one-period time discount factor, and  $1-\delta$  is the measure of relative risk aversion (with  $\delta < 1$ ). This equation reveals that the pricing kernel is a decreasing function of both the consumption jump size and the degree of risk aversion.

Given the asset price and pricing kernel processes, we are ready to price the one-trade bond using the Euler equation.

**<Theorem 1>** Given the trinomial asset price and pricing kernel processes, the price and required return of the one-trade bond is respectively

$$(2) \quad \beta = \frac{Qg}{1-q_n(1-g)}, \text{ and}$$

$$(3) \quad r_1 = \frac{r}{g} + \frac{1}{g} \left[ \frac{r}{(1+r)Q} - r \right],$$

where  $r$  is the calendar-time one-period riskless rate and  $Q = [hq_u + (1-h)q_d]$  is the expected one-period pricing kernel conditional upon trade arrival.

**<Proof>**

The Euler equation dictates that under no-arbitrage the pricing-kernel-scaled bond price process should follow a martingale with the bond's shadow price being equal to its expected pricing-kernel-scaled payoff:



$$(4) \quad \beta = E_T \left[ q_n^{t-1} Q \mid T = t \right],$$

where  $E_T$  denotes the expectation over the bond's maturity domain  $T$ ,  $T \in [1, \infty)$ , and  $q_n^{t-1} Q$  is the expected  $t$ -period pricing kernel-scaled-payoff conditional upon no trade arrival until period  $t$ . In other words,  $q_n^{t-1}$  is the  $(t-1)$ -period pricing kernel corresponding to no trade arrival in the first  $(t-1)$  periods,  $Q = hq_u + (1-h)q_d$  is the expected one-period pricing kernel conditional upon trade arrival in period  $t$ , and the payoff is \$1 when a trade arrives and zero otherwise. This bond has random maturity since the timing of the next trade arrival,  $T$ , is uncertain. Given our Bernoulli trade arrival structure, the first-order inter-arrival time distribution is geometric with probability mass function  $(1-g)^{t-1}g$ . Consequently, the bond price per Eq. (4) can be rewritten as the sum of an infinite geometric series:

$$(5) \quad \beta = \sum_{t=1}^{\infty} Qg([q_n(1-g)]^{t-1}).$$

Solving the sum yields Eq. (2).

In contrast, the one-period default-free bond price is

$$(6) \quad B = Qg + q_n(1-g).$$

Since by definition  $r_1 = \frac{1-\beta}{\beta}$  and  $r = \frac{1-B}{B}$ ,

$$(7) \quad \frac{r}{r_1} = \frac{\beta(1-B)}{B(1-\beta)}.$$

Substituting Eqs. (5) and (6) into Eq. (7) and simplifying yields

$$(8) \quad r_1 = \frac{r}{g\phi},$$

where

$$(9) \quad \phi = (1+r)Q,$$

is a random maturity adjustment factor for the equivalent martingale trade arrival probability.

Finally it is quickly checked that Eq. (8) can be rearranged to obtain Eq. (3).

**Q.E.D.**

Eq. (3) reveals that  $r_1$ , the one-trade riskless rate, has two components. The first component is the riskless component  $r/g$  as the riskless rate per expected trade arrival. The second is a random maturity risk

premium as the product of  $1/g$  and  $\left(\frac{r}{(1+r)Q} - r\right)$ , with the former being the standard deviation of the exponentially distributed inter-arrival time, a measure of maturity risk, and the latter being the trade arrival risk premium.

Finally the equilibrium bond price and one-trade rate can be determined by substituting Eq. (1) into Eqs. (2) and (3). The random maturity adjustment factor can also be rewritten in terms of the size of consumption jump and the degree of risk aversion by substituting Eq. (1) into Eq. (9) as shown below:

$$(10) \quad \phi = (1+r)E(A^{\delta-1}e^{-\theta}) = (1+r)e^{-\theta} [hA_u^{\delta-1} + (1-h)A_d^{\delta-1}].$$

For a risk-neutral investor, we have  $\delta = 1$  and  $(1+r)=e^{\theta}$  that leads to  $\phi = 1$ , i.e. no adjustment is needed. For a risk-averse investor however, the higher is the degree of risk-aversion the larger is the adjustment.

### C. The Calendar-Time Trinomial Option Pricing Model

This section prices a call option with maturity  $T$  in calendar-time ( $t$ ) using an  $n$ -step set-up. In other words, the option maturity  $T$  is equally divided into  $n$  steps (or time periods) where in each step a first Bernoulli trial determines trade arrival, with a positive outcome leading to an independent second trial on the up/down price movement. To span the additional trade arrival uncertainty, the shadow price (the equilibrium value) of the one-trade bond is used to facilitate pricing in this incomplete market. To apply the discrete-time martingale pricing methodology of Huang and Litzenberger (1988), we normalize the one-trade bond and the asset price using the matching bond - the calendar-time numeraire bond that has an identical maturity to that of the option. The derived option-pricing model is referred to as the ‘‘TRI’’ model.

Given no-arbitrage within our trinomial setting, the normalized prices of the one-trade bond and the underlying asset must be dictated by the following martingale representations:

$$(11) \quad \frac{\beta}{B} = pm + (1-m)\beta + (1-p)m,$$

$$(12) \quad \frac{S}{B} = pmuS + (1-m)S + (1-p)m dS,$$

where  $p$  and  $1-p$  are the respective one-step equivalent martingale probability measures for the asset price to move up and down; and  $m$  and  $1-m$  are the respective one-step equivalent martingale probability measures for trade arrival and non-arrival. Eqs. (11) and (12) essentially state that, with respect to the martingale probability measures, the normalized price must be equal to the expected value at the end of the period.

Solving Eq. (11) for the one-period trade arrival martingale probability measure yields

$$(13) \quad m = \frac{\frac{1}{\beta} - 1}{\frac{1}{\beta} - 1} = \frac{r}{r_1},$$

which shows that the interest rate relative between calendar-time and trading-time identifies the equivalent trade arrival martingale probability measure. When a trade always arrives at the end of a period without arrival uncertainty, the two interest rates command the same value and thus the probability measure becomes one.

Using Eq. (8), Eq. (13) can be further simplified to

$$(14) \quad m = g\phi,$$

where  $\phi$ , the random maturity adjustment factor for the equivalent martingale trade arrival probability measure as shown in Eq. (10), is determined by the investor's degree of risk aversion among others. For a risk-averse investor, the more risk averse the investor the larger the value of  $\phi$  ( $>1$ ) and thus the higher the ratio  $m/g$ . This is intuitive since, *ceteris paribus*, the more risk-averse the investor the higher should be the arrival probability in order to compensate the investor with a higher expected return. For a risk-neutral investor with  $\delta = 1$  however,  $\phi = 1$  and thus

$$(15) \quad m = g.$$

In other words, the equivalent martingale measure is identical to the true probability measure because risk-adjustment is no longer necessary.

To implement this trinomial representation as a standard  $n$ -step model, we divide the option maturity ( $T$ ) into  $n$  equal calendar-time periods such that the length of each period is  $\Delta t = T/n$ . Let  $j$  be the trade arrival intensity parameter, i.e. the average annual number of trade arrivals. The true probability of trade arrival in a calendar-time period  $\Delta t$  is thus:

$$g = j\Delta t,$$

with the regularity condition  $g \leq 1$ . The corresponding equivalent martingale probability is thus

$$m = j\phi\Delta t,$$

where the term  $j\phi$  serves as a risk-adjusted or pseudo trade arrival intensity parameter. We see that the trade arrival martingale probability is utility dependent and inversely related to the number of steps. The larger the number of steps in the trinomial implementation, the shorter the length of a time period  $\Delta t$ , and consequently the lower the probability of a trade arrival in a time period.

Next, substituting Eq. (13) into Eq. (12) and simplifying, we obtain the following one-period asset price equivalent martingale probability measure:

$$(16) \quad p = \frac{R_1 - d}{u - d},$$

where  $R_1 \equiv 1 + r_1$  is the gross return on the one-trade bond and  $u = \exp(\sigma_1)$  and  $d = 1/u$  are the respective gross up/down movements of the asset price. Although this probability measure resembles the original CRR measure, there are two fundamental differences because the two are defined over different time dimensions. First,  $p$  is utility-dependent here since  $R_1$  is a risky rate. Secondly, the  $u$  and  $d$  here are independent of  $\Delta t$  but they depend on  $\Delta t$  in CRR. Again this is because in TRI the asset price will only jump when a trade arrives, irrespective of the passage of calendar-time. Consequently in Eq. (16),  $u = \exp(\sigma_1)$  and  $d = 1/u$ , where the per-trade volatility  $\sigma_1$  replaces CRR's calendar-time volatility.

To summarize, the no-arbitrage martingale trinomial tree is obtained by superimposing a jump process on a standard binomial process such that the movement of the underlying asset at calendar-time  $t$  over a calendar-time period is modeled by the i.i.d. trinomial setup illustrated in Figure 4, where in each period a trade arrives with probability  $m$  and does not arrive with probability  $1-m$ . When a trade does arrive, asset price either increases from  $S_t$  to  $uS_t$  with probability  $p$  or decreases from  $S_t$  to  $dS_t$  with probability  $1-p$ . The model links option values to not only the per-trade asset price change but also the intensity of trade arrival. As the intensity increases, the tree grows faster with the option price moves higher to reflect the larger expected total price volatility, and vice versa.

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Insert Figure 4 About Here

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Option pricing proceeds in the usual backward recursive fashion: the normalized option price being equal to the expected payoff from the option under the martingale probability measures. By applying backward induction, we obtain the following  $n$ -period calendar-time trinomial option pricing formula:

$$(17) \quad C(n) = B_T \sum_{k=0}^n M_k \sum_{i=0}^k P(i) (u^i d^{k-i} S - X)^+ \\ = B_T \sum_{k=0}^n \sum_{i=0}^k \frac{n!}{(n-k)!(k-i)!i!} m^k (1-m)^{n-k} p^i (1-p)^{k-i} (u^i d^{k-i} S - X)^+,$$

where  $M_k$ , the  $n$ -period trade arrival equivalent martingale probability measure of  $k$  trade arrivals in  $n$  periods, is

$$(18) \quad M_k = \frac{n!}{k!(n-k)!} m^k (1-m)^{n-k},$$

and  $P(i)$ , the asset price equivalent martingale probability measure that the ending stock price is  $u^i d^{k-i} S$  when  $k$  trades arrive, is

$$(19) \quad P(i) = \frac{k!}{i!(k-i)!} p^i (1-p)^{k-i}.$$

Notably the larger is the trade arrival probability  $m$  the larger should be the option price to account for a larger expected number of expected price changes. When this probability approaches one, it is straightforward to show that Eq. (17) degenerates into the standard  $n$ -period CRR formula.

## II. Trading-Time Binomial Option Pricing and Convergence to CCL

We next derive the isomorphic option-pricing formula in the trading-time ( $k$ ) counterpart, prove it contains the information-time option-pricing model of CCL as a risk-neutral continuous-time limit, and demonstrate convergency using simulation. This version is referred to as the ‘‘BIN’’ model.

### A. The Trading-Time Binomial Option Pricing Model

Recall that a simple change of time scale from calendar-time to trading-time reduces the trinomial process to a stationary binomial one. Thus pricing options in trading-time has the benefit of working with the simpler standard binomial tree. However this isomorphic option has random maturity because the number of trade arrivals prior to its expiration is uncertain.

Since in an  $n$ -step TRI model, the number of trade arrivals over the life of an option may vary from a minimum of zero to a maximum of  $n$ , the option with random maturity can be valued using the Euler equation as a conditional expectation over the trade arrival uncertainty. More specifically, the normalized price of an  $n$ -period call option with random maturity in trading-time can be solved as the sum of the arrival-probability-weighted normalized prices of  $n+1$   $k$ -trade fixed-maturity options, denoted as  $C_k$ , i.e. options with a fixed number of  $k$  trade arrivals, where  $k=[0,n]$ :

$$(20) \quad \frac{C(n)}{B_T} = \sum_{k=0}^n M_k \frac{C_k}{B_T},$$

or equivalently,

$$(21) \quad C(n) = \sum_{k=0}^n M_k C_k,$$

where  $M_k$  is the trade arrival martingale probability measure of  $k$  trade arrivals in  $n$  periods as shown in Eq. (18) and  $B_T$  is the price of the matching bond.

Since the maturity of  $C_k$  is fixed in trading-time with  $k$  trades, the underlying asset price shall evolve over a trading-time binomial tree with fixed length  $k$ . Therefore we can price  $C_k$  using a general  $N$ -step standard binomial model such that

$$(22) \quad C_k = B_T \sum_{i=0}^N P_k(i) (u_k^i d_k^{N-i} S - X)^+,$$

where,

$$(23) \quad P_k(i) = \frac{N!}{i!(N-i)!} p_k^i (1-p_k)^{N-i},$$

is the  $N$ -period martingale probability that the ending stock price is  $u_k^i d_k^{N-i} S$  and

$$u_k = e^{\sigma_1 \sqrt{\Delta k}}, d_k = \frac{1}{u_k}, R_k = e^{r_1 \sqrt{\Delta k}},$$

$$p_k = \frac{R_k - d_k}{u_k - d_k},$$

where  $R_k$  is the gross riskless rate over the trading-time interval  $\Delta k$ , defined as  $k/N$ , and  $p_k$  is the corresponding one-step asset price change equivalent martingale probability. Again we require  $u_k > R_k > d_k$  to prevent arbitrage.

Similar to the TRI model, here the per-trade volatility  $\sigma_1$  replaces the calendar-time volatility. Since  $C_k$  is priced using the standard binomial model defined over trading-time, the up/down price movements now depend on the length of the trading-time interval  $\Delta k$ , measured in number of trades. Clearly now the length of a trading-time interval dictates the size of price movement over the interval. We are free to specify the length of  $\Delta k$  by choosing an appropriate value for  $N$  - the number of step in the model. For example in the next section, we shrink the trading-time interval by increasing the number of binomial step to achieve diffusion price changes.

Comparing Eqs. (21) - (23) to Eqs. (17) - (19), it is straightforward to see the isomorphism between TRI and BIN. The TRI model becomes a special case of the BIN model when we convolute the asset price change and the trade arrival probability measures into a trinomial probability distribution as shown in Eq. (17) and let  $N$  equal  $k$  such that there is one trade per step, i.e.  $\Delta k = 1$ . This is because in our setup of the TRI model only one trade may arrive in each step (or time period). Conversely, as we condition the trinomial probability distribution in Eq. (17) on the number of trade during the option's maturity and let  $\Delta k = 1$ , Eqs. (17) - (19) lead to Eqs. (21) - (23).

### B. Convergence and Comparison to CCL

Chang, Chang and Lim (1998) apply the concept of a stochastic time change in a continuous-time setting to derive an information-time European call option pricing formula as a risk-neutral Poisson sum of Merton's (1973) prices over the option's maturity domain. They further apply the Barone-Adesi and Whaley (1987) analytic approximation to the Merton prices to extend the formula to American options. Although both CCL and this research assume a subordinated asset price process with an embedded time change, there are major

differences in research methodology and assumptions. CCL is derived in information-time using the continuous-time pricing methodology by basing a diffusion process over a discrete time space. It is also preference-free by assuming and using the existence of a traded information-time riskless bond to hedge the information arrival uncertainty. In contrast, our main result is derived in calendar-time using the discrete-time pricing methodology and is utility-dependent, as we recognize the unavailability of the trading-time riskless asset. As such, our model allows for risk aversion and is applicable to American and path-dependent options exactly. Nonetheless, we will show next that by applying suitable parameterization and invoking risk-neutrality, BIN degenerates into CCL.

To mimic CCL's approach of defining a diffusion over a discrete-time space, we first need to shrink the trading-time intervals so as to converge the binomial price changes onto diffusions. This is tantamount to dividing each trade into infinitesimally small ones by letting  $N \rightarrow \infty$ . One way consistent to the  $n$ -step TRI model is to divide each trade into  $n$  smaller trades. This is accomplished by letting  $N$  equal to  $nk$  such that  $\Delta k = k/(nk) = 1/n$ . All trading-time binomial trees now have the same interval of  $1/n$  trade. Then as  $n$  approaches infinity and with suitable parameterization as in CRR or Jarrow and Rudd (1983), the geometric trade arrival process will converge onto a Poisson process with all binomial price changes simultaneously converging onto diffusions. This approach, however, requires an expensive  $n^2$ -step binomial tree for a fixed-maturity  $n$ -trade option, and thus can only be accomplished at a high computation cost.

To alleviate the computational burden, a feasible alternative is to price all fixed-maturity  $k$ -trade options, where  $k \in [0, n]$ , using an  $n$ -step binomial model. In other words, we let  $N=n$  such that

$$\Delta k = \frac{k}{n}.$$

This is equivalent to splitting each trade into  $n/k$  smaller trades such that consistency is maintained in all binomial trees having the same  $n$  number of steps albeit with different trade sizes (or trading-time intervals). The longer is the original tree, i.e. when there are more trade arrivals, the larger is the trade size after the split. For the  $n$ -trade option, the trade size is one trade, but for the one-trade option, the size is  $1/n$  trade. Although the time step, i.e. the trade size, is different for each fixed-maturity option, the computational complexity of this alternative convergence approach is much lower than that of the approach mentioned above. The numerical analysis in the next section shows that the resulting BIN price converges rapidly to the CCL price. It also ensures that with suitable parameterization, as the geometric trade arrival process in calendar-time converges onto a Poisson process with  $n$  approaching infinity, all of the binomial price changes converge onto diffusions.

<**Theorem 2**> Given  $\Delta k = k/n$ , the BIN formula as specified in Eqs. (21) – (23) shrinks to the CCL formula as  $n$  approaches infinity and when investors are risk-neutral.

<**Proof**> See the Appendix.

### C. Simulation

In this section, we numerically examine the convergence properties of TRI and BIN in relation to CCL with the use of a set of parameters consistent with their empirical estimates. Their study is based on transactions data on futures options on the S&P 500 index, the Deutschemark, and the Japanese Yen during the period from June 1, 1994 to December 19, 1994. We select the following parameter values, consistent with the average estimates from CCL:

$$j_0 = 100, \sigma_1 = 0.01, S = 100, X = 95, 100, 105, T = 3 \text{ months}, r = 0.08,$$

where  $j_0$  is the expected number of trade arrivals in a calendar year and  $\sigma_1$  is the per-arrival price change volatility. To examine numerical convergence, we compute option prices using number of time periods ( $n$ ) ranging from 50 to 1,000. In order to compare with CCL, we assume that the jump risk is diversifiable ( $\phi = 1$ ). Under this assumption, risk-neutral valuation is valid in both models.

Table I illustrates convergence properties of TRI and BIN when the underlying asset does not pay dividend and the jump risk is diversifiable. Prices of European call and put options are reported in the Table. Because jump risk is assumed to be diversifiable, the BIN prices should converge to the CCL price, which is reported in Table I for comparison. As expected, the BIN prices converge rapidly to the CCL prices. The pricing difference is less than a penny with only  $n = 50$  time periods. In addition, the TRI prices appear to converge fairly quickly as well. However, there is no closed-form solution to compare with in this case. More importantly, prices from the TRI and BIN models are quite close, with pricing difference being less than a penny when 1,000 time periods are used. This is very reassuring since, although the two versions appear to be isomorphic in terms of a stochastic time change, the implementation of the BIN model requires shrinking not only time period but also trade size. As  $n$  approaches infinity, the TRI price approaches a Poisson sum of binomial prices while the BIN price approaches a Poisson sum of BS prices, i.e. the CCL price. Since the TRI and BIN prices are actually fairly close, the simplicity of the TRI model makes it more appealing to implement than the BIN model.

Computation time further reveals the advantage of the TRI model over the BIN model. We measure the CPU time (in seconds) of TRI and BIN when they are implemented in Fortran 77 using a 500 MHz Pentium III processor (not reported here). The CPU time of the TRI model is negligible, with less than a second used even when 1,000 time steps are implemented. The BIN version, on the other hand, is much more expensive in computation time. When 200 and 1,000 time steps are used to compute the binomial price, nearly 3 seconds and six minutes are needed, respectively. Therefore for the extension in the next section, we will focus on TRI.

To further explore the convergence properties of our model, we replicate the results in Table I when the underlying asset pays a continuous dividend and also with other values of trade arrival intensity parameter ( $j$ ) and per trade volatility rate ( $\sigma_1$ ). The results show similar robustness and therefore are omitted here for brevity.



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Insert Table I About Here

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### **III. Extended Trinomial Model with Correlated Trade Arrivals and Price Changes: Modeling and Empirical Tests**

Bakshi, Cao and Chen (1997, 2000) have reported that when volatility and asset price changes are uncorrelated with each other, the alternative stochastic volatility models to the BS that ignore the leverage effect are unlikely to generate the levels of return skewness and kurtosis necessary to reconcile the BS implied-volatility smiles. This is especially true for index data where the smile is deep. The implication is that Hull and White (1987) and Stein and Stein (1991) that assume uncorrelated volatility and asset price changes and Merton (1976) and CCL that assume uncorrelated jump return and trade arrival intensity should not be expected to perform much better than the BS. TRI also falls into this category, because, like Merton (1976) and CCL, it also assumes i.i.d. price changes everywhere on the price path. This motivates us to generalize TRI by introducing correlation between asset return and trade arrival intensity. We shall call this extended model “ETRI”.

Black (1976) has suggested that when the stock price goes up, the level of volatility trends to go down, and vice-versa. This stylized inverse time-series relation between stock return and volatility changes has become known as the “Fischer Black effect” or as the leverage effect, and has been supported extensively in the empirical literature. For the S&P 500 index, Dumas, Fleming, and Whaley (1998) find this correlation to be  $-0.570$  during the period June 1, 1988 through December 21, 1993. Schwert (1989) finds this correlation to be asymmetric - the increase in volatility for decreases in the index tends to be larger than its counterpart. In this vein and for parsimony, we assume that the trade arrival intensity in ETRI that induces stochastic volatility in the model is a deterministic function of stock return. When the asset price goes up, the trade arrival intensity goes down to indicate a lower expected volatility and vice-versa.

The deterministic structure is chosen over a stochastic intensity structure for two reasons. First, it requires neither additional securities to hedge the intensity risk nor the intensity risk to be priced. Second, it is parsimonious with only one more parameter needed than the original pricing formula to control for the speed and direction of intensity change. Incorporating stochastic intensity will certainly free out the correlation, but will also lead to a more complex state space structure with at least four states in each period and possibly several additional parameters to estimate. Since ETRI is a tree model with computing time growing exponentially with the number of parameters required, a more general structure as such will be computationally unbearable when brought to extensive empirical testing. Therefore we focus on a deterministic structure in this study. In the following we describe and empirically test the ETRI model.

### A. Model

Consider the generalized subordinated binomial tree structure shown in Figure 5, where  $k$  is the net up moves from the initial asset price at node  $(0,0)$  to the current asset price at node  $(i, k)$  for  $0 \leq i \leq n, -i \leq k \leq i$  and  $S(k) = S_0 u^k$ .

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Insert Figure 5 About Here

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Stock price movement is still modeled by trade arrivals. If a trade does not arrive, the stock price remains unchanged. If a trade does arrive, the stock price may jump up by a factor of  $u$  or jump down by a factor of  $d$ . However, trade arrival intensity is no longer constant in the generalized subordinated binomial model. Specifically, jump intensity at node  $(i, k)$  is modeled as:

$$J(k) = \begin{cases} \lambda \exp\left[\theta \sqrt{k \cdot \Delta t}\right] & \text{if } k \geq 0, \\ \lambda \exp\left[-\theta \sqrt{-k \cdot \Delta t}\right] & \text{if } k < 0, \end{cases}$$

where  $\lambda = j_0 \phi$  is the initial jump intensity at initial stock price level (when  $k = 0$ ). This square-root jump intensity structure, motivated by that volatility moves in square-root of time, is asymmetric and may induce positive or negative correlation between jump intensity and stock price when  $\theta > 0$  or  $\theta < 0$ . Constant jump intensity modeled previously is a special case when  $\theta = 0$ . Numerical validation not reported here for brevity shows this square-root deterministic structure is robust in producing wider ranges of asset return skewness and kurtosis than a linear structure. With the new jump intensity structure, the martingale probabilities can be quickly shown to be

$$\begin{cases} p_1(k) = p(k) \cdot m(k), \\ p_2(k) = 1 - m(k), \\ p_3(k) = [1 - p(k)] \cdot m(k), \end{cases}$$

where

$$\begin{aligned} m(k) &= J(k) \Delta t, \\ p(k) &= \frac{R_1(k) - d}{u - d}, \\ R_1(k) &= 1 + r_1(k), \\ r_1(k) &= \frac{r \Delta t}{J(k)}. \end{aligned}$$

To implement this generalized version of the subordinated binomial model, three parameters need to be estimated: per trade volatility ( $\sigma_1$ ) and trade arrival parameters ( $\lambda$  and  $\theta$ ). Subsequent empirical tests are all based on this version of the subordinated binomial model.

### *B. Data Description*

Intra-day data on S&P 500 index options are used for our empirical study. Consistent with previous research (e.g. Bakshi, Cao and Chen (2000)), we choose to study put options in the period from September 1, 1993 to August 31, 1994. Given put-call parity and the fact that S&P 500 options are European, examining put options alone is sufficient. We also replicate the analysis using the corresponding call options and the results are not materially different. Hence, we concentrate on put options.

The intra-day option price series are obtained from the Chicago Board Options Exchange (CBOE). In addition, daily dividend series are obtained from Standard and Poor's Corporation through the DRI database while daily T-bill yields are obtained from the Wall Street Journal. Consistent with previous research, we concentrate on the mid bid-ask quote instead of actual transaction prices in order to avoid the bid-ask bounce problem. To alleviate computational burden, only the last reported bid-ask quote each day prior to 3:00 pm Central Standard Time is used in our empirical tests. The advantage of the intra-day data is that the option price and index value are recorded at the same time and thus avoid the problem of nonsynchronous trading. Since the S&P 500 index options are European, the present value of all dividends prior to the maturity of the option is removed from the index value before an option-pricing model is employed. Standard data filters are applied to clean up the option price data. First, observations with less than a week to maturity are eliminated because near maturity options may induce liquidity-related biases. Second, price quotes lower than \$3/8 are excluded from the sample. These prices may not reflect true option value due to proximity to tick size. Third, option quotes that are time-stamped later than 3:00 pm Central Standard Time are excluded. The stock market closes at 3:00 pm but the options market continues to trade for another 15 minutes. Nonsynchronous trading problem is avoided if index and option data are matched prior to 3:00 pm. In addition, deep-in-the-money options are eliminated from the sample because these options are expensive and rarely traded. In this study, we define deep-in-the-money options as options that are priced at 10% of the index value or higher. Finally, option quotes violating the boundary condition are eliminated from the sample. Altogether, the data filters have eliminated 41% of observations in the sample. The final sample consists of 22,055 put prices over 253 trading days.

### *C. Empirical Methodology and Results*

To empirically test the performance of ETRI, we apply it to S&P 500 index options and compare its empirical performance against competing models. To ensure the robustness of the results, we examine and contrast competing models with respect to in-sample fit, out-of-sample pricing performance and dynamic hedging performance. While in-sample and out-of-sample pricing errors reflect a model's *static* performance,

hedging errors reflect the model's *dynamic* performance in how well the model captures the dynamic properties of option and the underlying security prices. The competing models are the BS model and the CEV model, which are specified, respectively, by the following stochastic processes (stated under the risk-neutral probability measure):

$$dS(t) = rS(t)dt + \sigma S(t)dz ,$$

$$dS(t) = rS(t)dt + \sigma_0 [S(t)]^{\alpha_0} dz .$$

### C.1. Structural Parameter Estimation and In-Sample Test

All models tested in this study have one or more unobservables. These so-called structural parameters must be estimated before the model can be applied to option data. As is standard in the literature, we adopt the implied parameter estimation approach. Specifically, our estimation procedure solves for the structural parameter(s) that minimize aggregate pricing errors for each model over all options on a given day. The procedure is described below for our generalized subordinated binomial model. It is similar for other models examined in this study.

The ETRI model has three structural parameters: per trade volatility ( $\sigma_1$ ), initial trade arrival intensity ( $\lambda$ ) and a second trade arrival parameter controlling the correlation between trade arrival and stock price ( $\theta$ ). Following Dumas, Fleming and Whaley (1998), Bates (1996), Bakshi, Cao, and Chen (1997) among others, we estimate these parameters for each trading day by minimizing the sum of squared errors (SSE hereafter) between model price and market price using all options on that day. In other words, we solve the following minimization problem:

$$\min_{\sigma_1, \lambda, \theta} \sum_{i=1}^n [V_i(\sigma_1, \lambda, \theta) - V_i]^2$$

where  $V_i(\sigma_1, \lambda, \theta)$  and  $V_i$  are model price and market price of option  $i$ . This minimum SSE is calculated over all options available on a given trading day. In addition, the minimum SSE also serves as a measure of in-sample fit for the option model tested. We use the average minimum SSE over the entire sample period to compare and contrast the in-sample performance of competing models.

Table II summarizes the results from the structural parameter estimation for the three competing models. Parameter estimates shown in the table are sample averages with standard errors reported in parentheses. The in-sample fit is reported in the last column of the table. As measured by the average SSE, the ETRI model provides better in-sample fit than either of the two competing models. As shown in the table, the average SSE is 180.76 for the ETRI model, which is 35.8% and 15.8% lower than the corresponding SSE for the BS and CEV model, respectively. This means that in terms of incremental improvements over CRR, the leverage effect accounts for 20%, while random trade arrivals specification accounts for 15.8%. Furthermore, the ETRI model not only

provides better in-sample fit (lower SSE) than either of the two competing models on average over the sample period but it also does so on every day from September 1, 1993 to August 31, 1994. This result is expected as the ETRI model has three structural parameters while the other two models have only one or two such parameters. With one or two extra free parameters to fit the model, it is not surprising that the ETRI model provides better in-sample fit than the other two models. It is also interesting to note that the estimated leverage parameter ( $\theta$ ) in the ETRI model is negative. This implies that a price increase (decrease) in the underlying asset tends to reduce (increase) the volatility of the underlying asset, consistent with the leverage effect first described by Black (1976). It is, of course, another story whether the better in-sample fit can translate into superior out-of-sample performance.

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Insert Table II About Here

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### *C.2. State-Price Density and Internal Consistency*

Once the structural parameters are estimated, the state-price density (SPD) implied by option prices can be calculated for a given model. Take the subordinated binomial model for instance. The three structural parameters and initial stock price fully specify the probability distribution of future stock prices. For a given maturity, we can build a subordinated binomial tree using the estimated structural parameters. As describe previously, these parameters are estimated by minimizing the SSE using all option prices available on that day. By tracing the tree from the initial node to maturity, we calculate the probability that the stock price ends up at a given terminal node (nodal probability). The SPD is straightforward to determine once nodal probabilities at maturity are known. Alternatively, the SPD can be determined from the second partial derivative of option price with respect to strike price (Breedon and Litzenberger (1978)). However, we find that the nodal probability approach is both computationally less demanding and also more accurate.

Because the SPD captures all the essential pricing structure of an option pricing model, we compare and contrast SPDs implied by structural parameters estimated from each model using option prices. Any differences in in-sample pricing performance among competing models should manifest itself in differences between SPDs. To illustrate typical SPDs from our competing models, we select a representative trading day during our sample period. On September 22, 1993, the BS implied volatility is 12.7% representing the median volatility in our sample period. We use this particular day to show the differences in SPDs calculated using different option pricing models. Structural parameters are estimated as before by minimizing the sum of squared errors across all options on that date. The estimated structural parameters are  $\sigma = 0.1265$  (BS),  $\sigma_0 = 0.1279$  and  $\alpha_0 = 0$  (CEV), and  $\sigma_1 = 0.0230$ ,  $\lambda = 30.3633$  and  $\theta = -2.5938$  (ETRI). With these parameters, SPDs for a given maturity can be calculated for each model.

Consider first short-term SPDs implied by the three models. The SPDs are calculated with a 30-day maturity using the structural parameters estimated on September 22, 1993 and are illustrated in Figure 6. The BS and CEV models have nearly identical SPDs while the ETRI model provides a slightly more peaked SPD. To verify the visual differences observed in Figure 6, we calculate the skewness and kurtosis of each SPD with the understanding that the normal density (the BS model) has a skewness of zero and a kurtosis of 3. We find that the SPD from the CEV model has a skewness of  $-0.1106$  and a kurtosis of  $3.0196$ , with both measures only slightly different from the corresponding measures of the normal density. In comparison, the SPD from the ETRI model has a skewness of  $-0.0691$  and a kurtosis of  $3.4001$ . Again, the ETRI's skewness is not much different from the BS's, but its kurtosis is much higher than the BS's. These calculations confirm the differences illustrated in Figure 6 and show that the SPDs from the CEV and ETRI models are slightly negatively skewed and more leptokurtotic than that from the BS.

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Insert Figure 6 About Here

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Consider next long-term SPDs implied by the same structural parameter estimates. Using a 270-day maturity, Figure 7 plots the SPDs from the BS, CEV and ETRI models. The long-term SPDs from the BS and the CEV model are visibly more different than the short-term SPDs shown in Figure 6, but the differences are still not particularly large. In contrast, the SPD from the ETRI is clearly different from the SPD of either the BS model or the CEV model. It is significantly more skewed and more peaked than either competing SPD. To verify these differences, we again calculate the skewness and kurtosis for each SPD. They are  $-0.3415$  and  $3.2564$  for the CEV model and  $-0.5124$  and  $3.4811$  for the ETRI model, respectively. These statistics confirm the visual differences observed in Figure 7 and further show that the SPDs for the CEV and ETRI exhibit a much higher level of kurtosis with more weight concentrated in the middle as well as a much greater degree of negative skewness.

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Insert Figure 7 About Here

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The results presented in Figures 6 and 7 are representative of SPDs on most days in our sample period from September 1, 1993 to August 31, 1994. The SPDs for the CEV model and the ETRI model are negatively skewed and are more peaked than the corresponding SPD for the BS model. The difference is particularly pronounced for the ETRI model. If the ETRI model were the true model in our sample period, the BS model would have exhibited a larger implied volatility for options with a lower strike price than for options with a higher strike price. This is because the ETRI density is more peaked in the middle pushing more mass to the tails, more so to the left tail due to the negative skewness. This is consistent with the “volatility skew” observed

during our sample period. In addition, the ETRI model is capable of providing different levels of skewness and kurtosis across maturities. These features of the ETRI model captured by the SPD are internally consistent with empirical regularities observed in option prices.

### *C.3. Out-of-Sample Pricing Performance*

Good in-sample fit does not necessarily translate into superior out-of-sample pricing performance. The presence of more structural parameters may actually lead to overfitting and penalizes a model's out-of-sample performance if the extra parameters do not improve the structural fit. To ensure that the better in-sample fit of the ETRI model is meaningful, we next investigate the ETRI model's out-of-sample performance in comparison with competing models.

We begin with a one-day out-of-sample pricing test where an option pricing model is tested on a given trading day using the structural parameters estimated from the previous trading day. As described before, the structural parameters are estimated by minimizing the SSE between model price and market price for each model considered. On the next trading day, the estimated parameters are then used to price all available options. The resulting model price of the option is compared to the market price to calculate the pricing error. The pricing error is aggregated across all options to calculate the total SSE, mean squared errors (MSE), mean errors (ME) and mean absolute errors (MAE). The results are summarized in Table III. Since structural parameters may change over time, it is expected that the out-of-sample pricing performance may not be as good as the in-sample pricing performance. This is indeed the case when we use SSE to compare the out-of-sample pricing performance in Table III with the in-sample fit in Table II. For example, the average SSE for the ETRI model is 188.58 in the out-of-sample test, which is slightly larger than the 180.76 in the in-sample test of the same model. Similar results hold for the BS and CEV out-of-sample tests.

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Insert Table III About Here

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More interestingly, the ETRI model continues to perform better out-of-sample than the CEV model while the CEV model is in turn superior to the BS model. Panel I of Table III reports the out-of-sample pricing errors. The SSEs are 288.94, 222.26 and 188.58 for the BS, CEV and ETRI model, respectively. Measured by the SSE, the ETRI's out-of-sample pricing performance is 15% better than the CEV model and 35% better than the BS model, similar to the in-sample results. The conclusion is similar but less in magnitude when other measures of pricing errors are used to evaluate out-of-sample performance, with the exception of ME in which case the CEV model dominates. These results confirm that the ETRI's superior in-sample fit does to a large extent translate into better out-of-sample pricing performance.

### *C.4. Dynamic Hedging Test*

Empirical tests performed so far are all static in nature. In other words, these tests are all one-period tests that require the calibration of the option pricing model in one day and the application of the calibrated model in another day. Many real-world applications such as delta hedging, however, require the dynamic application of an option pricing model. To examine an option pricing model's dynamic fit to option prices over time, we conduct a dynamic hedging test of the three competing models. As is frequently used in previous studies, we consider a delta hedging experiment with rebalancing at fixed time intervals. The average hedging errors will be used to assess the dynamic fit of different models. Bakshi, Cao, and Chen (1997) have reported that in sharp contrast with that obtained on out-of-sample pricing, delta hedging performance seems to be insensitive to model misspecification. Regardless of hedge rebalancing frequency, the real significant improvement by the stochastic volatility models over the BS occurs only when OTM calls are being hedged. The fundamental benefit of the stochastic volatility models in hedging is thus that they provide a consistent theoretical framework to delta-vega-neutral hedge. It will be interesting to see if similar findings will be obtained in our experiment.

The hedge portfolio is constructed as follows. Consider an option writer who wishes to use the underlying asset (i.e., the S&P 500 index) to hedge an option position. Delta hedging is a simple and yet effective strategy to use in this case. Suppose the dealer wishes to hedge a short position in a put option with  $\tau$  periods to maturity and strike price  $X$ . The minimum-variance hedge of the put at period  $t$  requires  $H_s(t)$  units of the underlying asset and  $H_0(t)$  dollars invested in T-bills, where

$$H_s(t) = \delta(t),$$

$$H_0(t) = P(t, \tau) - H_s(t) \cdot S(t),$$

and  $\delta(t)$ ,  $P(t, \tau)$  and  $S(t)$  are option delta, option price and index value, respectively. The hedge portfolio constructed in period  $t$  is thus  $H_s(t) \cdot S(t) + H_0(t)$ . For each option pricing model tested, option delta is calculated separately using the structural parameters estimated from all option prices on the previous trading day. The constructed hedge portfolio is held until the next period. The hedging error is calculated as:

$$E(t+1) = H_0(t) \cdot R - P(t+1, \tau-1) - H_s(t) \cdot S(t+1),$$

where  $R$  is the plus 1 interest rate over the rebalancing period. The hedge portfolio is then rebalanced to maintain the delta neutral position. Option delta is recalculated using the structural parameters estimated on the day prior to the rebalancing day (period  $t+1$ ). This procedure is repeated until option maturity. The option delta thus obtained varies according to the model specification. In the ETRI model, since volatility is stochastic and correlated with stock returns, the position to be taken in the stock must control not only for the direct impact of the underlying price risk but also for the indirect impact of that part of volatility risk that is correlated with stock price fluctuations. In contrast, in the BS model the option delta needs only to control for the impact of the underlying price risk.

The mean dollar hedging error (ME) and the mean absolute hedging error (MAE) are calculated as:



$$\text{ME} = \frac{1}{\tau} \sum_{t=1}^{\tau} E(t),$$

$$\text{MAE} = \frac{1}{\tau} \sum_{t=1}^{\tau} |E(t)|,$$

respectively. We also calculate the mean absolute deviation (MAD) to measure the variability of hedging errors:

$$\text{MAD} = \frac{1}{\tau} \sum_{t=1}^{\tau} |E(t) - \text{ME}|.$$

Because there are many options traded on any given day, we calculate these mean values across all time periods and options. For simplicity, we abuse the notation by continuing to use ME, MAE and MAD to denote the corresponding mean values across time periods and options.

Table IV reports the hedging errors from the delta hedging test with a one-week rebalancing period. Other rebalancing periods (such as one day and one month) are also considered, but the results are similar and hence not reported. Note that Table IV presents hedging errors relative to the BS model. For example, the ME for the BS and ETRI model is 2.4120 and 2.4605, respectively. The ratio of hedging errors of ETRI over BS is thus 1.0073, which is reported in the last row of Table IV under the heading “Overall”. All three measures of hedging errors indicate that on average the three competing models exhibit nearly identical delta hedging performance. The superior static pricing performances across moneyness of the ETRI model does not necessarily translate into superior dynamic pricing performances.

Table IV also reports hedging errors for three maturity groups and six moneyness categories. The maturity groups represent short-term (less than 60 days), medium-term (between 60 and 180 days) and long-term (over 180 days) options. Option moneyness is defined as the ratio of option strike price over index value. To adjust for interest rate and dividends, we discount the strike price using the risk-free rate and subtract the present value of dividends from the index value. As shown in the table, hedging performance varies across the maturity and moneyness groups.. Both the CEV and the ETRI models tend to perform better than the BS model when the option is either deep in or out of the money, especially for short-term options. Between the CEV and the ETRI models, the ETRI model clearly dominates the CEV model for either deep in or out of the money short-term options. For other moneyness levels, the CEV and the ETRI models perform either on par with the BS model or slightly worse. These results are therefore quite consistent with the findings from Bakshi, Cao, and Chen (1997). The ETRI model only offers incremental improvements in hedging when short-term in or out of money options are hedged.

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Insert Table IV About Here

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Finally, our data and sample period are consistent with those adopted by Bakshi, Cao and Chen (2000). Due to differences in estimation methods, their results are not directly comparable to ours. They employ an internally consistent methodology to fit the structural parameters of the model once *ex post* while we employ the typical implied method to re-fit the model daily *ex ante*. Because both studies use the BS model as a competing model, however, model performances relative to the BS model may be discussed between the two studies. We find that the performance of the ETRI model is comparable to that of the stochastic volatility models in Bakshi, Cao and Chen (2000) in out-of-sample pricing tests. The advantage of the ETRI model is of course that much fewer structural parameters need to be estimated in comparison with stochastic volatility models. On the other hand, Bakshi, Cao and Chen (2000) show that the stochastic volatility models dominate the BS model by a significant margin in delta hedging tests across moneyness level while the ETRI model does not. This lack of significant improvement in hedging performance perhaps points to the potential need to further extend the ETRI model to incorporate seasonality and clustering in the trade arrival dynamics. However, incorporating stochastic trade arrival intensity may destroy the parsimonious properties of the ETRI model. This is because the ETRI model is a tree-based model with computing time growing exponentially with the number of structural parameters. The stochastic structure requires either additional securities to hedge the intensity risk or the intensity risk to be priced, which will lead to a more complex state space structure with at least four states in each period and possibly several additional parameters to estimate. This more general structure, though theoretically appealing, may be computationally too demanding when brought to extensive empirical testing. This tradeoff between model complexity and computational efficiency is left for future research.

#### IV. Concluding Remarks

We have developed a unique tree structure with random move time by randomizing trade arrival, motivated by the findings in the microstructure literature that price and trade commove with random inter-arrival time, and in the subordinated process modeling literature that subordinating asset price changes to random trade arrival significantly reduces asset return leptokurtosis.

In calendar-time, random trade arrival leads to a trinomial tree structure that embeds stochastic volatility, while in trading-time the isomorphic tree structure is binomial with random maturity. We further extend the trinomial model to allow for negatively correlated trade arrival intensity and asset price changes to address the leverage effect. By benchmarking to the one-parameter BS and the two-parameter CEV models, we demonstrate that this parsimonious structure with only three parameters offers significant incremental improvements in pricing and hedging.

Several extensions are possible. The first is to further extend the structure to incorporate seasonality and clustering in the trade arrival process, which might bring into further increment improvements. Given the tree nature of the model however, this extended structure might be computationally unbearable in empirical

testing. This motivates the next extension. Since the trading-time binomial version contains CCL as a risk-neutral limiting case, its extension to incorporate a stochastic intensity arrival process promises one way to extend Chang, Chang, and Lim (1998) as well as an analytic solution to the empirical obstacle mentioned above. Third, since the calendar-time trinomial model promises a unique and parsimonious approach to option pricing with stochastic volatility, applications to exotic and embedded options may also be promising.

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## Footnotes

<sup>1</sup> Boyle (1986) developed a trinomial pricing model and demonstrated numerically that it converges faster than the CRR model. Madan, Milne and Shefrin (1989) studied multinomial generalizations of the original model and proved that the convergence properties remain unchanged. Breen (1991), Tian (1993), Kim and Byun (1994), Curran (1995), Leisen and Reimer (1995), and Heston and Zhou (2000) examined and provided some suggestions for improving the rate of convergence of binomial and/or trinomial models. Boyle (1988), Boyle, Evnine and Gibbs (1989), He (1990), Kamrad and Ritchken (1991), Amin (1991), Wei (1993), Ho, Stapleton and Subrahmanyam (1995), and Hilliard and Schwartz (1996, 1997) researched on tree-based models for asset prices with two- or multi-state variables. Hull and White (1990), Nelson and Ramaswamy (1990), and Tian (1994) studied binomial or trinomial models when the underlying asset follows more complex stochastic processes such as those suitable for the evolution of interest rates. No-arbitrage interest rate models are developed in such a way that they are automatically consistent with the observed term structure of interest rates and/or interest rate volatilities.

<sup>2</sup> The basic approach is to first find a unique risk-neutral probability distribution of ending asset prices at the maturity of options implied by, or consistent with, currently observed option prices. Under some reasonable economic and/or technical assumptions, the risk-neutral probability distribution leads to a unique binomial or trinomial tree. Implied binomial or trinomial models include Rubinstein (1994), Dupire (1994), Derman and Kani (1994), Derman, Kani, and Chriss (1996), Jackwerth and Rubinstein (1995), and Brown and Toft (1996), Rubinstein (1998), among others.

<sup>3</sup> A subordinated process  $\{X[T(t)]\}$  is obtained by randomizing the time clock  $t$  in a stationary Markov process  $\{X(t)\}$  by a new time clock  $\{T(t)\}$ . Thus constructed, the resulting nonstationary process  $\{X[T(t)]\}$  is said to be subordinated to  $\{X(t)\}$ , the stationary parent process, and directed by  $\{T(t)\}$ , the directing process, which is also called the subordinator. For more details, see Feller (1971).

<sup>4</sup> For finding supports of the subordinated process model, see Epps and Epps (1976), Westerfield (1977), Kon (1984), Harris (1986) and Callaway (1989) on daily stock prices, Harris (1987) on transaction data, Ané and Geman (2000) on high frequency data, and Tauchen and Pitts (1983) and Hall, Brorsen and Irwin (1989) on futures.

<sup>5</sup> Clark (1973) and others have introduced the concept of trading-time or transaction-time to refine subordinated process models. Trading-time refers to a time scale where price changes are measured from trade to trade rather than from time to time as in the calendar-time norm. Trading-time subordinated models predict that the variability of security returns positively relates to the number of trades during a given calendar-time period, so the leptokurtosis of return distribution can be significantly reduced if we measure time from trade to trade. In other words, the total price change in a given calendar-time period, such as a day, is the sum of a "random" number of independent price changes due to random information arrival. Subordinated process models describe price evolution with the trade arrival process as the randomizing subordinator, and thus asset returns become stationary when measured in trading-time.

<sup>6</sup> Huang and Litzenberger (1988) rederived the Cox-Ross-Rubinstein's binomial option pricing model by using a discrete-time version of the martingale valuation theory that can be conveniently applied to an incomplete market.

<sup>7</sup> It is commonly known that the BS model exhibits the "volatility smile/sneer" pricing bias when brought to data, that the model's implied volatilities tend to differ across exercise prices and times to expiration. Rubinstein (1985) presents some early evidence. He shows that the model consistently undervalues out-of-the-money, short-maturity options traded on the Chicago Board Options Exchange. Rubinstein (1994) shows that the S&P 500 option-implied volatilities forms a "smile" pattern prior to the October 1987 market crash. Options that are



deep ITM or OTM have higher implied volatilities than ATM options. After the crash, a “sneer” appears – the implied volatilities decrease monotonically as the exercise price rises relative to the index level, with the rate of decrease increasing for options with shorter time to expiration.

## Appendix

<**Theorem 2**> Given  $\Delta k = k/n$ , the BIN formula as specified in Eqs. (21) – (23) shrinks to the CCL formula as  $n$  approaches infinity and when investors are risk-neutral.

<**Proof**>

First, from Eq. (18),  $M_k$ , the  $n$ -period trade arrival equivalent martingale probability measure of  $k$  trade arrivals in  $n$  periods, is

$$M_k = \frac{n!}{k!(n-k)!} m^k (1-m)^{n-k},$$

where  $m = j\phi T/n$  with  $j\phi$  being the risk-adjusted arrival intensity parameter and  $T$  being the option's maturity. Since as  $n$  approaches infinity, the geometric interarrival distribution converges to a Poisson distribution, we have

$$(24) \quad \lim_{n \rightarrow \infty} M_k = \Gamma(k, j\phi T),$$

where

$$\Gamma(k, j\phi T) = e^{-j\phi T} \frac{(j\phi T)^k}{k!},$$

is a Poisson distribution with arrival intensity parameter  $j\phi$ .

Next, from Eqs. (22) and (23) and with  $N=n$ , we have the  $N$ -step binomial price of a  $k$ -trade option being

$$(25) \quad C_k = B_T \sum_{i=0}^n P_k(i) (u_k^i d_k^{n-i} S - X)^+,$$

where,

$$P_k(i) = \frac{n!}{i!(n-i)!} p_k^i (1-p_k)^{n-i},$$

is the  $n$ -period martingale probability that the ending stock price is  $u_k^i d_k^{n-i} S$ .

To obtain the diffusion limit, we apply the following CRR parameterization:

$$u_k = e^{\sigma_1 \sqrt{\Delta k}}, d_k = \frac{1}{u_k}, R_k = e^{r_0 \sqrt{\Delta k}},$$

$$p_k = \frac{R_k - d_k}{u_k - d_k},$$

where  $r_0$  is the instantaneous trading-time riskless rate and  $R_k$  is the trading-time riskless rate over the trading-time interval  $\Delta k = k/n$ . Then as  $n$  approaches infinity with  $\Delta k$  approaching zero, the  $n$ -period binomial price converges to the BS price:

$$(26) \quad \lim_{n \rightarrow \infty} C_k = C_{BS}(S, X, r_0, \sigma_1, k) = SN(d_1) - Xe^{-r_0 k} N(d_2),$$

where  $C_{BS}(S, X, r_0, \sigma_1, k)$  is the trading-time BS price of a call option with asset price  $S$ , strike price  $X$ , instantaneous trading-time riskless rate  $r_0$ , volatility  $\sigma_1$ , and maturity  $k$ , with

$$d_1 = \frac{\ln(S/X) + \left(r_0 + \frac{1}{2}\sigma_1^2\right)k}{\sigma_1\sqrt{k}}, \quad \text{and} \quad d_2 = d_1 - \sigma_1\sqrt{k}.$$

Next by letting  $n$  approaches infinity in Eq. (21) and by substituting  $M_k$  and  $C_k$  in Eq. (21) using Eqs. (24) and (26), we obtain the continuous-time limit:

$$(27) \quad C(\infty) = \sum_{k=0}^{\infty} \Gamma(k, j\phi T) C_{BS}(S, X, r_0, \sigma_1, k),$$

where  $\Gamma(k, j\phi T) = e^{-j\phi T} \frac{(j\phi T)^k}{k!}$  is a Poisson distribution with pseudo arrival intensity parameter  $j\phi$ , and  $C_{BS}(S, X, r_0, \sigma_1, k)$  is the trading-time BS price.

$C(\infty)$  is utility-dependent because parameters  $r_0$  and  $\phi$  are. It is an increasing function of  $j\phi$ , the risk-adjusted or the pseudo trade arrival frequency parameter, in that higher values of  $j$  and  $\phi$  lead to more frequent trade arrivals and thus command higher option values.

Finally, it is straightforward to see that when investors are risk-neutral with  $\phi = 1$ , Eq. (27) simplifies to the following information-time option pricing formula of CCL when applying to an equity option:

$$(28) \quad C_{ccl} = \sum_{k=0}^{\infty} \Gamma(k, jT) C_{BS}(S, X, r_0, \sigma_1, k),$$

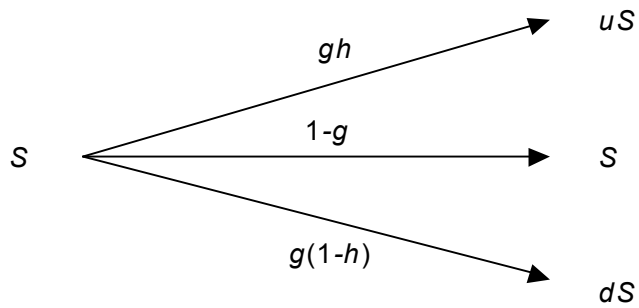
where  $\Gamma(k, jT) = e^{-jT} \frac{(jT)^k}{k!}$  is a Poisson distribution with arrival intensity parameter  $j$ .

**Q.E.D.**

**Figure 1**

**A Subordinated Binomial Tree under Actuarial Probabilities.**

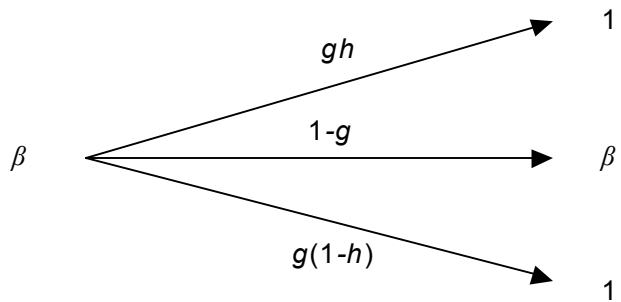
The asset price is  $S$  at the beginning of the period. The change of price during the period depends on whether or not a trade arrives and whether an up or down jump takes place if a trade does arrive. In the figure,  $u$  and  $d$  denote the constant up and down gross jump size, respectively,  $g$  denotes the probability of one trade arrival during a calendar-time period,  $1-g$  the probability of zero arrival,  $h$  the up probability, and  $1-h$  the down probability.



**Figure 2**

**The Evolution of the One-Trade Bond under Actuarial Probabilities.**

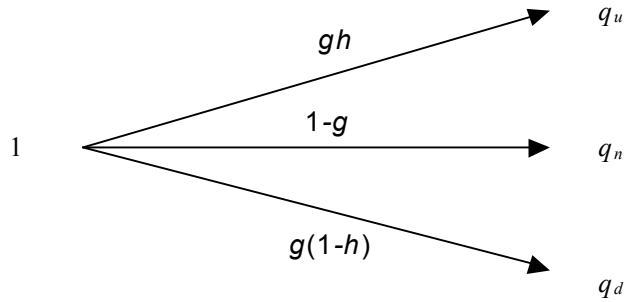
The one-trade bond price is  $\beta$  at the beginning of the period. The bond price remains unchanged if no trade arrives during the calendar-time period and increases to 1 if a trade does arrive. In the figure,  $g$  denotes the probability of trade arrival,  $1-g$  the probability of no trade arrival,  $h$  the probability of an up jump in stock price if a trade does arrive, and  $1-h$  the probability of a down jump in stock price if a trade does arrive.



**Figure 3**

**The Evolution of the Pricing Kernel over a Calendar-Time Period.**

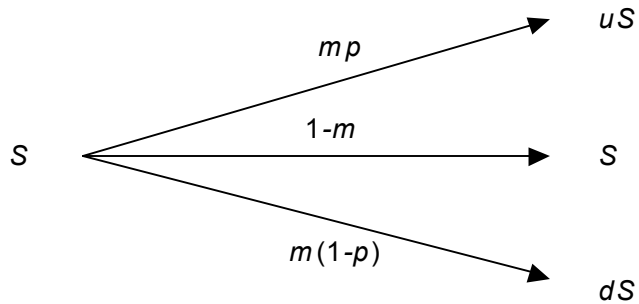
The pricing kernel is denoted by  $q_i$  where the subscript indicates one of the three possible states ( $u$ ,  $n$  and  $d$ ). In the figure,  $g$  denotes the probability of one trade arrival during a calendar-time period,  $1-g$  the probability of zero arrival,  $h$  the up probability, and  $1-h$  the down probability.



**Figure 4**

**A Subordinated Binomial Tree under Risk-Neutral Probabilities.**

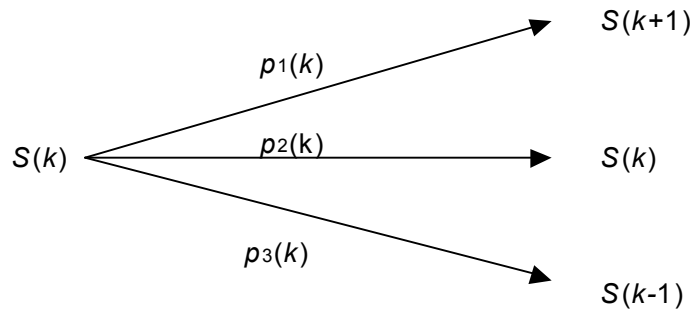
The asset price is  $S$  at the beginning of the period. The change of price during the period depends on whether or not a trade arrives and whether an up or down jump takes place if a trade does arrive. In the figure,  $u$  and  $d$  denote the constant up and down gross jump sizes,  $m$  denotes the probability of one trade arrival during a calendar-time period,  $1-m$  the probability of zero arrival,  $p$  the up probability, and  $1-p$  the down probability.



**Figure 5**

**A Generalized Subordinated Binomial Tree.**

The asset price is  $S(k)$  at the beginning of the period where  $k$  is the net up moves from the initial asset price at node  $(0, 0)$  to the current asset price at node  $(i, k)$ . The change in asset price during the period depends on whether or not a trade arrives and whether an up or down jump takes place if a trade does arrive.

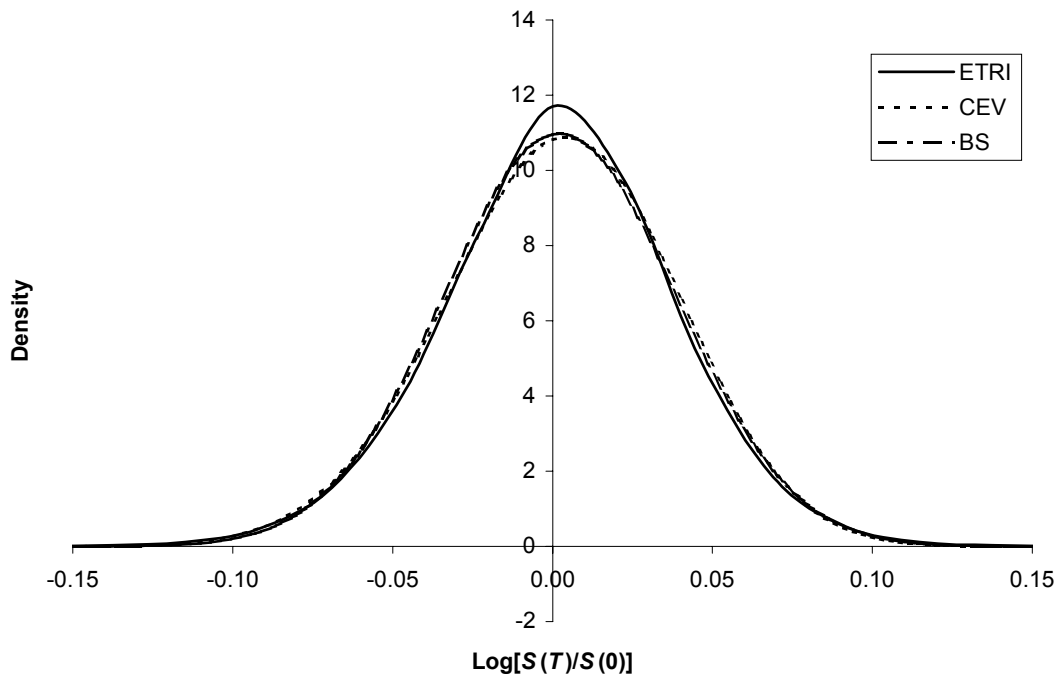




**Figure 6**

**30-Day State-Price Density**

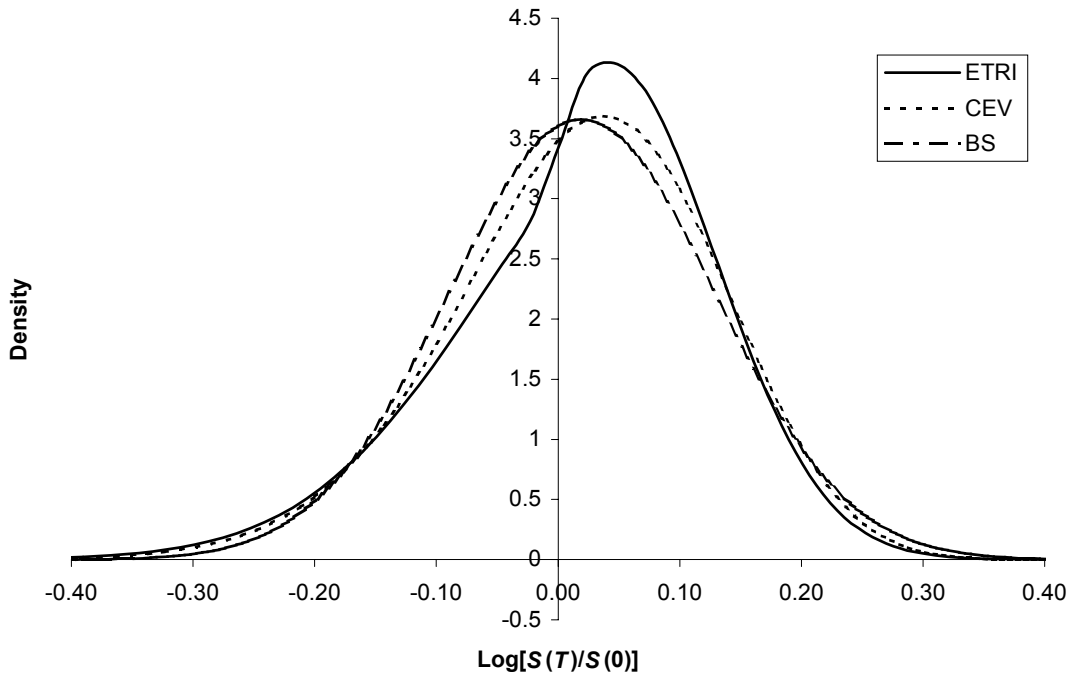
The 30-day state-price density function is plotted for the BS, CEV and ETRI model, respectively. Structural parameters used in the calculation are estimated from option data from September 22, 1993. The BS implied volatility is 12.7% on that date, the median volatility in the sample period.



**Figure 7**

**270-Day State-Price Density**

The 270-day state-price density function is plotted for the BS, CEV and ETRI model, respectively. Structural parameters used in the calculation are estimated from option data from September 22, 1993. The BS implied volatility is 12.7% on that date, the median volatility in the sample period.



**Table I**  
**Numerical Convergence of the TRI and the BIN Models**

The subordinated binomial prices are computed for the number of time periods ( $n$ ) ranging from 50 to 1,000. Both the trinomial version (TRI) and the sum of binomials version (BIN) are implemented. The parameter values used for per trade volatility, trade arrival intensity per year, option maturity, riskless rate, asset price, and strike prices are  $\sigma_1 = 0.01$ ,  $j_0 = 100$ ,  $T = 0.25$ ,  $r_0 = 0.08$ ,  $S = 100$ , and  $X = 95, 100, 105$ , respectively. Prices from the binomial version are compared with the continuous-time limit of Chang, Chang, and Lim (1998).

# of periods ( $n$ )	Call options						European put options					
	$X = 95$		$X = 100$		$X = 105$		$X = 95$		$X = 100$		$X = 105$	
	BIN	TRI	BIN	TRI	BIN	TRI	BIN	TRI	BIN	TRI	BIN	TRI
50	7.048	7.047	3.106	3.114	0.892	0.885	0.167	0.166	1.126	1.134	3.813	3.805
60	7.046	7.047	3.107	3.114	0.892	0.886	0.166	0.166	1.127	1.133	3.813	3.806
70	7.046	7.047	3.108	3.113	0.891	0.886	0.166	0.166	1.128	1.133	3.812	3.806
80	7.047	7.047	3.108	3.113	0.892	0.886	0.166	0.165	1.129	1.132	3.814	3.807
90	7.046	7.047	3.109	3.113	0.893	0.887	0.166	0.165	1.129	1.132	3.815	3.807
100	7.046	7.047	3.109	3.113	0.894	0.887	0.166	0.165	1.129	1.132	3.815	3.807
150	7.046	7.047	3.110	3.112	0.895	0.888	0.165	0.165	1.130	1.131	3.816	3.808
200	7.046	7.047	3.110	3.112	0.895	0.888	0.165	0.165	1.131	1.131	3.817	3.808
300	7.046	7.047	3.111	3.112	0.896	0.888	0.165	0.165	1.132	1.131	3.817	3.808
400	7.046	7.047	3.111	3.111	0.896	0.889	0.165	0.165	1.132	1.131	3.818	3.809
500	7.046	7.047	3.111	3.111	0.896	0.889	0.165	0.165	1.132	1.130	3.818	3.809
600	7.046	7.047	3.112	3.111	0.896	0.889	0.165	0.165	1.132	1.130	3.818	3.809
700	7.045	7.047	3.112	3.111	0.896	0.889	0.165	0.165	1.132	1.130	3.818	3.809
800	7.045	7.047	3.112	3.111	0.897	0.889	0.165	0.165	1.132	1.130	3.818	3.809
900	7.045	7.047	3.112	3.111	0.897	0.889	0.165	0.165	1.132	1.130	3.818	3.809
1,000	7.045	7.047	3.112	3.111	0.897	0.889	0.165	0.165	1.132	1.130	3.818	3.809
Closed-form	7.045	n/a	3.112	n/a	0.897	n/a	0.165	n/a	1.133	n/a	3.819	n/a

**Table II**  
**Structural Parameter Estimates and In-Sample Fit**

The structural parameters are volatility ( $\sigma$ ) for the Black-Scholes (BS) model, volatility ( $\sigma_0$ ) and leverage ( $\alpha_0$ ) for the constant elasticity of variance (CEV) model, and per trade volatility ( $\sigma_1$ ), trade arrival intensity ( $\lambda$ ) and leverage ( $\theta$ ) for the extended subordinated binomial (ETRI) model. These structural parameters are estimated for each trading day by minimizing the sum of squared errors (SSE) between model price and market price using all options on that day. Numbers in parentheses are standard errors.

Model	Parameter Estimate						SSE
	$\sigma$	$\sigma_0$	$\alpha_0$	$\sigma_1$	$\lambda$	$\theta$	
BS	0.1210 (0.0145)						281.51 (124.06)
CEV		0.1345 (0.0143)	0.0000 (0.0000)				214.80 (108.06)
ETRI				0.0253 (0.0025)	28.1127 (5.0048)	-2.8691 (0.2266)	180.76 (104.20)

**Table III**  
**One Day Out-of-Sample Pricing Errors**

To measure a model's out-of-sample performance, we price all options on a trading day using the structural parameters estimated from the previous trading day. The model price is then compared to the market price to calculate the pricing error for each option. The pricing error is aggregated across options to calculate the total sum of squared errors (SSE), mean squared errors (MSE), mean errors (ME) and mean absolute errors (MAE). The three models compared are the Black-Scholes (BS) model, the constant elasticity of variance (CEV) model, and the extended subordinated binomial (ETRI) model. Panel I reports the results for the three models while Panel II presents the pricing errors relative to the BS model. Numbers in parentheses are standard errors.

Model	SSE	MSE	ME	MAE
Panel I: Out-of-sample pricing errors				
BS	288.94 (128.53)	1.7795 (0.2877)	-0.1632 (0.2757)	1.4718 (0.1914)
CEV	222.26 (111.84)	1.5595 (0.2706)	-0.0609 (0.2807)	1.2940 (0.1617)
ETRI	188.58 (108.35)	1.4292 (0.2811)	-0.1438 (0.2842)	1.1790 (0.1705)
Panel II: Pricing errors relative to the BS model				
CEV	0.7692	0.8764	0.3732	0.8792
ETRI	0.6527	0.8031	0.8811	0.8011

**Table V**

**Delta Hedging with One-Week Rebalancing**

In a delta hedging experiment with one-week rebalancing, the hedge portfolio is constructed with a long position in the underlying asset and a short position in a put option. The hedge ratio is chosen such that the hedge portfolio is delta neutral. Option delta is calculated separately for each model using the structural parameters estimated from all option prices on the previous trading day. On the next rebalancing day (after 7 days), the hedging error is calculated and recorded. The hedge portfolio is then rebalanced to maintain a delta neutral position. This procedure is repeated until the option expires. The average hedging error across all options over the sample period is used to compare competing models. Three hedging errors are considered: mean dollar hedging errors (ME), mean absolute dollar hedging errors (MAE), and mean absolute deviation (MAD). The three competing models are the Black-Scholes (BS) model, the constant elasticity of variance (CEV) model, and the extended subordinated binomial (ETRI) model. The table reports the hedging error of the CEV and ETRI models relative to the BS model.

Maturity (days)	Money ( $X/S$ )	ME		MAE		MAD	
		CEV	ETRI	CEV	ETRI	CEV	ETRI
< 60	< 0.94	0.9634	0.9572	0.9589	0.9487	0.9647	0.9484
	0.94 – 0.97	1.0086	1.0143	1.0050	1.0089	0.9774	0.9601
	0.97 – 1.00	1.0083	1.0159	1.0022	1.0058	0.9722	0.9617
	1.00 – 1.03	1.0055	1.0177	0.9965	1.0028	0.9838	0.9783
	1.03 – 1.06	1.0262	1.0709	0.9828	0.9836	0.9800	0.9728
	≥ 1.06	0.8279	0.7625	0.9050	0.8767	0.9204	0.8859
60 – 180	< 0.94	0.9830	0.9819	0.9647	0.9617	0.9669	0.9627
	0.94 – 0.97	1.0578	1.0643	1.0295	1.0326	1.0343	1.0394
	0.97 – 1.00	1.0252	1.0302	1.0254	1.0321	1.0192	1.0261
	1.00 – 1.03	1.0045	1.0118	0.9994	1.0041	1.0117	1.0202
	1.03 – 1.06	1.0250	1.0424	0.9998	1.0044	0.9962	0.9967
	≥ 1.06	1.0587	1.0872	0.9787	0.9746	0.9909	0.9882
≥ 180	< 0.94	0.9419	0.9483	0.9136	0.9212	0.9146	0.9214
	0.94 – 0.97	0.9803	0.9857	0.9384	0.9436	0.9111	0.9171
	0.97 – 1.00	1.0147	1.0265	1.0102	1.0224	0.9979	1.0073
	1.00 – 1.03	1.0258	1.0316	1.0062	1.0127	0.9872	0.9891
	1.03 – 1.06	1.0511	1.0706	1.0414	1.0590	1.0387	1.0584
	≥ 1.06	1.0651	1.0768	1.0502	1.0590	1.0574	1.0672
Overall		1.0073	1.0171	0.9833	0.9859	0.9830	0.9829