Fractional Integration at a Seasonal Frequency with an Application to Quarterly Unemployment Rates *

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June 2003

Abstract

This paper utilises a general class of long memory model that allows a time series process to be fractionally integrated at both the zero and seasonal frequencies. This model is advantageous because it avoids the need for seasonally adjusting data and allows a wider range of long run behaviour to be incorporated into the modelling process. It is shown that a frequency domain maximum likelihood estimator easily and adequately estimates all of the parameters in this model. An important application is made by estimating seasonal ARFIMA models for quarterly US and UK unemployment rates. It is found that the data appear to be non-mean reverting when taking into consideration fractional levels of integration at the zero and seasonal frequencies, thus supporting the hysteresis hypothesis.

Keywords: Spectrum, Small Sample Properties, Natural Rate Hypothesis, Hysteresis Hypothesis.

JEL Classification: C22, E24.

*The author acknowledges comments and suggestions made on an earlier version of this paper by Vance Martin, Nilss Olekalns and Ping Wu.

1 Introduction

The emergence of the autoregressive fractionally integrated moving average (ARFIMA) class of models has lead to a widespread re-examination of many empirical relationships in economics; some well known applications being, for example, Diebold and Rudebusch (1989) who examine persistence in aggregate output; Diebold, Husted and Rush (1991) and Cheung and Lai (1993) who find that purchasing power parity is a valid long run concept; Diebold and Rudebusch (1991) who examine the excess smoothness of income within the context of the permanent income hypothesis; Cheung (1993) who finds foreign-exchange rates are fractionally integrated; and Hassler and Wolters (1995) who find that inflation can be characterised as a fractionally integrated variable. A comprehensive survey of empirical applications and recent developments of the ARFIMA model can be found in Baille (1996).

These studies share a common theme in that they examine long run relationships at the zero frequency. The existence and widespread use of quarterly and monthly data in empirical work suggests that some account should be made of the seasonal nature of many economic data sets. Wilkins (1998) finds the existence of a seasonal frequency and or the seasonal adjustment of a time series process can distort the estimates obtained using the variance ratio and range over standard deviation tests for long memory. This follows work by Ghysels (1990) and Ghysels and Perron (1993) who find seasonality and seasonal adjustment adversely affects unit root tests; see also Olekalns (1994) for further evidence of this.

Rather than try to remove the impact of any seasonality from a time series process before analysis takes place, this paper explicitly incorporates the seasonal frequency into the estimation procedure and allows it to be integrated of a fractional order, the same way that behaviour at the zero frequency is allowed to be fractionally integrated in the ARFIMA model. This follows the earlier work of Gray, Zhang and Woodward (1989) and Viano, Deniau and Oppenheim (1995) who generalize the fractional model to allow long memory relationships at frequencies other than zero and also Porter-Hudak (1990) who applies a seasonal fractional differencing model to monetary aggregates. Other recent empirical applications using the fractionally integrated seasonal model include Franses and Ooms (1997) who examine UK inflation, Gil-Alana and Robinson (2001) who examine UK and Japanese consumption and income and Gil-Alana (2002) who examines national output. See also Lildholdt (2002) who provides a justification for seasonal long memory based on an argument similar to that of Granger (1980) for long memory at the zero frequency.

Estimation of the seasonal ARFIMA model can be achieved easily by specifying the likelihood function in the frequency domain, and thus, the estimation procedure is an extension of the likelihood function in Fox and Taqqu (1986). To judge the performance of this maximum likelihood estimator, a simulation experiment is undertaken to obtain its small sample properties. The results indicate that the frequency domain maximum likelihood estimator is able to differentiate between the fractional differencing parameters at the different frequencies as well as being able to identify any autoregressive or moving average parameters in the data generating process.

In the second half of the paper, an application is made to quarterly US and UK unemployment rates to help distinguish between the natural rate and hysteresis hypotheses of unemployment rate behaviour. The natural rate of unemployment, or the nonaccelerating inflation rate of unemployment (NAIRU), is that rate of unemployment that is consistent with an inflation rate that displays no tendency to change. This is the underlying rate of unemployment that the economy gravitates towards after an exogenous shock pushes it away. In contrast, hysteresis describes path dependency in the dynamic behaviour of a time series process. The hysteresis hypothesis asserts that there is no constant long run natural rate of unemployment and that the effect of an exogenous shock does not dissipate within some finite time horizon.

The traditional methodology of analysing unemployment rates has relied on the non-rejection of the unit root model as an indicator of the hysteresis hypothesis, while rejection of the unit root model has been used as an indicator of the natural rate hypothesis. In time series terms, it has been presupposed that the level of integration, d, is integer valued and that shocks either die away almost instantaneously; $d = 0$, or infinitely persist; $d = 1$. The fractional model allows a compromise between these two extremes by avoiding the $d = 0, 1$ dichotomy. When $d < 1$ the natural rate hypothesis is supported as the process obeys mean reversion, whereas for $d \geq 1$, the hysteresis hypothesis is supported as no mean reversion is displayed. Given the strong seasonal nature of unemployment rates, it seems sensible to incorporate, rather than remove, the seasonality present in quarterly or monthly unemployment time series data. Therefore the seasonal ARFIMA model is a suitable choice of model to use when examining the long run behaviour of quarterly unemployment rates.

The rest of this paper is structured as follows. Section 2 presents the seasonal ARFIMA model and briefly discusses some of its time and frequency domain properties. Section 3 reports the results of a simulation experiment to obtain the small sample properties of the frequency domain maximum likelihood estimator for the seasonal ARFIMA model. Section 4 applies the methodology to estimating long memory at the zero and seasonal frequencies in quarterly US and UK unemployment rates. Section 5 concludes the paper.

2 The SARFIMA Model

The seasonal ARFIMA (SARFIMA) model is a straightforward extension of the non-seasonal ARFIMA model, see Granger and Joyeux (1980) and Hosking (1981) for the original contributions on the ARFIMA model. The zero mean SARFIMA(p, d_0, d_k, q) model is most simply expressed as

$$\Phi(L)(1 - L^k)^{d_k}(1 - L)^{d_0} y_t = \Theta(L)\varepsilon_t \tag{1}$$

where $\Phi(L) = \sum_{j=0}^{p} \phi_j L^j$ and $\Theta(L) = \sum_{j=0}^{q} \theta_j L^j$ are lag operator polynomials in L of order p and q respectively with $\phi_0 = \theta_0 = 0$, $(1 - L)^{d_0}$ is the zero frequency fractional differencing filter, ε_t is a $T \times 1$ sequence of independently and identically distributed random numbers with distribution $(0, \sigma^2)$ and $(1 - L^k)^{d_k}$ is the seasonal frequency fractional differencing filter, where k denotes the periodicity of the possibly greater than one seasonal components. For example, monthly data may have cycles at both the quarterly and monthly frequencies so that $k = 4$ and 12. For simplicity, it will generally be assumed that only one seasonal frequency operates. The expression for the seasonal frequency fractional differencing filter can be obtained from that for the zero frequency;

$$(1 - L^k)^{d_k} = \sum_{j=0}^{\infty} \frac{\Gamma(j - d_k)}{\Gamma(j + 1)\Gamma(-d_k)} L^{k \cdot j} \tag{2}$$

where the lag operator $L^{k \cdot j}$ ensures that only the kth lag of y_t is fractionally differenced.

When a time series process has only a long memory component at the seasonal frequency, that is $d_0 = 0$, then (1) will be called a pure SARFIMA model and be denoted as SARFIMA$(p, 0, d_k, q)$. A general SARFIMA model has $d_0 \neq 0$ and, of course, when $d_k = 0$, (1) simplifies to the ARFIMA(p, d, q) model. The SARFIMA model therefore is more general than the ARFIMA model because it is now possible to look at cases where long memory is present in the zero and non-zero frequencies. To a certain extent, capturing long memory at a seasonal frequency will require the seasonal component to behave in a manner similar to that of a deterministic cycle. This should not cause too much of a problem because of the regularity of the seasonal frequency. It is worth noting however, that capturing long memory over a business cycle is problematic in theory because this cycle is more stochastic in nature, as well as being problematic in practice because of limitations in available sample sizes.

For covariance stationarity of (1) to hold, a necessary but not sufficient condition is $d_0, d_k < 0.5$. Setting the ARMA lag operator polynomials to zero for simplicity and assuming only one seasonal frequency at, for example, $k = 4$ allows the model to be written as

$$(1 - L^4)^{d_4}(1 - L)^{d_0} y_t = \varepsilon_t \tag{3}$$

This can be rearranged to give

$$(1 - L)^{d_0 + d_4}(1 + L)^{d_4}(1 + L^2)^{d_4} y_t = \varepsilon_t \tag{4}$$

from which it can be seen that the behaviour of y_t at a frequency of zero is determined not only by d_0 but by the quantity $d_0 + d_4$. For a more general SARFIMA model, the exponent becomes $d_0 + \sum_i d_{k_i}$ where i indexes seasonal frequency. Accordingly, the necessary and sufficient conditions for covariance stationarity are now

$$\max\{d_0, d_{k_i}\} < 0.5 \quad \text{and} \quad d_0 + \sum_i d_{k_i} < 0.5 \tag{5}$$

while the conditions for mean reversion are

$$\max\{d_0, d_{k_i}\} < 1 \quad \text{and} \quad d_0 + \sum_i d_{k_i} < 1 \tag{6}$$

2.1 Time and Frequency Domain Properties

Theoretical results for related fractional processes can be found in Gray, Zhang and Woodward (1989) and Viano, Deniau and Oppenheim (1995). This section outlines some results relevant for the SARFIMA model currently being considered.

2.1.1 Autocorrelation Function

The autocorrelation function for the pure SARFIMA$(0, 0, d_k, 0)$ model can be obtained directly from the autocorrelation function for the conventional ARFIMA model but with zero coefficients at the non-seasonal lags. The autocorrelation function for the general SARFIMA(p, d_0, d_k, q) model is a more complicated expression containing the long memory parameters for more than one frequency as well as the autoregressive and moving average parameters representing the "short memory" characteristics of the process. The spectral representation of the autocovariance function for such a model is

$$\gamma_j = \frac{\sigma^2}{\pi} \int_0^{\pi} e^{ij\lambda} |1 - e^{-i\lambda}|^{-2d_0} |1 - e^{-ik\lambda}|^{-2d_k} |\theta(e^{-i\lambda})|^2 |\phi(e^{-i\lambda})|^{-2} d\lambda \tag{7}$$

which is a complex function of d_0, d_k, ϕ_i and θ_i for $i = 0, \ldots, p$ or q. For illustrative purposes, the averaged estimated autocorrelation function (ACF) for 100 simulated quarterly processes of sample size $T = 200$ observations are graphed in Figure 1a for $d_0 = 0.0$, $d_4 = 0.4$ and all ARMA parameters set equal to zero. The dashed horizontal line is an approximate 95% confidence interval. The long memory at the seasonal frequency is clearly evident as every fourth lag is quite significant.

2.1.2 Impulse Response Function

The fractional cumulative impulse response function (CIRF) gives an indicator of the time required for a time series process to revert to its mean value after an exogenous shock perturbs the series. Given that the SARFIMA model contains two parameters that determine long range behaviour, interpretation of the fractional levels of integration is most clearly made with respect to the CIRF. For the zero mean SARFIMA process, the CIRF can be formally defined as

$$(1-L)y_t = \Phi^{-1}(L)(1-L^k)^{-d_k}(1-L)^{1-d_0}\Theta(L)\varepsilon_t = \mathbf{\Psi}(L)\varepsilon_t \tag{8}$$

where the moving average coefficients $\mathbf{\Psi} = \{\Psi_1, \Psi_2, \ldots, \Psi_\infty\}$ yield the entire history of responses with Ψ_i denoting the individual cumulative response of y_t in the ith period to a shock in period 0. The limiting behaviour of $\mathbf{\Psi}$ is determined by the large lag values in the expansion of $(1-L)^{1-(d_0+d_k)}$ which is obtained using (4).

2.1.3 Spectrum

The spectrum for the SARFIMA(p, d_0, d_k, q) model with fractional levels of integration at the zero and kth seasonal frequencies is

$$I(\lambda_j) = \frac{\sigma^2}{2\pi}\frac{|\theta(e^{-i\lambda_j})|^2}{|1-e^{-i\lambda_j}|^{2d_0}|1-e^{-ik\lambda_j}|^{2d_k}|\phi(e^{-i\lambda_j})|^2} \tag{9}$$

where $\lambda_j = 2\pi j/T$ for $j = 1, 2, \ldots, T/2$ is the jth Fourier frequency. The existence of fractional integration at a seasonal frequency implies that the spectrum may now have more than one point of discontinuity.

Panel b of Figure 1 graphs an estimate of the theoretical spectrum for a quarterly SARFIMA process where $d_0 = 0.0$ and $d_4 = 0.4$. This figure clearly suggests the discontinuity at a frequency of 0.25, as well as the existence of a harmonic at a frequency of 0.5. Unfortunately, it is difficult to see clearly what is happening in the spectrum when d_4 is positive due to the spike at the 0.25 frequency overwhelming all other frequencies. Panel c of Figure 1 illustrates the allowable behaviour in the model by graphing the theoretical spectra for four quarterly SARFIMA processes with $d_0 = 0.0$ and $d_4 = -0.1, -0.2, -0.3$ and -0.4. Negative values for d_4 clearly illustrate the behaviour possible in the spectrum that is not attainable in the ARFIMA or ARMA class of models. As a further illustration, Panel d of Figure 1 graphs the spectra for a number of monthly SARFIMA$(0, 0, d_{12}, 0)$ processes with d_{12} ranging between -0.4 and 0.0.

2.2 Estimation

The simplest estimation procedure for (1) is maximum likelihood estimation in the frequency domain, see Fox and Taqqu (1986). This is because the frequency domain likelihood function can easily accommodate fractional integration at a non-zero frequency. Equation (9) allows specification of the log likelihood function

$$\ln L(\mathbf{\Pi}; p(\lambda_j)) = -\frac{1}{2\pi} \sum_{j=1}^{T/2} \frac{p(\lambda_j)}{I(\lambda_j)} \tag{10}$$

where $\mathbf{\Pi} = \{d_0, d_k, \phi_1, \phi_2, \ldots, \phi_p, \theta_1, \theta_2, \ldots, \theta_q, \sigma^2\}$ is a $3 + p + q \times 1$ parameter vector and $p(\lambda_j)$ is the periodogram estimated using conventional methods. As shown in Fox and Taqqu (1986), maximisation of (10) results in asymptotically normal estimates of $\mathbf{\Pi}$. Cheung and Diebold (1994) show that the small sample properties obtained using (10) and the small sample properties obtained using the exact time domain maximum likelihood estimator due to Sowell (1992) are similar in samples of the size considered here. However, the small sample properties of (10) when there are two fractional integration parameters are unknown. The next section investigates this issue using simulation methods.

3 Monte Carlo Experiment

3.1 Simulation Design

The Monte Carlo experiment investigates the adequacy of the likelihood function in (10), specified using (9), when estimating the parameters of the pure and general SARFIMA models. Small sample bias and root mean squared error (RMSE) statistics are reported. A sample size of $T = 200$ observations is implemented which corresponds to 50 years of quarterly data. This is approximately equivalent to the available sample sizes for the quarterly US and UK unemployment rates investigated in Section 4. A total of $R = 1000$ replications are performed. The data are simulated by expanding the fractional differencing operators $(1-L^k)^{d_k}$ and $(1-L)^{d_0}$ up to a lag of $l = 1000$ and then simulating an AR$(l+p)$ process, using if necessary an MA(q) error process, $\theta(L)\varepsilon_t$; see Martin and Wilkins (1999) for more details on a long lag simulator for the ARFIMA(p,d,q) model.

Two data generating processes (DGPs) are considered. The DGP for the SARFIMA$(0, d_0, d_4, 0)$ experiment is

$$(1-L^4)^{d_4}(1-L)^{d_0}y_t = \varepsilon_t \tag{11}$$

with parameter values $d_0 = \{-0.4, -0.2, 0.2, 0.4\}$ and $d_4 = \{0.0, 0.2, 0.4\}$. Positive correlation at seasonal frequencies suggests that only positive values for d_4 should be considered, but negative values for d_0 are included because it is commonplace to first difference data before estimation. (The argument here is if the level of integration of the process at the zero frequency is less than one, then the corresponding integration level of the first differenced series will be negative.) The second DGP is the SARFIMA$(1, d_0, d_4, 1)$ model, which can be expressed as

$$(1-\phi_1 L)(1-L^4)^{d_4}(1-L)^{d_0}y_t = (1+\theta_1 L)\varepsilon_t \tag{12}$$

with parameter values $d_0 = \{-0.3, 0.3\}$, $d_4 = \{0.2, 0.4\}$, $\phi_1 = 0.7$ and $\theta_1 = 0.3$. All computations are undertaken using Gauss version 3.2. The Gauss code is available from the author on request.

3.2 Simulation Results

The results for the twelve different combinations of d_0 and d_4 are reported in Table 1 for the simple fractional seasonal noise model in (11). For all values of the fractional differencing parameter (at the zero or quarterly frequencies) the small sample bias levels are below an absolute level of 0.09 while the RMSE values are below 0.12, and most are below 0.09. This clearly shows that (10) has no difficulty identifying two fractional differencing parameters. Note that these results are quite consistent with the results obtained using the frequency domain maximum likelihood estimator for the non-seasonal ARFIMA$(0, d, 0)$ model, see Cheung and Diebold (1994) for example.

Table 2 reports the results for the four different combinations of parameter values for the more complicated SARFIMA$(1, d_0, d_4, 1)$ model in (12). Once again, the small sample properties of the frequency domain maximum likelihood estimator are quite good, and are comparable to the small sample properties obtained for the simpler non-seasonal ARFIMA$(1, d, 1)$ model, see Martin and Wilkins (1999) for example. There are one or two concerns however, and these principally revolve around the small sample bias on d_0, which remains greater than -0.15 for all four specifications. The small sample bias levels for the ARMA parameters are much lower in comparison and the RMSE values for all parameters seem reasonable. To investigate the bias levels on the fractional parameters in more detail, the simulations were re-performed using a sample size of $T = 2000$ observations with $R = 100$ replications. A smaller number of replications were performed because of the increased computational time associated with the larger sample. For this new specification, the bias did not exceed an absolute value of 0.04, while the RMSE only exceeded 0.10 once with a value of 0.114 for d_0 in the $d_0 = 0.3, d_4 = 0.4, \phi_1 = 0.7, \theta_1 = 0.3$ specification. These results indicate that the larger sample behaviour of the frequency domain maximum likelihood estimator for the SARFIMA model approaches the asymptotic results in Fox and Taqqu (1986).

In general, the results of the Monte Carlo experiments suggest that the frequency domain maximum likelihood estimator is a consistent estimator of the parameters in the SARFIMA model. The small sample properties of the estimator would also appear to be quite acceptable, given the complexity of the model. With these results in mind, the SARFIMA model is now applied to modelling long run behaviour in unemployment rates.

4 Long Memory in Unemployment Rates

4.1 Time Series Modelling of Unemployment Rates

Despite the volume of research that has been undertaken into unemployment rate behaviour over the recent past, there is still much debate as to the validity of the natural rate and hysteresis hypotheses; see for example the symposium on the natural rate of unemployment in a recent issue of The Journal of Economic Perspectives (Winter 1997). This debate can be traced back to Blanchard and Summers (1986, 1987) who develop models for hysteresis in unemployment rates using the paradigm of wage setting in insider-outsider models before finding evidence of hysteresis effects in European unemployment rates. More recently, the existence of hysteresis is one interpretation that can be placed on the results of Crosby and Olekalns (1996) who find that the NAIRU for Australia has consistently increased since 1959. Mitchell and Wu (1995) find that quarterly unemployment rates for the OECD countries exhibit behaviour that is consistent with fractionally integrated variables. However, their estimates of the level of integration differ from country to country and there is no clear support for either hypothesis.

Consider the stationary filtered time series representation of the unemployment rate, U_t^*, that can be modelled as the ARMA(p, q) process

$$U_t^* = \Phi'(L)^{-1}(\alpha + \Theta'(L)\epsilon_t) \tag{13}$$

where $\Phi'(L)$ and $\Theta'(L)$ are lag operator polynomials of degree p' and q' respectively, α is a constant and $\epsilon_t \sim \text{iid}(0, \sigma^2)$. In its original form, unemployment, U_t, follows a difference stationary model if

$$\Delta U_t = \Phi'(L)^{-1}(\alpha + \Theta'(L)\epsilon_t) \tag{14}$$

where $U_t^* = \Delta U_t$ with $\Delta = (1-L)$ denoting integer differencing; and a trend stationary model if

$$U_t = \Phi'(L)^{-1}(\alpha + \beta t + \Theta'(L)\epsilon_t) \tag{15}$$

where $U_t^* = U_t - \Phi'(L)^{-1}\beta t$ with t denoting a linear time trend. Combining (14) and (15) and assuming that the level of integration of unemployment rates is not $I(2)$ or greater, it is possible to obtain the more general model

$$U_t = \left(1 - I_{(0,1)}\right)U_{t-1} + \Phi'(L)^{-1}\left(\alpha + I_{(0,1)}\beta t + \Theta'(L)\epsilon_t\right) \tag{16}$$

where $I_{(0,1)}$ is the indicator function, defined according to

$$I_{(0,1)} = \begin{cases} 1 & d = 0 \\ 0 & d = 1 \end{cases} \tag{17}$$

Equation (16) represents a compact formulation of the two models that can be verified empirically through integer based integration testing. This approach has the inherent restriction that the level of integration is restricted to being either $I(0)$ or $I(1)$. As noted in the introduction, these two cases exhibit the extreme behaviour where shocks either die out almost instantaneously or persist with a unit effect into the infinite future. In contrast to this, the ARFIMA and SARFIMA models allow varying "shades of stationarity" to be modelled through d_0 and d_k and therefore varying responses to once off exogenous shocks, while simultaneously modelling short run deviations around the long run trajectory through the ARMA parameters. For the ARFIMA model, the empirical model for unemployment rates can be expressed in a manner similar to that in (13) but now $U_t^* = \nabla^d U_t$ where ∇^d is taken to represent the zero frequency fractional differencing filter. For the SARFIMA model, two fractional levels of integration must be taken into account. The stationary filtered time series representation for unemployment rates is now $U_t^* = \nabla^{d_k}\nabla^{d_0}U_t$ where ∇^{d_k} is the seasonal frequency fractional differencing filter. The advantage of these specifications is that more flexibility is allowed into the modelling and estimation procedure.

4.2 Empirical Results

The empirical analysis uses seasonally unadjusted and adjusted quarterly US and UK unemployment rates for the period 1947:Q1 to 2002:Q2 ($T = 222$ observations) obtained from the Yearbook of Labour Statistics, The International Labour Office. The seasonally adjusted data were obtained using the ratio to moving average de-seasonalisation procedure. The seasonally unadjusted time series for both countries are graphed in panels a and b of Figure 2. Panels c and d of Figure 2 contain the periodograms of the first differenced series. Note that both periodograms have significant spikes at a frequency of 0.25, corresponding to the seasonal cycle of 4 quarters.

A visual inspection of Figure 2 does appear to suggest that shocks to unemployment rates take a while to dissipate. For example, apart from the obvious seasonality, the US time series is strongly characterised by what looks like a business cycle that consistently causes a sharp rise in unemployment rates before a more gradual fall. For the UK, there was a very sharp rise in unemployment rates in the early 1980s, followed by a sharp fall in the late 1980s. This type of long run behaviour is consistent with the behaviour that can be captured with the fractional model.

4.2.1 Estimation of ARFIMA and SARFIMA Models

Both ARFIMA and SARFIMA models are estimated for US and UK unemployment rates to facilitate comparison between the seasonal and non-seasonal models. In all cases, estimation is conducted for the first differenced series, thus avoiding the problems of estimating nonstationary models. Hypothesis testing concerning the level of integration at the zero frequency is also conducted on the first differenced series. For example, to test the hypothesis that $d_0 = 1$ in the original series involves testing the hypothesis that $d_0 - 1 = 0$ in the first differenced series. This avoids the problem of a non-standard limiting distribution in the hypothesis testing procedure. The results obtained using the ARFIMA models are discussed first.

The estimated ARFIMA(p, d, q) models for $p, q = 0, 1$ are reported in Table 3, while the maximised value of the log likelihood function and the Akaike (AIC) and Schwarz (SIC) information criteria are reported in Table 4. The conventional frequency domain likelihood function was implemented for the ARFIMA models. For reasons of comparison, the ARFIMA models were estimated for both the seasonally unadjusted and adjusted series. Concentrating first on the unadjusted series, the AIC chosen model for US unemployment rates is the ARFIMA$(0, d, 1)$ model with a d estimate of 0.635 that is significantly less than one at conventional levels of significance. The AIC chosen model for UK unemployment rates is the ARFIMA$(1, d, 0)$ model with a d estimate of 1.517 that is significantly greater than one at conventional levels of significance. These estimates would appear to be consistent with a visual inspection of the data and are also consistent with the Geweke and Porter-Hudak (GPH) (1983) estimates of d which are (using the first $m = 15$ frequencies and with estimated standard errors in parentheses); 0.571 (0.181) for the US and 1.334 (0.245) for the UK. Also note that all point estimates of d for US unemployment rates are below one and all point estimates for UK unemployment rates, with one exception, are above one.

For the seasonally adjusted data, the estimate of d in the AIC chosen ARFIMA$(1, d, 0)$ model for US unemployment rates is 0.227, and the estimate of d for the seasonally adjusted UK unemployment rates in the AIC chosen ARFIMA$(0, d, 1)$ model is 1.028. The GPH estimates of d are remarkably similar to those obtained for the unadjusted series, and are; 0.571 (0.178) for the US and 1.332 (0.253) for the UK. The seasonal adjustment procedure appears to have neither increased or decreased the estimates of d uniformly across model or country. There are however, one or two noticeable differences in parameter values, the ARFIMA$(1, d, 1)$ model being the best example of this.

The estimated SARFIMA(p, d_0, d_4, q) models for $p, q = 0, 1$ for US and UK unemployment rates are reported in Table 5, and the maximised value of the log likelihood function and the Akaike and Schwarz information criteria are reported in Table 6. SARFIMA models were estimated only for the seasonally unadjusted series. For the US, the AIC chosen model is the SARFIMA$(1, d_0, d_4, 0)$ model with a level of integration at the zero frequency, 0.463, that is reasonably consistent with the level of integration obtained using the AIC chosen model for the unadjusted data in Table 3. The level of integration at the seasonal frequency for this model is 0.428, and this is very consistent across model. This is no doubt due to the fact that d_0 and the ARMA parameters are largely independent of this band of the spectrum and so the specification of these parameters has little impact on d_4. In contrast, it is now well known that the estimate of the level of integration at the zero frequency is sensitive to the specification of the ARMA parameters and this is why the estimates of d_0 fluctuate more widely than the estimates of d_4. Invoking the mean reversion condition in (6) yields $d_0 + d_4 = 0.891$, which is insignificantly less than one. Thus, the conditions for mean reversion are violated when incorporating long memory at a quarterly frequency in US unemployment rates.

The AIC chosen model for UK unemployment rates is also the SARFIMA$(1, d_0, d_4, 0)$ model. Here the point estimates of $d_0 = 0.492$ and $d_4 = 0.496$ are both individually well below one, but once again the combined effect of $d_0 + d_4 = 0.988$ implies a non-mean reverting DGP. This result, unlike that for the US, is consistent with the ARFIMA estimates in Table 3 which also imply a non-stationary DGP for the UK.

As a final estimation check, a variant of the GPH estimator is applied to the data. This involves estimating the GPH regression over the quarterly seasonal band of the spectrum only. The regression model can be expressed as

$$\ln p(\lambda_j) = \delta - d_4 \ln\{|1 - e^{-i4\lambda_j}|^2\} + \varepsilon_j \tag{18}$$

where j indexes frequency over the seasonal band. The determination of the length of this band is, of course, quite arbitrary. Different bandwidths were experimented with but the results for only two are presented as these are representative of the results in general. When $j = 43, 44, \ldots, 57$, representing a truncation parameter of approximately $m = T^{0.5}$, the estimates obtained using (18) are; $d_4 = 0.765$ (0.104) for the US and $d_4 = 0.511$ (0.136) for the UK.

Expanding the bandwidth to $j = 40, 41, \ldots, 60$, the estimates are; $d_4 = 0.673$ (0.111) for the US and $d_4 = 0.475$ (0.128) for the UK. Note that the UK estimates of d are very similar to those obtained using the frequency domain maximum likelihood estimator.

Unfortunately, these GPH estimates must be interpreted with some caution. While the asymptotic properties will follow from those established in earlier research, see for example Geweke and Porter-Hudak (1983), the small sample properties are unknown, although Porter-Hudak (1990) presents some results for a small scale Monte Carlo experiment on a related issue. In addition, there is potential for any ARMA parameters in the DGP to bias the estimate of d_4. This might be the case for the US for example. The GPH results do however reinforce the estimation results presented earlier for the frequency domain maximum likelihood estimator.

The negative small sample biases on the fractional parameters reported in the simulation results in Table 2 suggest that the maximum likelihood estimation procedure tends to underestimate the true level of integration at the zero and seasonal frequencies for a sample size of $T = 200$ observations. Using the small sample bias results in Table 2 and the estimation results in Table 5 as a guide, another simulation experiment was performed for the SARFIMA$(1, d_0, d_4, 0)$ model with parameter values $d_0 = 0.6, d_4 = 0.5$ and $\phi = 0.674$ corresponding approximately to the US parameter estimates and $d_0 = 0.6, d_4 = 0.6$ and $\phi = 0.858$ corresponding approximately to the UK. The DGP parameter values for d_0 and d_4 have been increased slightly with respect to the estimates in Table 5 to allow for any possible small sample bias when estimating the parameters of this particular model. As the values for d_0 imply covariance non-stationary DGPs, the simulated processes were first differenced before estimation. All other details are the same as in Section 3. For the first "US" DGP, the bias (RMSE) values are, for ϕ_1; 0.059 (0.143), for d_0; -0.113 (0.193) and for d_4; -0.026 (0.075). For the second "UK" DGP, the bias (RMSE) values are, for ϕ_1; -0.012 (0.086), for d_0; -0.031 (0.130) and for d_4; -0.017 (0.075). This suggests that the estimates in Table 5 only marginally understate the true level of long memory at the zero and seasonal frequencies and certainly less than that implied by Table 2 for the

SARFIMA$(1, d_0, d_4, 1)$ model.

Overall, the evidence suggests that US and UK unemployment rates are both non-mean reverting, although this appears to be somewhat stronger for the UK. For the AIC chosen SARFIMA models, the individually estimated levels of integration are significantly different from zero and one at conventional levels, thus supporting the proposition that unit root type specifications are mis-specified. More long memory behaviour is found to be present in the estimated SARFIMA models than in the ARFIMA models. This result is not surprising given the strong seasonality of the time series. Finally, the results imply that the hysteresis hypothesis is a more empirically valid model of unemployment rate behaviour than the natural rate hypothesis.

4.2.2 Impulse Response Functions

Given that there are two long memory parameters in the DGPs for US and UK unemployment rates, statements about the long run behaviour of the processes and the degree of mean reversion present can be made more clearly by examining the fractional impulse response functions. The AIC and SIC chosen models of the previous section are used to calculate CIRFs for US and UK unemployment rates. The impulse responses are calculated over 40 quarters, or 10 years and are graphed in Figure 3.

Panels *a* and *c* of Figure 3 graph the impulse responses for the US, while Panels *b* and *d* graph the impulse responses for the UK. Note that because the SARFIMA models were estimated only for the seasonally unadjusted data, the impulse responses using the SARFIMA model for the US in Panels *a* and *c* are the same, as are the SARFIMA impulse responses for the UK in Panels *b* and *d*. They are included in both panels for each country for reasons of comparison. The ARFIMA based impulse responses differ in each panel due to the seasonally adjusted or unadjusted data being used. Lastly, where there is disagreement between the model selection criteria in Tables 4 and 6, the impulse responses are appropriately labelled to indicate whether the CIRF is based on the AIC or the SIC chosen model.

The following points can be made. For the US there is somewhat conflicting evidence as to the mean revertibility of unemployment rates. The SARFIMA based CIRF clearly suggests the US data is non-mean reverting, whereas two of the three ARFIMA based CIRFs suggest mean reversion. Of course, this just reflects the conflicting results in Tables 3 and 5 between the ARFIMA and SARFIMA models for the US unemployment data. For the UK, the picture is far clearer. All of the calculated CIRFs indicate the non-mean reverting nature of the DGP for UK unemployment rates. In particular, for the SARFIMA-SIC chosen model, the DGP appears to be quite explosive. This is not surprising, as for this model $d_0 + d_4 = 1.847$.

5 Conclusion

This paper has addressed the issue of obtaining an estimate of the level of integration of the DGP for a time series process at a frequency other than that of zero. It has been justified that given the widespread use of seasonal data in empirical applications, it is only natural that long memory relationships at other frequencies be examined because of the useful economic information they may contain. A general model has been utilised, a straightforward estimation procedure using frequency domain tools has been suggested and small sample results reported for a range of DGPs. Due to the strong seasonality present, an application has also been made to modelling US and UK unemployment rates. Recent applied time series work has attempted to distinguish between hysteresis and natural rate models of unemployment rate behaviour using the integer based integration dichotomy. This paper has improved upon the methodological approach by allowing a wider range of long run behaviour to be modelled by using the ARFIMA and SARFIMA methodologies.

It has been found that the SARFIMA model is a useful one for capturing long memory relationships in time series data. The estimates of the level of integration in US and UK unemployment rates have been found to be significantly different from the relatively restrictive $I(0)$ and $I(1)$ cases at the zero and seasonal frequencies. This result has implications for the policy management of unemployment in these two countries. Under the natural rate model, unemployment is able to revert to its long run equilibrium level, and so, government intervention to assist the adjustment process is unnecessary from a long run point of view. The hysteresis model asserts that the short run equilibrium level is directly dependent on past actual levels of unemployment. Increases in actual levels of unemployment increase the equilibrium level, with no tendency for reversion to the original level. Accordingly, the finding of hysteresis effects in unemployment rates supports the argument for more activist government policies to reduce actual unemployment, and therefore, the equilibrium level. The findings in this paper support the proposition that US and UK unemployment rates are best characterised within this hysteresis framework.

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Table 1: Small Sample Summary Statistics for the SARFIMA Maximum Likelihood Estimator: SARFIMA$(0, d_0, d_4, 0)$ DGP with $T = 200$ and $R = 1000$

DGP	$d_0 = -0.4$	$d_4 = 0.0$	$d_0 = -0.4$	$d_4 = 0.2$	$d_0 = -0.4$	$d_4 = 0.4$
Bias	-0.021	0.025	-0.005	-0.070	-0.007	-0.076
RMSE	0.066	0.031	0.064	0.092	0.066	0.106
	$d_0 = -0.2$	$d_4 = 0.0$	$d_0 = -0.2$	$d_4 = 0.2$	$d_0 = -0.2$	$d_4 = 0.4$
Bias	-0.024	0.025	-0.007	-0.071	-0.007	-0.076
RMSE	0.067	0.030	0.064	0.092	0.066	0.106
	$d_0 = 0.2$	$d_4 = 0.0$	$d_0 = 0.2$	$d_4 = 0.2$	$d_0 = 0.2$	$d_4 = 0.4$
Bias	-0.025	0.025	-0.004	-0.070	0.004	-0.075
RMSE	0.067	0.030	0.064	0.092	0.068	0.106
	$d_0 = 0.4$	$d_4 = 0.0$	$d_0 = 0.4$	$d_4 = 0.2$	$d_0 = 0.4$	$d_4 = 0.4$
Bias	-0.022	0.025	0.005	-0.070	0.028	-0.083
RMSE	0.066	0.030	0.067	0.092	0.084	0.112

Note: The simulation design is outlined in Section 3.1.

Table 2: Small Sample Summary Statistics for the SARFIMA Maximum Likelihood Estimator: SARFIMA$(1, d_0, d_4, 1)$ DGP with $T = 200$ and $R = 1000$

DGP	$d_0 = -0.3$	$d_4 = 0.2$	$\phi_1 = 0.7$	$\theta_1 = 0.3$
Bias	-0.177	-0.086	0.092	0.057
RMSE	0.267	0.104	0.180	0.112
	$d_0 = 0.3$	$d_4 = 0.2$	$\phi_1 = 0.7$	$\theta_1 = 0.3$
Bias	-0.180	-0.097	0.098	0.026
RMSE	0.274	0.110	0.187	0.105
	$d_0 = -0.3$	$d_4 = 0.4$	$\phi_1 = 0.7$	$\theta_1 = 0.3$
Bias	-0.166	-0.100	0.082	0.054
RMSE	0.273	0.128	0.196	0.112
	$d_0 = 0.3$	$d_4 = 0.4$	$\phi_1 = 0.7$	$\theta_1 = 0.3$
Bias	-0.178	-0.158	0.093	-0.009
RMSE	0.301	0.183	0.237	0.125

Note: The simulation design is outlined in Section 3.1.

Table 3: Frequency Domain Maximum Likelihood Estimates of ARFIMA(p, d, q) Models for Unemployment Rates

ARFIMA Model	$\widehat{d}$	$\widehat{\phi}_1$	$\widehat{\theta}_1$	$\widehat{\sigma}^2$
		Unadjusted: United States		
$(0, d, 0)$	0.873 (0.097)	-	-	0.541 (0.050)
$(1, d, 0)$	0.626 (0.178)	0.331 (0.187)	-	0.516 (0.049)
$(0, d, 1)$	0.635 (0.095)	-	0.404 (0.087)	0.490 (0.047)
$(1, d, 1)$	0.612 (0.136)	0.042 (0.216)	0.389 (0.122)	0.490 (0.047)
		Unadjusted: United Kingdom		
$(0, d, 0)$	1.193 (0.072)	-	-	0.178 (0.029)
$(1, d, 0)$	1.517 (0.115)	-0.456 (0.112)	-	0.164 (0.027)
$(0, d, 1)$	1.507 (0.180)	-	-0.381 (0.167)	0.171 (0.028)
$(1, d, 1)$	0.623 (0.227)	0.939 (0.060)	-0.448 (0.174)	0.166 (0.028)
		Adjusted: United States		
$(0, d, 0)$	1.052 (0.093)	-	-	0.309 (0.038)
$(1, d, 0)$	0.227 (0.197)	0.837 (0.135)	-	0.293 (0.036)
$(0, d, 1)$	0.947 (0.127)	-	0.128 (0.123)	0.305 (0.037)
$(1, d, 1)$	0.344 (0.472)	0.789 (0.244)	-0.079 (0.266)	0.293 (0.037)
		Adjusted: United Kingdom		
$(0, d, 0)$	1.204 (0.083)	-	-	0.216 (0.031)
$(1, d, 0)$	1.116 (0.125)	0.144 (0.149)	-	0.213 (0.031)
$(0, d, 1)$	1.028 (0.085)	-	0.354 (0.103)	0.205 (0.036)
$(1, d, 1)$	1.073 (0.104)	-0.158 (0.213)	0.440 (0.145)	0.204 (0.031)

Note: Estimated standard errors are in parentheses.

Table 4: The Akaike and Schwarz Information Criteria and Maximised Value of the Log Likelihood Function for the ARFIMA(p, d, q) Models in Table 3

ARFIMA Model	$L(\mathbf{\Pi}; P(\lambda_j))$	AIC	SIC
	Unadjusted: United States		
$(0, d, 0)$	-22360.915	44725.830	44732.627
$(1, d, 0)$	-22358.757	44723.513	44733.708
$(0, d, 1)$	-22356.002	44718.004†	44728.199‡
$(1, d, 1)$	-22355.983*	44719.967	44733.560
	Unadjusted: United Kingdom		
$(0, d, 0)$	-22298.989	44601.978	44608.775
$(1, d, 0)$	-22293.718*	44593.435†	44603.630‡
$(0, d, 1)$	-22296.085	44598.171	44608.365
$(1, d, 1)$	-22295.686	44599.373	44612.965
	Adjusted: United States		
$(0, d, 0)$	-22329.567	44663.135	44669.931‡
$(1, d, 0)$	-22327.991	44661.981†	44672.176
$(0, d, 1)$	-22329.171	44664.343	44674.537
$(1, d, 1)$	-22327.935*	44663.870	44677.463
	Adjusted: United Kingdom		
$(0, d, 0)$	-22309.434	44622.867	44629.664
$(1, d, 0)$	-22308.903	44623.806	44634.000
$(0, d, 1)$	-22306.915	44619.831†	44630.025‡
$(1, d, 1)$	-22306.612*	44621.225	44634.817

Note: The AIC is calculated according to $-2 \ln L(\cdot) + 2n$ and the SIC is calculated according to $-2 \ln L(\cdot) + 2n \ln T$ where n denotes the number of model parameters equal to $2+p+q$. $\star$ denotes maximised value of the log likelihood function, $\dagger$ denotes minimum AIC and $\ddagger$ denotes minimum SIC.

Table 5: Frequency Domain Maximum Likelihood Estimates of SARFIMA(p, d_0, d_4, q) Models for Unemployment Rates

SARFIMA Model	$\widehat{d}_0$	$\widehat{d}_4$	$\widehat{\phi}_1$	$\widehat{\theta}_1$	$\widehat{\sigma}^2$
		United States			
$(0, d_0, d_4, 0)$	1.184 (0.111)	0.407 (0.066)	-	-	0.314 (0.038)
$(1, d_0, d_4, 0)$	0.463 (0.180)	0.428 (0.067)	0.674 (0.134)	-	0.280 (0.036)
$(0, d_0, d_4, 1)$	0.904 (0.146)	0.404 (0.069)	-	0.294 (0.122)	0.299 (0.037)
$(1, d_0, d_4, 1)$	0.276 (0.246)	0.432 (0.070)	0.716 (0.143)	0.164 (0.146)	0.275 (0.035)
		United Kingdom			
$(0, d_0, d_4, 0)$	1.377 (0.088)	0.470 (0.076)	-	-	0.106 (0.022)
$(1, d_0, d_4, 0)$	0.492 (0.130)	0.496 (0.078)	0.858 (0.085)	-	0.100 (0.021)
$(0, d_0, d_4, 1)$	1.447 (0.185)	0.464 (0.077)	-	-0.079 (0.178)	0.106 (0.022)
$(1, d_0, d_4, 1)$	0.721 (0.259)	0.480 (0.078)	0.815 (0.104)	-0.216 (0.196)	0.100 (0.021)

Note: Estimated standard errors are in parentheses.

Table 6: The Akaike and Schwarz Information Criteria and Maximised Value of the Log Likelihood Function for the SARFIMA(p, d_0, d_4, q) Models in Table 5

SARFIMA Model	$L(\mathbf{\Pi}; P(\lambda_j))$	AIC	SIC
	United States		
$(0, d_0, d_4, 0)$	-22329.115	44664.230	44674.425
$(1, d_0, d_4, 0)$	-22324.090	44656.180‡	44669.773‡
$(0, d_0, d_4, 1)$	-22327.020	44662.040	44675.633
$(1, d_0, d_4, 1)$	-22323.539*	44657.078	44674.069
	United Kingdom		
$(0, d_0, d_4, 0)$	-22268.648	44543.295	44553.490‡
$(1, d_0, d_4, 0)$	-22266.907	44541.814‡	44555.406
$(0, d_0, d_4, 1)$	-22268.547	44545.093	44558.686
$(1, d_0, d_4, 1)$	-22266.247*	44542.495	44559.486

Note: The AIC is calculated according to $-2 \ln L(\cdot) + 2n$ and the SIC is calculated according to $-2 \ln L(\cdot) + 2n \ln T$ where n denotes the number of model parameters equal to $3+p+q$. $\star$ denotes maximised value of the log likelihood function, † denotes minimum AIC and ‡ denotes minimum SIC.

Figure 1: Time and Frequency Domain Properties of the Pure SARFIMA$(0, 0, d_k, 0)$ Model

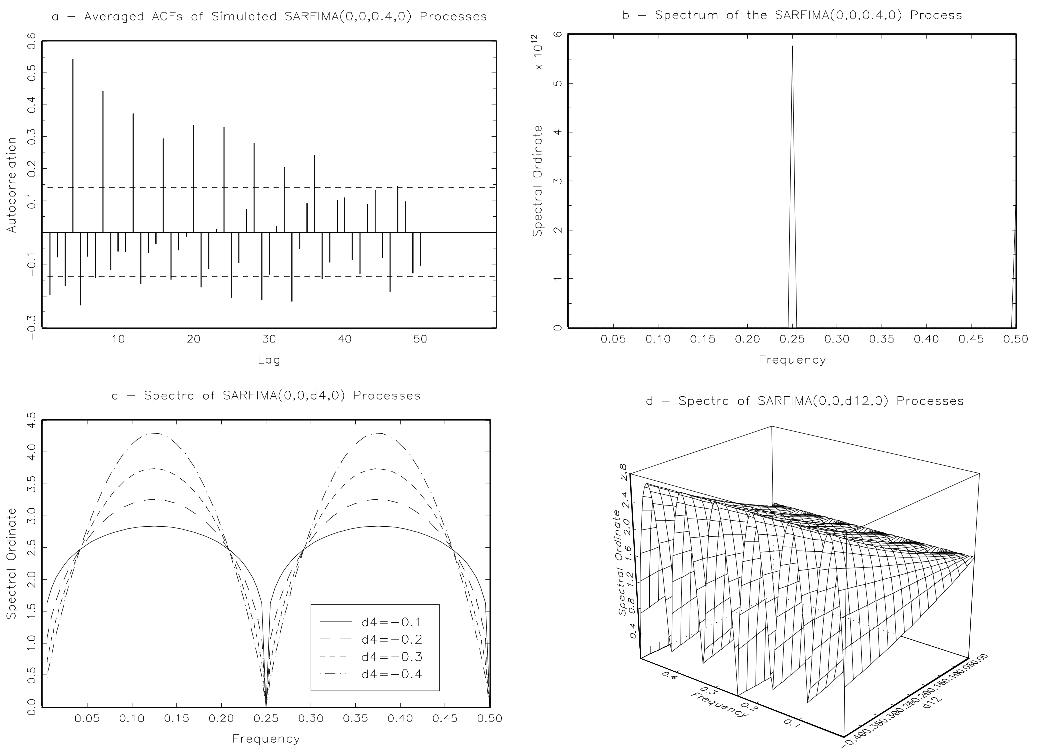

Figure 2: US and UK Unemployment Rates and Periodograms of the First Differences: Seasonally Unadjusted Data

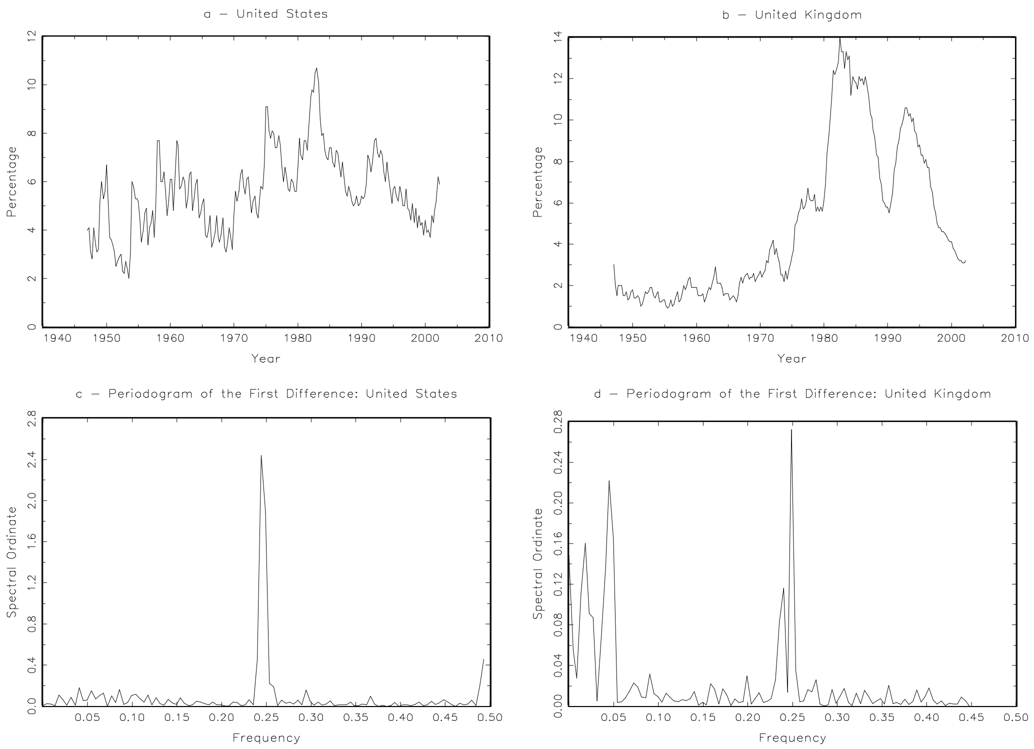

Figure 3: Fractional Impulse Response Functions for US and UK Unemployment Rates