A Structural Model for Credit Migrations

By

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Credit Ratings and Transition Matrices

- Ratings are conducted using traditional actuarial methods by rating agencies, mainly Standard & Poors, Moody's, Fitch, and Duff & Phelps.
- Transition matrices are estimated based on actuarial methods. From these matrices, estimated default probabilities (default rates) are inferred.
- They are used in JP Morgan's Credit Metrics and McKinsey's Credit Portfolio View, … etc.

Credit Ratings

Explanation	Standard & Poors	Moody's
Investment grade		
Highest grade	AAA	Aaa
High grade	AA	Aa
Upper medium grade	A	A
Medium grade	BBB	Baa
Speculative grade		
Lower medium grade	BB	Ba
Speculative	B	B
Poor standing	CCC	Caa
High speculative	CC	Ca
Lowest quality, no interest	C	C
In default	D	

Transition Matrix

Initial rating	*Rating at year-end*							
	AAA	*AA*	*A*	*BBB*	*BB*	*B*	*CCC*	*Default*
AAA	93.66	5.83	0.40	0.09	0.03	0.00	0.00	0.00
AA	0.66	91.72	6.94	0.49	0.06	0.09	0.02	0.01
A	0.07	2.25	91.76	5.18	0.49	0.20	0.01	0.04
BBB	0.03	0.26	4.83	89.24	4.44	0.81	0.16	0.24
BB	0.03	0.06	0.44	6.66	83.23	7.46	1.05	1.08
B	0.00	0.10	0.32	0.46	5.72	83.62	3.84	5.94
CCC	0.15	0.00	0.29	0.88	1.91	10.28	61.23	25.26
Default	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Source: Standard & Poor's, January 2001.

Migrating probability from BBB to A

Traditional Ratings

- Traditional rating method relies on analyzing financial statements. Rating transition probabilities are based on average historical frequencies of defaults and credit migrations. One assumes that actual default rates and credit changes equal to their historical counterparts.
- In practice, credit migrations and default rate changes are taken to be equivalent. However, default rates evolve continuously, but credit migrations are reported in a discrete fashion.
- Traditional rating method classifies firms into credit categories. But how do the rating agencies classify firms according to default probabilities?

- Kealhofer et. al (1998) found a high degree of overlaps of estimated default probabilities across different letter grades according to the S&P ratings.
- In addition to EDP, traditional ratings incorporate other information to rate a firm. But what and how?
- The structural model (option theoretic) approach used by KMV monitors the default probability continuously. In principle, one could rate each company based on default probabilities. But this is difficult to implement
- Moody's acquired KMV in 2002. It now becomes feasible to examine credit migration from the option-theoretic point of view.

Merton's Approach

- Let V denote the value of the firm's asset, B the market value of the loan, and F the face value of the loan at maturity T.
- Credit risk exists as long as $P(V_T < F) > 0$.
- What can a bank do to eliminate the credit risk?
- Bank longs a put option on the asset value (stochastic) of the obligor, at a strike price F, maturing at time T.
- If the bank purchases such an option, it would eliminate the credit risk associated with the loan completely.

The bank's payoff matrix at times 0 and T with the purchase of the put option.

Time	0	T	
Value of assets	V_0	$V_T <= F$	$V_T > F$
Bank's Position:			
(a) Make a loan	$-B_0$	V_T	F
(b) Long a put	$-P_0$	$F-V_T$	0
Total	$-B_0 - P_0$	F	F

- The value of the put is the cost of eliminating the credit risk, which by means of the Black-Scholes formula is a function of the default probability.
- One is then interested in the default probability (DF), $P(V_T < F)$.
- The KMV approach derives the estimated default probability (EDP) based on Merton's structural model approach.
- Assume the firm asset value V_t follows a geometric Brownian motion. Then the distance to default is defined as:

KMV Approach

- Distance to default, DD_t:

$$X_t = \frac{\ln V_t - \ln F_t}{\sigma_{V_t}},$$

where $F_t = CL_t + LTD_t / 2$ is the default point.

An alternate definition of *DD* is

$$DD_t = \frac{\text{Firm Asset Value}\,(V_t)\text{ - Default Point}\,(F_t)}{\text{Firm Asset Value}\,(V_t)\times\text{Asset Volatility}(\sigma_{\mathrm{V_t}})}$$

EDPs

- No inter-temporal default

$$EDP_1(t,T) = P(X_T \le 0 \mid X_t)$$

- Inter-temporal default

$$EDP_2(t,T) = P(\min_{t \le s \le T} X_s \le 0 \mid X_t)$$

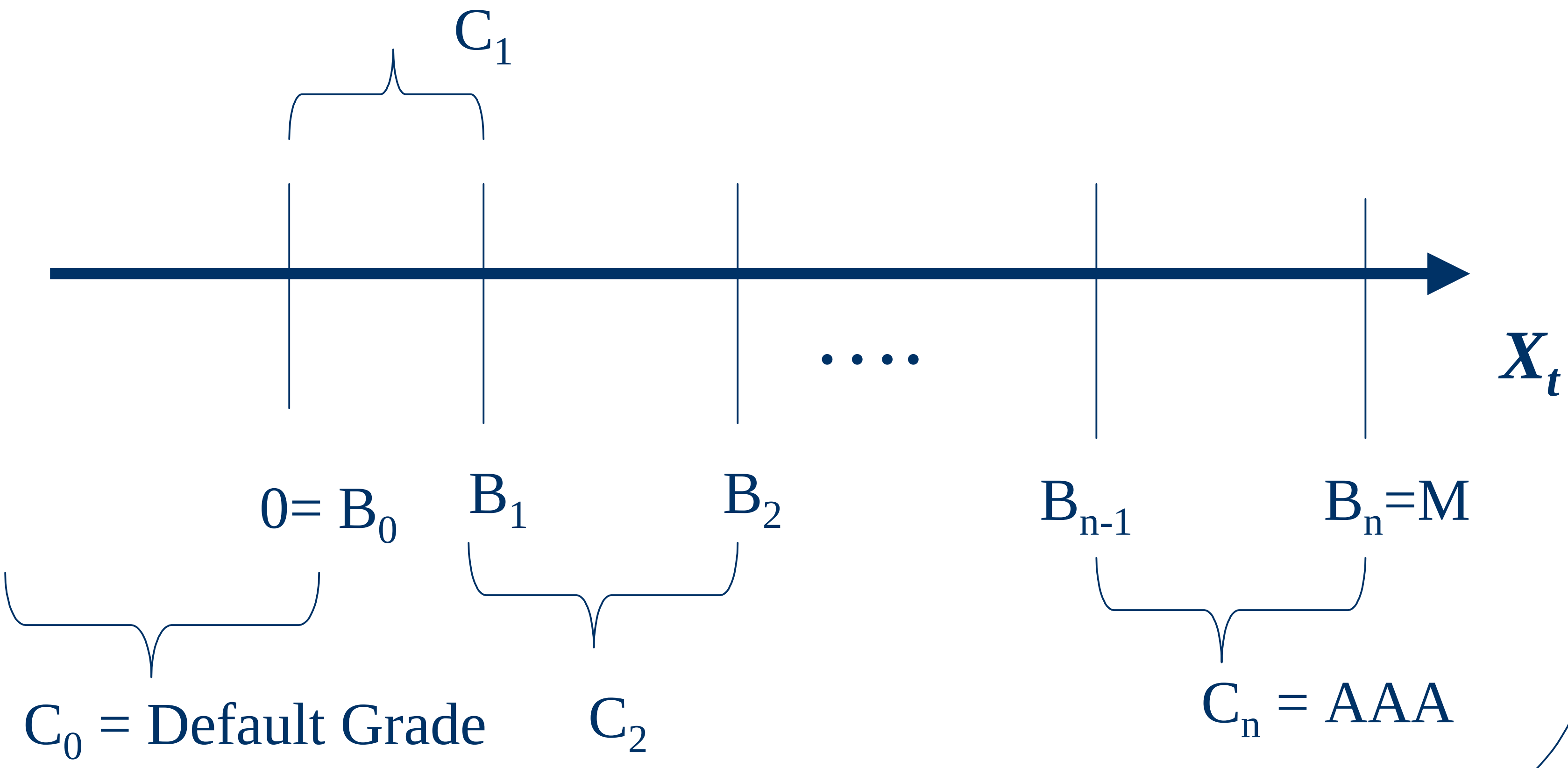
Diffusion model

- Under this model, X_t follows a Brownian motion $$dX_t = \mu dt + \sigma dW_t,$$ where μ and σ are constants.
- EDPs become:

$$EDP_1(t,T) = N\left(-\frac{X_t + \mu(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$EDP_2(t,T) = N\left(-\frac{X_t + \mu(T-t)}{\sigma\sqrt{T-t}}\right) + e^{\frac{2\mu X_t}{\sigma^2}} N\left(\frac{X_t - \mu(T-t)}{\sigma\sqrt{T-t}}\right)$$

Credit Classes



Migrating Signals

- X_t = Distance to Default
- c_t is the credit rating variable; $c_t = k$ if the firm is rated in class k (like a BB-firm) at time t.
- Migrating signals: $X_t \notin (B_{k-1}, B_k]$ when $c_t = k$
  - Upgrade signal: If $X_t > B_k$ given $c_t = k$
  - Downgrade signal: If $X_t \leq B_{k-1}$ given $c_t = k$
- Migrating signal Durations:
  - Upgrading duration (Γ_k^+): the total time of an upgrading signal.
  - Downgrading duration(Γ_k^-): the total time of a downgrading signal.

- Mathematically,

$$\Gamma_k^{+}(t,T) = \int_t^T I(X_s > B_k)\,ds$$

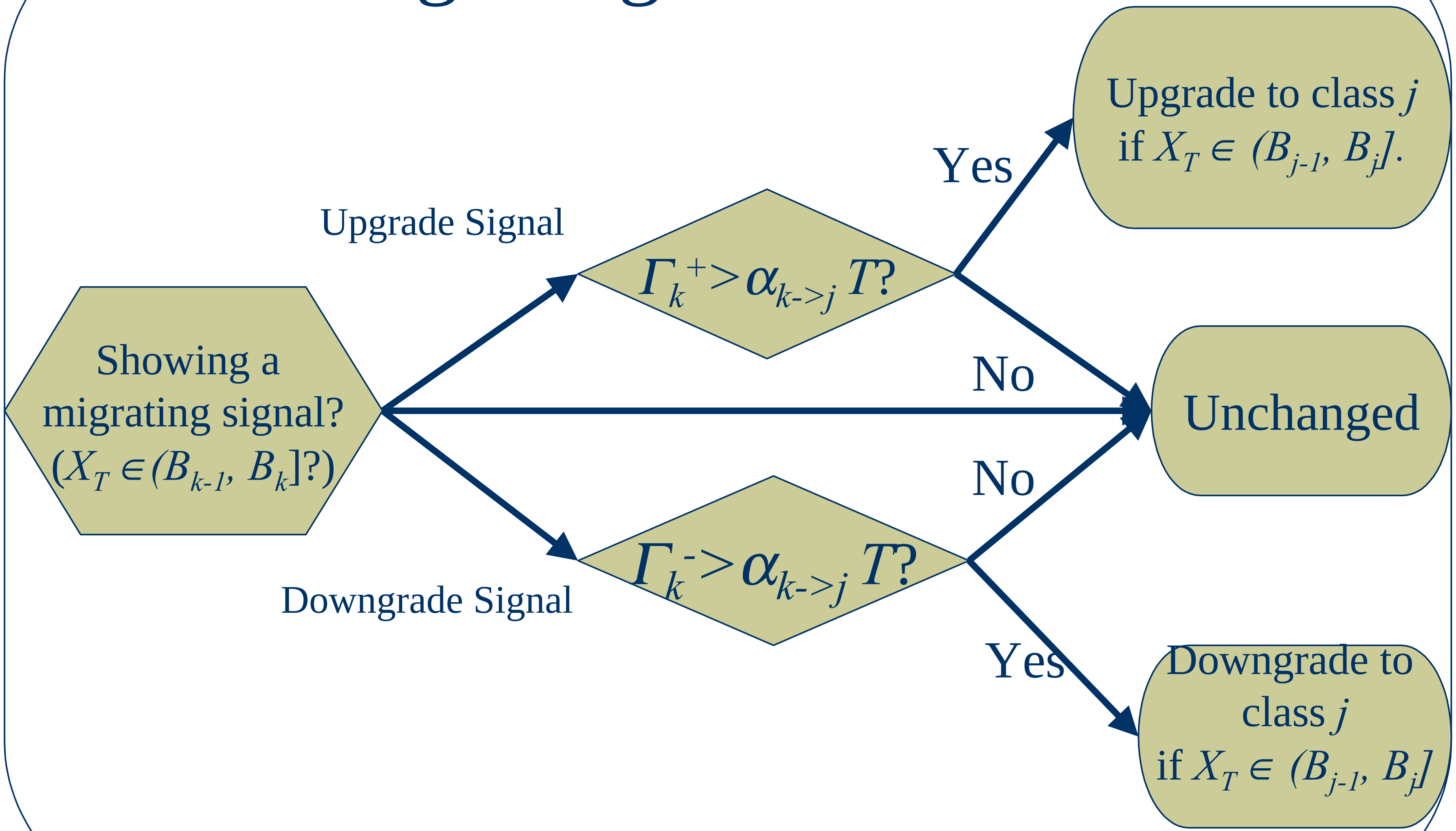
$$\Gamma_k^{-}(t,T) = \int_t^T I(X_s \le B_{k-1})\,ds$$

where $I(\bullet)$ is the indicator function.

- Migration Probability:

$p_{kj}(0,T) = P(c_T = j \mid c_0 = k)$

Migrating Flow Chart



Interpretation of $\alpha_{k->j}$

- A highly rated firm enjoys the privilege that it would not be downgraded due to a short term decrease in credit quality. ($\Gamma_k^- < \alpha_{k->j} T$, $j< k$).
- A poorly rated firm needs to create a positive credit profile in order to be upgraded. ($\Gamma_k^+ > \alpha_{k->j} T$, $j > k$).
- $\alpha_{k->0} = 0$ because the firm has no reputation when defaults.

Migrating Probability

- Assumptions:
  - X_t follows a Brownian motion:

$$dX_t = \mu dt + \sigma dW_t$$

  - No inter-temporal default in the period $(0, T)$, i.e. the firm can only default at $t = T$.

$$p_{k0}^{1}(0,T) = P(c_T = 0 \mid c_0 = k) = P(X_T \leq 0, \Gamma_k^{-} > 0 \mid X_0)$$
$$= P(X_T \leq 0 \mid X_0) = EDP_1.$$

- To evaluate the migrating probability, one needs the joint density function, ψ, of (X_t, Γ_k).

Occupation Time density

- The joint density function of (X_t, Γ_k^-) satisfies the Fokker-Planck equation:

$$\frac{\partial \psi}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 \psi}{\partial x^2} - \mu \frac{\partial \psi}{\partial x} - I(x < B_{k-1})\frac{\partial \psi}{\partial \Gamma_k^-} = 0 \quad \text{for } t > 0, \Gamma_k^- > 0$$

$$\psi\,|_{t=0} = \delta(x)\delta(\Gamma_k^-),$$

$$\psi\,|_{\Gamma_k^-=0} = P(x \in dx, \min_{t \le s \le T} X_s > B_{k-1})$$

- The solution ψ can be found in Linetsky (1999).
- $p_{kj}(0,T) = E\{I(X_T \in (B_{j-1}, B_j])\, I(\Gamma_k^- > \alpha_{k->j}\, T) \mid X_0\}$ for $j<k$.

Downgrading Probability

- Downgrading probability:

If $X_0 < B_{k-1}$, then

$$p_{kj}(0,T) = N\left(\frac{y_2 - x - \mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{y_1 - x - \mu T}{\sigma\sqrt{T}}\right)$$

$$- e^{-\frac{2\mu x}{\sigma^2}}\left[N\left(\frac{y_2 + x - \mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{y_1 + x - \mu T}{\sigma\sqrt{T}}\right)\right]$$

$$+ \int_0^T F(t,T)\left[h(y_1,x,t) - h(y_2,x,t)\right]dt.$$

If $X_0 \geq B_{k-1}$, then

$$p_{kj}(0,T) = \int_0^T F(t,T)\left[g(y_1,x,t) - g(y_2,x,t)\right]dt.$$

Upgrading Probability

- Upgrading probability:
  - If $X_0 \geq B_k$, then

$$p_{kj}(0,T) = N\left(\frac{z_2 - \hat{x} - \mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{z_1 - \hat{x} - \mu T}{\sigma\sqrt{T}}\right)$$

$$- e^{-\frac{2\mu\hat{x}}{\sigma^2}}\left[N\left(\frac{z_2 + \hat{x} - \mu T}{\sigma\sqrt{T}}\right) - N\left(\frac{z_1 + \hat{x} - \mu T}{\sigma\sqrt{T}}\right)\right]$$

$$+ \int_0^T F(t,T)\left[h(z_2,\hat{x},t) - h(z_1,\hat{x},t)\right]dt.$$

  - If $X_0 < B_k$, then

$$p_{kj}(0,T) = \int_0^T F(t,T)\left[g(z_2,\hat{x},t) - g(z_1,\hat{x},t)\right]dt.$$

Fundamental Equations

$$F(t,T) = \begin{cases} (1-\alpha)T, & 0 \le t \le \alpha T \\ T\text{-}t, & \alpha T < t \le T \end{cases}$$

$$g(y,x,t) = \frac{n\left(\dfrac{x+\mu(T-t)}{\sigma\sqrt{T-t}}\right)}{(T-t)^{3/2}t}\left[D_1 n\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) + D_2 N\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right)\right],$$

$$D_1 = \frac{2x\mu}{\sigma^2} - \left(1 - \frac{x^2}{\sigma^2(T-t)}\right)\sqrt{t} + \frac{x(y-\mu t)}{\sigma^2\sqrt{t}},$$

$$x = X_0 - B_{k-1}, \quad \hat{x} = X_0 - B_k,$$

$$y_1 = B_{j-1} - B_{k-1} \quad \text{and} \quad y_2 = B_j - B_{k-1},$$

$$z_1 = B_{j-1} - B_k \quad \text{and} \quad z_2 = B_j - B_k,$$

$$h(y,x,t) = \frac{e^{-2\mu x/\sigma^2 + \mu^2(T-t)}}{\sqrt{2\pi}(T-t)^{3/2}}\left[\mu N\left(\frac{y+x-\mu t}{\sigma\sqrt{t}}\right) - \frac{1}{2\sqrt{t}} n\left(\frac{y+x-\mu t}{\sigma\sqrt{t}}\right)\right].$$

Applications

- Values of $\alpha_{k->j}$ for each k with $j \neq k$ can be approximated through a proxy method if the drift, the volatility, and the transition matrix are given. This value tells us for how long should a class k firm perform well to be upgraded (downgraded) to class j for $j > k$ ($j < k$).
- Conversely, given $\alpha_{k->j}$, the drift μ and the volatility σ of the Distance to Default, X_t, the transition vector for a specific firm can be obtained.

Example: Evaluation of $\alpha_{k->j}$

Consider the following transition matrix from S&P (1981-2000):

	Average One year Transition Matrix							
Initial rating	AAA	AA	A	BBB	BB	B	CCC	D
AAA	93.66	5.83	0.4	0.09	0.03	0	0	0
AA	0.66	91.72	6.94	0.49	0.06	0.09	0.02	0.01
A	0.07	2.25	91.76	5.18	0.49	0.2	0.01	0.04
BBB	0.03	0.26	4.83	89.24	4.44	0.8	0.15	0.24
BB	0.03	0.06	0.44	6.66	83.23	7.46	1.04	1.08
B	0	0.1	0.32	0.46	5.72	83.62	3.84	5.94
CCC	0.15	0	0.29	0.88	1.91	10.28	61.23	25.26

Assume $\mu = 0$ and $\sigma = 1$. Recall

$$EDP_1(t,T) = N\left(-\frac{X_t + \mu(T-t)}{\sigma\sqrt{T-t}}\right),$$

- If $c_0 = k$, then a typical default distance is given by $X_0^{(k)} = -N^{-1}$(Default probability of class k)
- Let the boundary $B_k = (X_0^{(k)} + X_0^{(k+1)})/2$. The range of Distance to Default, $(B_{k-1}, B_k]$, of each credit class can then be constructed.
- With given μ, σ, and the transition probability of credit migration, $p_{kj}(0,T)$, use the fundamental equations to compute values of $\alpha_{k->j}$ numerically.
- From the computed B_k, one can calculate the EDPs by the formula EDP = $N(- B_k)$.
- Results are given in the following table.

Rating	EDP range (%)	$(B_{k-1}, B_k]$	$X_0^{(k)}$
AAA	0.0035 – 0.007	3.9764 – 3.8048	3.8906
AA	0.007 – 0 .023	3.8048 – 3.5359	3.7190
A	0.023 – 0.101	3.5359 – 3.0865	3.3528
BBB	0.101 – 0.5253	3.0865 – 2.5587	2.8202
BB	0.5253 – 2.689	2.5587 – 1.9268	2.2973
B	2.689 – 13.28	1.9268 – 1.1131	1.5598
CCC	13.28 – 30.86	1.1131 – 0.4997	0.6663

Computed values of $\alpha_{k->j}$

	AAA	AA	A	BBB	BB	B	CCC
AAA	Nil	0.964	0.995	0.997	0.998	0.998	0.997
AA	0.995	Nil	0.906	0.983	0.989	0.987	0.987
A	0.991	0.968	Nil	0.879	0.968	0.974	0.977
BBB	0.988	0.978	0.892	Nil	0.888	0.958	0.963
BB	0.987	0.984	0.973	0.882	Nil	0.685	0.893
B	1	0.959	0.946	0.940	0.776	Nil	0.531
CCC	1	1	0.811	0.812	0.813	0.629	Nil

Observations

- Consider an A-rated firm. Based on a one year credit profile, an A-firm would be downgraded to BBB lower within a year only when it shows a downgrade signal more than ten months.
- On the other hand, it has to perform well for most of the year (11.9 months) to be upgraded to AAA within a year.

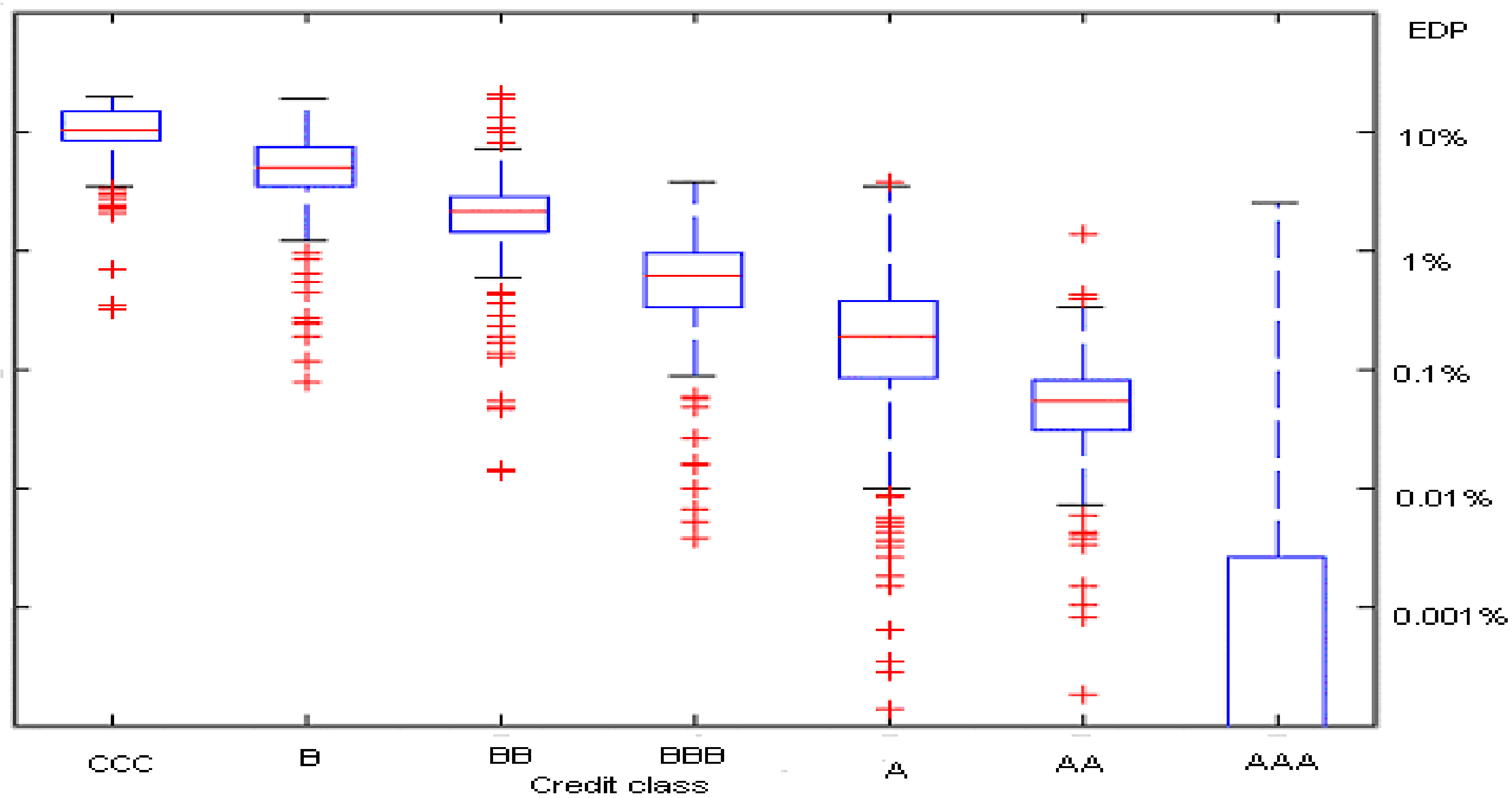
Example: Overlaps of EDPs

Ratings	EDP range	X_t-range $(B_{k-1}, B_k]$	No of firms
AAA	0.0035 – 0.007	3.9764 – 3 .8048	n_1
AA	0.007 – 0 .023	3.8048 – 3.5359	n_2
A	0.023 – 0.101	3.5359 – 3.0865	n_3
BBB	0.101 – 0.5253	3.0865 – 2.5587	n_4
BB	0.5253 – 2.689	2.5587 – 1.9268	n_5
B	2.689 – 13.28	1.9268 – 1.1131	n_6
CCC	13.28 – 30.86	1.1131 – 0.4997	n_7
NR(Not rated)	30.86 – 50	0.4997 – 0.0	n_8
D	50 – 100	0.0 – $(-\infty)$	0

$\alpha = 0.8$ for upgrade and $\alpha = 0.5$ for downgrade

- At $t = 0$, simulate the Distance to Default, $X_0^{(i)}$ according to the class variable $c_0^{(i)}$ of each firm as follows, $i = 1, \ldots, n_k$. If $c_0^{(i)} = k$, simulate the 1-year $X_0^{(i)}$ from $U(-B_k, -B_{k-1})$.
- Simulate a five-year sample path of $\boldsymbol{X}$ for each firm by the model $X_{t+\Delta t}^{(i)} = X_t^{(i)} + \varepsilon \sqrt{\Delta t}$, $\varepsilon \sim N(0,1)$, $t = 1, \ldots, T$.
- At the end of each year, the values of $c_t^{(i)}$ are reviewed according to the proposed criterion, for all $i = 1, 2, \ldots, n_k$.
- Given $c_t^{(i)} = k$, calculate the one year ahead EDP value of firm i by $\text{EDP}(i) = N(-X_T^{(i)})$, where $X_T^{(i)}$ is the simulated distance to default value of firm i at the end of the 5th year.

Results



Future Directions

- Calibrate bounds for each credit class. Two possible methods are:
  - Extending the method of calibrating default boundaries.
  - Using CART approach.
- Extend to inter-temporal default.
- Consider the jump-diffusion economy.
- Conduct statistical tests for the model.

Future Events

- Workshop in Risk Management on July 18, 2003 in Conrad Hotel.
- Two year part-time Master of Science Program in Risk Management starting this fall at CUHK.

Transition Matrix

Transition Matrix: Probabilities of Credit Rating Migrating From One Rating Quality to Another, Within One Year

Initial Rating	Rating at Year-End (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	1.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

Source: Standard & Poor's *CreditWeek* (April 15, 1996).

Migrating probability from BBB to A

Model Features

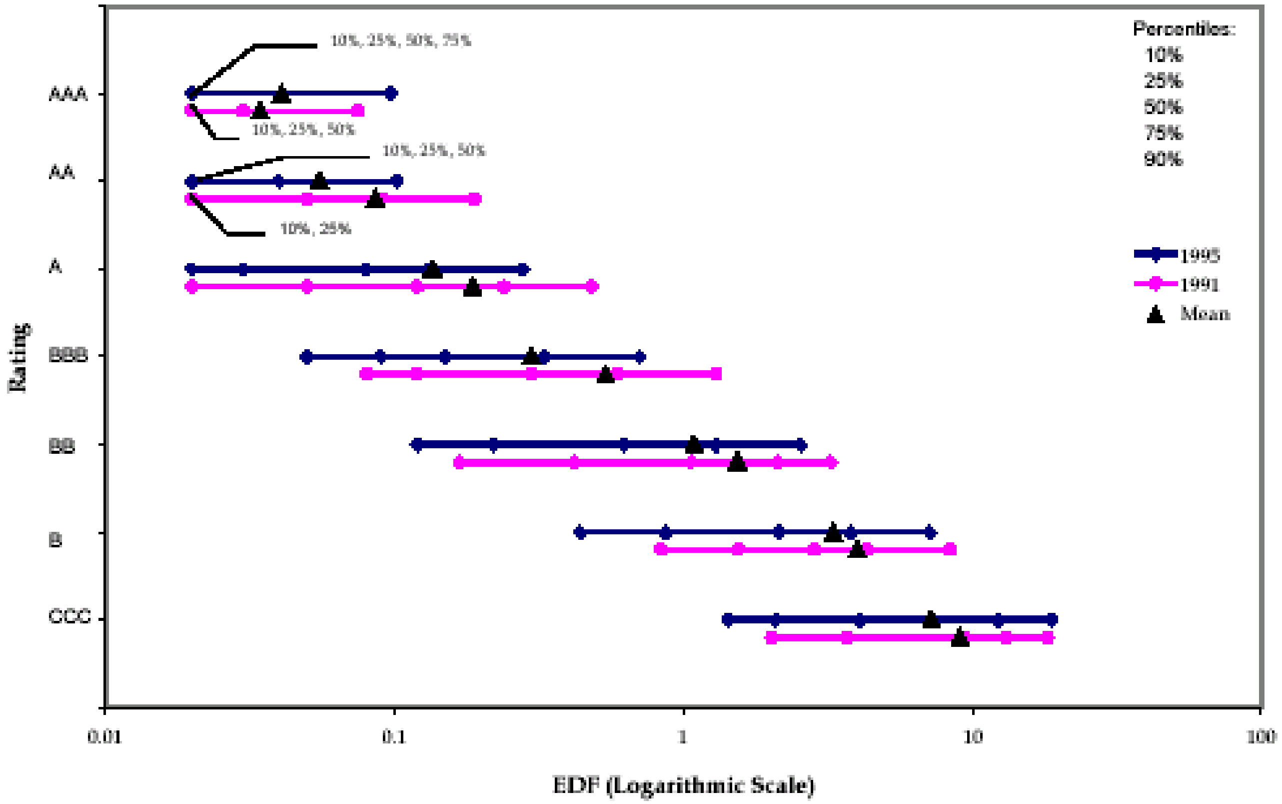
- This model matches the intuition that transition to state 0 is the same as the probability of default. Since $\alpha_{k->0} = 0,$

$$p_{k0}^{1}(0,T) = P(c_T = 0 \mid c_0 = k) = P(X_T \le 0, \Gamma_k^- > 0 \mid X_0)$$
$$= P(X_T \le 0 \mid X_0) = EDP_1,$$
$$p_{k0}^{2}(0,T) = P(c_T = 0 \mid c_0 = k) = P(\min_{0 \le s \le T} X_s \le 0, \Gamma_k^- > 0 \mid X_0)$$
$$= P(\min_{0 \le s \le T} X_s \le 0 \mid X_0) = EDP_2,$$

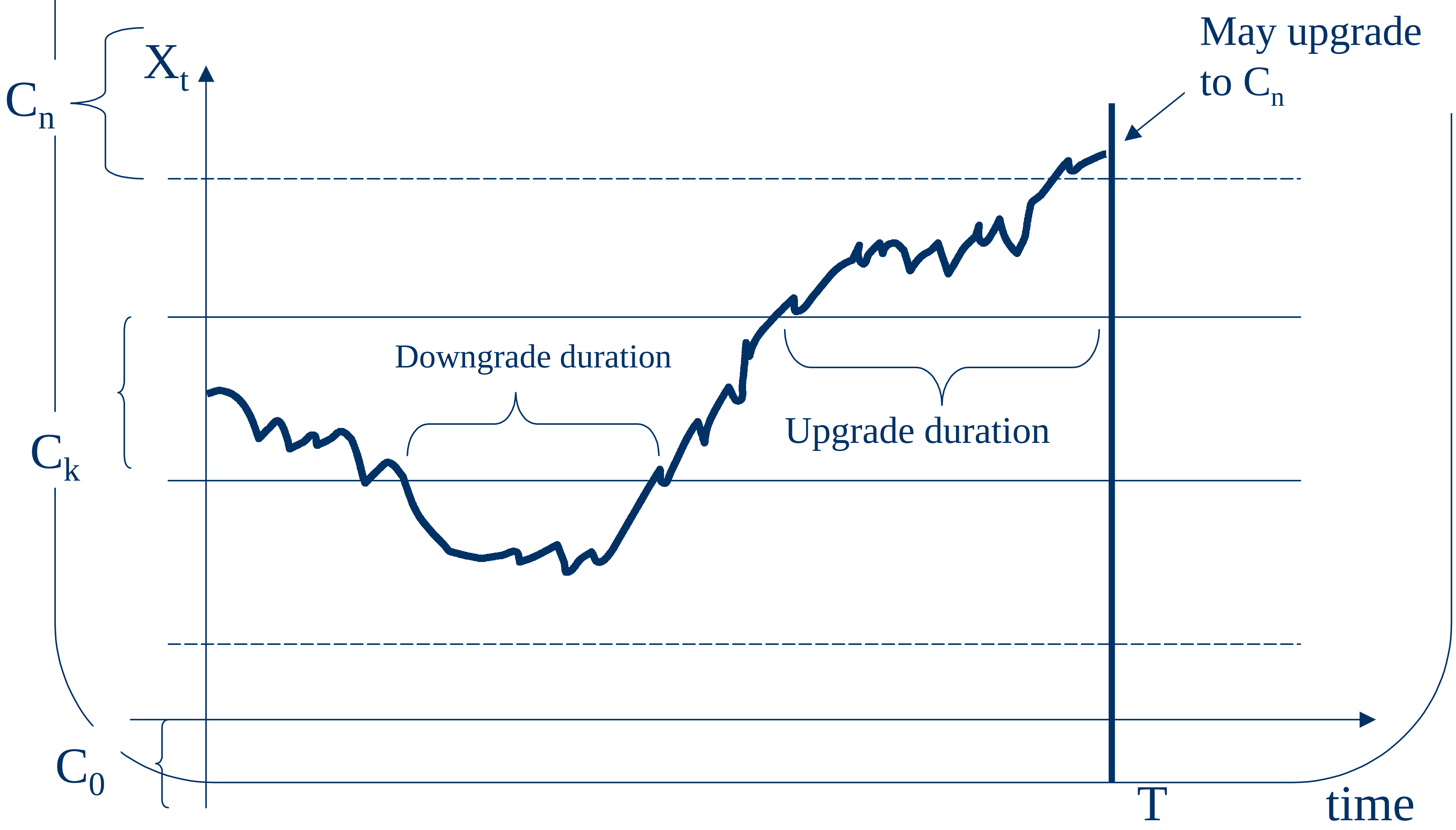
- This model allows for possible overlaps in EDP across different credit classes (letter grades).

Overlaps in EDP for different ratings

EDP (EDF), Expected Default Probability (Frequency)



Source: Kealhofer et. al (1998).


C_n
X_t
May upgrade
to C_n
Downgrade duration
Upgrade duration
C_k
C_0
T
time