

# Optimal Investment with Default Risk

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## Abstract

In this paper, we investigate how investors who face both equity risk and credit risk would optimally allocate their financial wealth in a dynamic continuous-time setup. We model credit risk through the defaultable zero-coupon bond and solve the dynamics of its price after pricing it. Using stochastic control methods, we obtain a closed-form solution to this investment problem and characterize its variation with respect to different factors in the economy. We find that non-zero recovery rate of the credit-risky bond affects investors' decision in a fundamental way. Because of this, investors try to time the market conditions in their decision making process. It also induces hedging term in this setup of otherwise deterministic investment opportunity set. Through numerical examples, we show that the inclusion of credit market is able to enhance investors' welfare.

*Key Words and Phrases:* credit risk, asset allocation, welfare analysis, corporate bond

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# 1 Introduction

Default risk is no stranger to investors. The past few years have seen many investment-grade companies slide into junk status in the U.S.. Default risk not only affects the corporate world, but also strikes the sovereign side. Japan was once downgraded three times by leading credit rating agencies and Argentina declared bankruptcy and defaulted on its debt obligations. On the other hand, during the years of bear equity market, despite the bleak investment climate worldwide, the fixed income markets (including both treasury bonds and corporate bonds) have shown robustness, giving “conservative” investors decent rates of return on their investment.

Surprisingly, finance theory says little about how to optimally allocate assets when default risk cannot be neglected anymore. The “classical” approach of dynamic asset allocation (or optimal consumption and portfolio choice) literature is to study a representative agent dynamically allocating his/her wealth into several asset categories, usually consisting of a (risk-free) bond, and equity (stock index). This is often done by specifying the dynamics of stock (index) price and other relevant variables, and by employing the Hamilton-Jacobi-Bellman (HJB hereafter) equation to solve the problem.<sup>1</sup> The main theme in this literature is how to efficiently allocate wealth among several financial assets in order to achieve utility maximization in the presence of different risk-return combinations.

Default risk, however, has not been studied in this context.<sup>2</sup> Default risk is an intrinsic factor in the fixed-income market. Literature on fixed-income market mainly deals with stochasticity of interest rate or its risk premium, not with default risk. Rational investors would hedge against or speculate on this risk in addition to exposing to equity markets. Thus incorporating fixed-income markets in the analysis not only recognizes the current reality of the financial world, but also contributes to the literature of asset allocation by explicitly investigating the impact of default risk on the investment decisions.<sup>3</sup> The traditional approach mainly focuses on the changes of investment opportunity as a result of market risk (or equity risk in the case of deterministic risk-free interest rate). During the past couple of years, sophisticated investors know well that bond markets are potentially profitable, especially in the downturn of stock markets. Investors look for returns from their corporate bond holdings that could be higher than from stock or money markets investment. Some institutional

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<sup>1</sup>Alternatively, such problems can be solved through martingale approach advanced by Cox and Huang (1989) and Karatzas *et al.* (1987).

<sup>2</sup>One exception is a paper by Walder (2001). He uses affine state variable techniques to investigate how to invest among a treasury bond and a portfolio of corporate zero-coupon bonds in addition to a money market account. However, he does not consider equity market while equity investment is an integral part of the portfolios of many institutional investors. We formally include equity market into the asset allocation model. Whereas Walder resorts to affine functionals of unidentified state variables, the current paper explicitly specifies the state variables and gives intuition for the impact of such tangible factors on investors’ portfolio decision.

<sup>3</sup>Default risk can be roughly thought of as synonym of credit risk in a simplified model where credit migration risk and correlation risk are absent.

investors, like pension funds, are also raising their allocations to this sector. For example, Calpers, the largest US pension fund, doubled the amount of money it invests in high-yield bonds during a market downturn.<sup>4</sup> On the other hand, the presence of credit risk can potentially expand investors' risk and return frontier, and provide an opportunity to enhance their economic welfare by achieving greater diversification.

To formally investigate this problem, we study the optimal investment decision by investors in the framework of Merton (1971). We model an investor with power utility function trying to maximize her terminal utility in a partial equilibrium setup. There are two kinds of financial markets she can invest in: one is the equity market, the other the credit market. This is to reflect the realistic investment opportunities investors face. To facilitate analysis, we focus on the corporate zero-coupon bonds market, even though credit markets include coupon bonds and various credit derivatives. The very nature of defaultable bonds makes modeling rather involved even with the zero-coupon bond, the simplest security in credit market. Following the so-called "reduced-form" approach, we start from specifying information structure of the economy and derive the pricing equation of the defaultable zero-coupon bond using the recovery scheme of market value (RMV) of Duffie and Singleton (1999).

For simplicity, we assume a constant riskless interest rate and model the credit spread as an Ornstein-Uhlenbeck process. As a result, the dynamics of the defaultable zero-coupon bond can be readily derived. Literature on corporate bond pricing usually assumes zero-recovery to simplify analysis.<sup>5</sup> We show that this modeling feature neglects important aspects in defaultable bond pricing. Non-zero recovery rate induces an adjustment in the drift term under the risk-neutral measure. It resembles a dividend rate process even though the security is a zero-coupon bond. More importantly, this adjustment term is stochastic and has similar dynamics as that of credit spread under the assumption of constant writedown rate. Consequently, it becomes one of the state variables in the investor's asset allocation problem. Besides this uncertainty, we assume there are two correlated risk factors in the form of Wiener process driving the two markets respectively.

We solve investor's optimal investment problem and derive the closed-form solution. The optimal investment weight of equity market consists of only the myopic demand, since the investment opportunity set in this economy is deterministic. It is not the case for the defaultable bond investment, however. Because of the stochasticity of the risk premium in defaultable bond, there is hedging demand by investors for that. If the recovery rate is zero, the risk premium will be deterministic and no hedging demand exists. This highlights the importance of recovery rate in the investment in defaultable securities. Unlike many papers in the literature, a state variable (besides time) also appears in the optimal weights. As a result, investors in this model try to time the market and make investment decisions accordingly.

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<sup>4</sup>*Financial Times*, Jan 14, 2002

<sup>5</sup>For example, Jarrow *et al.* (2005) and Walder (2001).

In order to get a sense of how large the effects of theoretical result are, we perform numerical analysis by specifying relevant parameter values. We either adopt other authors' estimates of some parameters or specify the rest of them by ourselves. In doing so, we also provide some robustness tests by computing interesting quantities over economically reasonable intervals of some parameters. We find that behaviors of myopic demands and those of hedging demands can be very different. For example, hedging demands may not decrease as investors' risk-aversion increases. We verify that by investing in credit market, investors can achieve significant welfare improvement. This result is quite robust with respect to various parameter profiles.

The rest of the paper is organized as follows. Section 2 presents the pricing result of the defaultable zero-coupon bond using RMV assumption, formally describes the risk structure of the economy and derives the SDEs of equity and defaultable bond under relevant measures. Section 3 solves the optimal investment problem in closed-form. Implications of the analytic result for asset allocation are discussed in Section 4. Section 5 carries out numerical exercises by taking parameter values from literature. Section 6 summarizes the results and concludes the paper. Proofs are collected in Section 7.

## 2 A Model with Defaultable Bond

In the extant asset allocation literature, only market risk or cognitive risk associated with market risk (in the incomplete information case) is considered. Credit risk is almost totally ignored. Credit risk is another kind of risk investors have to face on top of market risk. Simply put, credit risk can be identified with default risk, the possibility that a counterparty in a financial contract fails to fulfil a contractual commitment. Many financial instruments are credit-risk sensitive: corporate bonds, vulnerable claims, credit derivatives, some sovereign debts, and so on. We only consider corporate bonds in this asset allocation setup. Corporate bonds by definition bear credit risk (and possibly other risks such as liquidity risk), since the obligors (bond issuers) may fail to repay coupons and/or principals of the debt. Credit risk consists of many sub-risks, such as default risk, recovery risk, correlation risk (in portfolios of credits), migration risk etc. Default risk is the most fundamental one. In modeling default risk, we adopt the so-called "reduced-form" approach advocated by Jarrow and Turnbull (1995), Madan and Unal (1998) and Duffie and Singleton (1999), among others.

### 2.1 Information Structure

To begin with, we assume financial assets are traded continuously in a frictionless market. Investors in this economy are price takers, so that their individual decisions would not affect price formation in a direct or obvious way. Taken as primitive is a finite time span  $\mathcal{T} := [0, \bar{T}]$ , where  $\bar{T} \in (0, \infty)$ , and a complete probability space  $(\Omega, \mathcal{G}, Q)$ , endowed with a reference filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  which

satisfies the usual conditions.<sup>6</sup> Assume that  $\mathcal{F}_t \subseteq \mathcal{G}$  for any  $t \in \mathcal{T}$ . The probability measure  $Q$  is a martingale probability measure in this paper, which is assumed to be equivalent to the statistical (real-world) measure  $\bar{P}$ .<sup>7</sup> For the convenience of analysis in the following two sections, we will start with the measure  $Q$ . All the processes defined in this paper live on the probability space  $(\Omega, \mathcal{G}, Q)$ . Since we are also concerned with investors' welfare besides pricing, we will keep track of the physical probability measure  $\bar{P}$  as well in later sections.

Let  $\tau$  be a non-negative random variable on this space. It represents *default time* of the corporate bond considered in this paper. For the sake of convenience, assume  $Q(\tau = 0) = 0$  and  $Q(\tau > t) > 0$  for any  $t \in \mathcal{T}$ . Define a right-continuous process  $H$  with  $H(t) := 1_{\{\tau \leq t\}}$  where  $1_{\{\tau \leq t\}}$  is the indicator function. Denote by  $\mathbb{H}$  the associated filtration on the same probability space, with  $\mathcal{H}_t = \sigma(H(u) : u \leq t)$  for all  $t \in \mathcal{T}$ . Now, let  $\mathbb{G}$  be another filtration (satisfying the usual conditions as well) on the probability space such that  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ , that is,  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$  for any  $t \in \mathcal{T}$ . Such information structure is standard in the reduced-form approach literature. The default time  $\tau$  is a  $\mathbb{G}$ -stopping time, more precisely, a surprise stopping time (Duffie and Singleton (1999), Madan and Unal (1998)).<sup>8</sup> This is motivated by the argument that it is theoretically more desirable to characterize the default likelihood than to pinpoint when defaults would happen according to some easily misspecified conditions. In most cases, the filtration  $\mathbb{F}$  represents the information flow of (observable) state variables available to investors over time. By observing such information, investors can reach their judgement about the default likelihood of the corporate bonds concerned. Without additional information, represented by  $\mathbb{H}$ , however, they may not be able to tell if a default has happened or not. In mathematical terms, default time  $\tau$  is a  $\mathbb{G}$ -stopping time but may fail to be an  $\mathbb{F}$ -stopping time. Of course, when they are equipped with information set of  $\mathbb{G}$ , they can tell whether a default has happened. Therefore, the default time  $\tau$  is assumed to be outside of the span of  $\mathcal{F} := \mathcal{F}_{\bar{T}}$  but adapted to  $\mathbb{G}$ .<sup>9</sup>

## 2.2 A Defaultable Bond Pricing Model

Since this paper mainly deals with dynamic asset allocation, a standard pricing model of defaultable corporate bonds is presented briefly in this section. We assume that the only financial asset that is subject to default risk is a corporate (that is, defaultable) zero-coupon bond (or a portfolio of zero-coupon bonds issued by identical firms whose default times are independent). It is possible to include more general corporate bonds such as coupon bonds in this framework and derive similar pricing results. For the sake of clearest intuition and implications, however, we consider the simplest defaultable bond here.

<sup>6</sup>A filtration  $\mathbb{F}$  is said to satisfy the *usual conditions* if it is right-continuous and  $\mathcal{F}_0$  contains all the  $Q$ -negligible events in  $\mathcal{F} := \mathcal{F}_{\bar{T}}$ .

<sup>7</sup>A probability measure  $Q$  is said to be equivalent to another probability measure  $P$  if the two probabilities have the same measure zero sets.

<sup>8</sup> $\tau$  is called a stopping time with respect to a filtration  $\mathbb{G}$  if the event  $\{\tau \leq t\} \in \mathcal{G}_t$  for every  $t$ .

<sup>9</sup>A process  $X$  is said to be adapted to  $\mathbb{G}$  if  $X(t)$  is measurable with respect to  $\mathcal{G}_t$ ,  $\forall t \in \mathcal{T}$ .

The maturity date of the defaultable zero bond is  $T_1 \in \mathcal{T}$ . Other contractual features of this bond include: the promised principal,  $F$ ; default time  $\tau \in \mathcal{T} \cup (\bar{T}, \infty]$ , that is, if  $\tau \in (T_1, \infty]$ , by definition there is no default during the life time of the corporate bond; at the time of default, a payment  $z(\tau)$  is recovered in fulfillment of the corporate debt obligation. It is common that only a fraction of the promised amount will be recovered upon default. Since we consider the defaultable zero-coupon bond, the coupon process in this paper is identically zero. Formally, a defaultable zero-coupon bond can be defined as a vector  $DZB = (F, z, \tau, T_1)$ , components of which are presumed adapted to the filtration  $\mathbb{G}$ .

**Definition 1** *The cumulated cash-flow process  $D$  of a defaultable zero-coupon bond  $DZB = (F, z, \tau, T_1)$  is defined as*

$$D(t) = F \times 1_{\{t \geq T_1\}} + \int_{(0, t]} z(u) dH(u)$$

It is apparent that the second term in the above definition accounts for the recovery upon default, since  $\int_{(0, t]} z(u) dH(u) = z(\tau) \times 1_{\{t \geq \tau\}}$ .

There exists a money market account in this economy starting with \$1, represented by process  $b$ , given by

$$b(t) = e^{rt} \quad (1)$$

where the *short-term interest rate* process  $r$  is assumed to be a constant process.

It is well-known in finance theory (Duffie (1996), for example) that the absence of arbitrage opportunities holds when there exists a martingale measure  $Q$  equivalent to  $\bar{P}$  under which the discounted (using money market account) gains processes for all assets are martingales. One easily gets the following pricing formula for  $DZB = (F, z, \tau, T_1)$  whose price is denoted by  $p(t, T_1)$

$$\begin{aligned} p(t, T_1) &= e^{rt} \mathbb{E}^Q \left( \int_{(t, T_1]} e^{-ru} dD(u) \middle| \mathcal{G}_t \right) \\ &= e^{rt} \mathbb{E}^Q \left( \int_{(t, T_1]} e^{-ru} z(u) dH(u) + e^{-rT_1} (1 - H(T_1)) F \middle| \mathcal{G}_t \right) \end{aligned} \quad (2)$$

where  $\mathbb{E}^Q$  is the expectation operator under the probability measure  $Q$ .<sup>10</sup>

**Definition 2** *The  $\mathbb{F}$ -hazard rate (or intensity) process  $h$  is an  $\mathbb{F}$ -progressively measurable, non-negative stochastic process such that*

$$M(t) := H(t) - \int_0^t (1 - H(u-)) h(u) du$$

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<sup>10</sup>More precisely, equation (2) is a definition of the defaultable bond. The validity of arbitrage pricing relies crucially on the replicability (or attainability) of the contingent claim using primitive assets in the economy. Since in general, the default time  $\tau$  is not an  $\mathbb{F}$ -stopping time, where the filtration  $\mathbb{F}$  is generated by some tradable assets, the issue of replicability of defaultable bonds is not obvious *a priori*. Nonetheless, we assume the arbitrage pricing method is applicable and suitable self-financing trading strategies can replicate the defaultable bond.

is a  $\mathbb{G}$ -martingale under  $Q$ , where  $H(u-) := \lim_{s \uparrow u} H(s) = 1_{\{\tau < u\}}$ .<sup>11</sup>

It is understood that over a short period of time  $(t, t + dt)$  the probability of default is approximately  $h(t) dt$  provided that no default has yet occurred by time  $t$ .<sup>12</sup>  $h$  is a compensator to  $H$  only up to (and including) the default time, and  $h$  in this context is the risk-neutral hazard rate process. Artzner and Delbaen (1995) show that the default time  $\tau$  has a hazard rate process under  $Q$ . Then it is well-established that the price of  $DZB = (F, z, \tau, T_1)$  admits the following representation:<sup>13</sup>

$$\begin{aligned} p(t, T_1) &= 1_{\{\tau > t\}} \mathbb{E}^Q \left( \int_t^{T_1} \exp \left( - \int_t^u (r + h(s)) ds \right) z(u) h(u) du \middle| \mathcal{F}_t \right) \\ &\quad + 1_{\{\tau > t\}} \mathbb{E}^Q \left( \exp \left( - \int_t^{T_1} (r + h(s)) ds \right) \middle| \mathcal{F}_t \right) \times F \end{aligned} \quad (3)$$

It should be noted that compared to (2), (3) eliminates the jump terms associated with process  $H$  (except the obvious indicator function outside of expectations) and that the conditioning filtration is  $\mathbb{F}$  instead of  $\mathbb{G}$ . This simplifies the following analysis (and pricing in the first place) considerably.

To put the above results into perspective, we adopt the recovery of market value (RMV hereafter) assumption according to Duffie and Singleton (1999), that is,

$$z(t) = (1 - \omega(t)) p(t-, T_1) \quad (4)$$

where  $\omega$  is an  $\mathbb{F}$ -predictable process of the *write-down* proportion (or loss rate) of the debt and it is customary to assume  $\omega \in (0, 1]$   $Q$ -a.s. Under this convention, a neat result due to Duffie and Singleton (1999) is as follows:

$$\begin{aligned} p(t, T_1) &= 1_{\{\tau > t\}} \times F \times \mathbb{E}^Q \left( \exp \left( - \int_t^{T_1} (r + \underbrace{h(s)\omega(s)}_{=: \delta(s)}) ds \right) \middle| \mathcal{F}_t \right) \quad (5) \\ &= 1_{\{\tau > t\}} v(t, T_1) \times F \end{aligned}$$

where  $\delta(t)$  is the credit spread;  $v(t, T_1) := \mathbb{E}^Q \left( \exp \left( - \int_t^{T_1} (r + \delta(s)) ds \right) \middle| \mathcal{F}_t \right)$  is the *pre-default* value of the  $DZB = (1, z, \tau, T_1)$ .

<sup>11</sup>Such  $\mathbb{F}$ -adapted hazard rate process (also called intensity process) is called  $\mathbb{F}$ -martingale hazard rate process.

<sup>12</sup>It should be emphasized that such probability of default is under the risk-neutral probability measure  $Q$  and in general not equal to its counterpart under the physical probability measure  $\bar{P}$  unless the market prices of default risks (both default *timing* risk and default *recovery* risk) are zero.

<sup>13</sup>Heuristically, one may show as in Madan and Unal (1998) that  $\mathbb{E}^Q[1 - H(u) | \mathcal{F}_t]$  and  $\tau > t] = \exp \left( - \int_t^u h(s) ds \right)$  and then use iterated expectations in (2), (3) immediately follows. For rigorous proof, see Bielecki and Rutkowski (2001), for instance.

Apparently, RMV model fails to distinguish write-down rate from the intensity rate, as shown by the multiplicative term  $h \times \omega$  in expression (5). But if one is willing to parameterize the recovery rate, one can effectively differentiate the impacts of hazard rate and recovery rate from corporate bond data. Besides this shortcoming, there is yet another theoretical deficiency, pointed out by Madan (2000): RMV effectively transfers early dollars in default to terminal dollars at maturity using a risky bond with the same contractual features as the original corporate bond that has defaulted. Such method fails to replicate the exact cost of the original promise at maturity as there may be another default. In contrast, other recovery schemes, such as recovery of treasury (or RT for short, that is, transferring early dollars using default-free treasury bond with the same face value and maturity), are consistent in replicating the cost of the original promise at maturity. For the theoretical considerations and empirical performance of different recovery schemes, we refer interested readers to Bakshi *et al.* (2001) for detailed discussions. Despite the above-mentioned imperfections of RMV, we still adopt it in this paper for its simplicity in analysis.

### 2.3 Asset Universe in the Economy

To fully explore the impact of default risk on the investors' portfolio choice, we assume constant interest rate.<sup>14</sup> Given the theme of this paper, we note that Leland and Toft (1996), Longstaff and Schwartz (1995), Kim *et al.* (1993), among others, show that interest rate uncertainty can only affect credit spreads marginally in their theoretical models. As a result, defaultable bond prices should not be affected substantially by the absence of interest rate randomness. On the other hand, it is hoped that such abstraction will isolate the effects of default risk and give the cleanest intuitions about agents' economic behavior.

The critical variable in a defaultable zero-coupon bond's pricing formula (5) is the instantaneous credit spread  $\delta$ . Once we pin down the credit spread, the price of a DZB is determined correspondingly under the recovery of market value scheme. We assume that the instantaneous credit spread  $\delta$  follows Ornstein-Uhlenbeck process:

$$d\delta(t) = \kappa_\delta (\theta_\delta - \delta(t)) dt + \sigma_\delta dw_\delta(t) \quad (6)$$

where  $w_\delta$  is a standard Brownian motion under  $Q$ ;  $\kappa_\delta$ ,  $\theta_\delta$ ,  $\sigma_\delta$  are deterministic processes. Assume  $\kappa_\delta, \sigma_\delta > 0$ .  $\kappa_\delta$  is the reversion speed of credit spread towards its long-term mean  $\theta_\delta > 0$ . This is admittedly a rather simple model of default risk. The reason for this model is that it not only lends itself to simplifying analysis, but also is motivated by empirical evidence and modeling by other authors. For instance, the literature recognizes that credit spreads in general may depend on some firm-specific (or industry-specific) distress factors, such as book-to-market ratios, leverages, stock prices, profitabilities, and others. Bakshi *et al.* (2005), for example, model these firm-specific factors using

<sup>14</sup>Based on this paper's framework, one of the authors of this paper, Yuanfeng Hou, extends this research and takes into account of interest rate risk in one of his recent papers called "Integrating Market Risk and Credit Risk".



Ornstein-Uhlenbeck processes and find such modeling quite robust in empirical studies. Collin-Dufresne and Goldstein (2001) also assume the log-leverage ratio following such process in a structural credit spread model. Together with the Vasicek-type of interest rate dynamics and a linear model, one can derive a model for credit spread similar to (6).<sup>15</sup>

There is a non-dividend-paying equity in this economy, whose price process is  $S$ . Assume

$$dS(t) = rS(t)dt + \sigma_S(t)S(t)dw_S(t) \quad (7)$$

where  $w_S$  is another standard Brownian motion under measure  $Q$ ,  $\sigma_S$  the deterministic instantaneous volatility of diffusive equity returns. Brownian motion  $w_S$  (the equity risk) may correlate with default risk, that is  $dw_S dw_\delta = \rho dt$ , with deterministic correlation coefficient  $\rho$ .

Given the risk structure in this economy, dynamics of  $\delta$  and  $S$  under the physical measure  $\bar{P}$  can be derived given the specifications of the market prices of risks. Such transformation between risk-neutral measure and physical measure is necessary since investors in this paper are risk-averse and derive utility under the physical measure.

One can use Girsanov theorem to do the drift adjustment. Girsanov theorem originally applies to the case of independent Brownian motions. It can be easily extended to correlated cases, however. Given that the dynamics of credit spread  $\delta$ , and of equity price  $S$  under the risk-neutral probability measure  $Q$  are described as (6) and (7) respectively, with  $dw_\delta dw_S = \rho dt$ , the corresponding dynamics under the physical probability measure  $\bar{P}$  are given by:

$$d\delta(t) = [\kappa_\delta(\theta_\delta - \delta(t)) + \underbrace{\sigma_\delta(\lambda_\delta(t) + \rho\lambda_S(t))}_{=:\bar{\lambda}_\delta(t)}]dt + \sigma_\delta dw_\delta^{\bar{P}}(t) \quad (8)$$

$$dS(t) = [r + \underbrace{\sigma_S(\rho\lambda_\delta(t) + \lambda_S(t))}_{=:\bar{\lambda}_S(t)}]S(t)dt + \sigma_S S(t)dw_S^{\bar{P}}(t) \quad (9)$$

where  $\lambda_\delta$  is the deterministic market price of default risk,  $\lambda_S$  the deterministic market price of equity risk, and  $\mathbf{w}^{\bar{P}} = (w_\delta^{\bar{P}}, w_S^{\bar{P}})'$  is a vector Brownian motion under  $\bar{P}$  with the same correlation structure.

It should be emphasized that the change in the drift term in each SDE includes exposures to both risk sources in this economy provided the correlation is non-zero. This is intuitive since the risk drivers (represented by the Brownian motions) are in general correlated and exposure to one naturally induces exposure to others, hence additional risk premia are demanded by investors. These risks are systemic in that each of them cannot be diversified away by holding appropriate well-diversified portfolios. For simplicity, we assume all the

<sup>15</sup>One may add orthogonal jumps to (6) as in Akgun (2001) and Collin-Dufresne and Solnik (2001) to adapt sudden changes of credit spreads from turbulent market conditions or from credit rating changes. The jump feature is not considered in this paper, however.

Duffee (1999) models the risk-neutral hazard rate  $h$  as a translated single-factor square-root process plus two other components tied to the default-free interest rate factors.

market prices of risks are deterministic in this paper; as a result,  $\bar{\lambda}_\delta$  and  $\bar{\lambda}_S$  are deterministic too.<sup>16</sup>

In the context of asset allocation, the dynamics of the defaultable zero-coupon bond (that is, its SDE) must be derived. This is shown in the following proposition:

**Proposition 1** *Without loss of generality, set  $F = 1$ . Let the price of the defaultable zero coupon bond  $p(t, T_1)$  be given by (5). Then  $p(t, T_1)$  must satisfy the following SDE:*

$$dp(t, T_1) = p(t-, T_1) [(r + \delta(t)) dt + \sigma_P(t, T_1) dw_\delta] - v(t, T_1) dH_t \quad (10)$$

where  $\sigma_P(t, T_1) := -\frac{1 - \exp(-\kappa_\delta(T_1 - t))}{\kappa_\delta} \sigma_\delta$ .

**Proof.** See Appendix. ■

The last term in the SDE (10) accommodates the default event. In the event of default, the price of the defaultable bond drops to zero, which is exactly shown by  $v(t, T_1) dH_t$  evaluated at default time. This is true irrespective of recovery specification, that is, even in the case of non-zero recovery, the price of defaultable bond drops to zero, as evidenced by the indicator function in (5). This holds since upon default the corporate bond ceases to exist, though the partial recovery is assumed to be in the form of pre-default value of the bond. As the discussion after equation (5) shows, RMV is not perfect in transferring early dollars to maturity date. Such imperfection could cause misunderstanding of the above result. One key thing to understanding (10), however, is that the corporate bond vanishes upon default (hence its price drops to zero), no matter what recovery scheme (RMV, RT or others) or recovery specification (zero or non-zero) is used, as shown precisely by the last jump term. Except the last term, the defaultable bond behaves like a treasury bond (which is not introduced in this paper as interest rate is constant), except that: first, the drift term incorporates credit spread of the bond. The credit spread enters the drift term as we adopt RMV method. If there is zero recovery at default,  $h$  instead of  $\delta$  will appear in the drift. Second, the default risk (represented by the Brownian driver  $w_\delta$  in addition to  $dH$  term) shows its force in the dynamics. Note that (10) is specified under the risk-neutral measure  $Q$ , so one should expect the risk-premia associated with equity risk and credit risk would show up under the physical probability measure  $\bar{P}$  since there is non-zero correlation between the two risk drivers in general.

Before the dynamic asset allocation problem is formulated, another important issue concerning investors is: will some specific risk be compensated in the market? If some risk is not systemic and hence not priced in the economy, no one would like to take such risk as long as she could avoid it. Note that (10) can be written as

$$\frac{dp(t, T_1)}{p(t-, T_1)} = [(r + \delta(t) - h(t)) dt + \sigma_P(t, T_1) dw_\delta] - dM_t \quad (11)$$

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<sup>16</sup>Walder (2001) considers the time-varying risk premia of the CIR-type.

up to (and including) default time. In the case of zero recovery, Jarrow *et al.* (2005) show that when assuming there exist a countably infinite number of identical firms whose default times are independent of each other in the economy, the martingale term  $dM$  can be diversified away.<sup>17</sup> Using this argument, the price of a well-diversified portfolio of the defaultable bonds in this limit economy has the following dynamics

$$\frac{dP(t, T_1)}{P(t-, T_1)} = [r + \underbrace{\delta(t) - h(t)}_{=:-\eta(t)}]dt + \sigma_P(t, T_1) dw_\delta(t) \quad (12)$$

where  $P(t, T_1)$  is the price of this diversified portfolio. Note that the  $Q$ -martingale term  $dM$  disappears from (12) as a result of diversification. In other words, the *idiosyncratic* default risk is not priced in this limit economy under mild technical conditions. This result, though a bit stringent for its diversification argument, simplifies the analysis considerably.<sup>18</sup>

As (12) shows, taking recovery (not necessarily recovery *risk*) into account, the drift of the return of defaultable zero-coupon bond may deviate from  $r$  even under the risk-neutral probability measure  $Q$ ! Walder (2001) assumes zero recovery following Jarrow *et al.* (2005). Zero-recovery approach, however, may overlook this drift effect. In this paper, the defaultable zero-coupon bond (portfolio) has a distinct drift term and different risk exposure from other assets. This still holds when idiosyncratic default risk has been eliminated. (12) actually shows that under the current assumptions, systemic (not the idiosyncratic) default risk affects the defaultable bond portfolio in a rather different way from other risks. One may view  $\eta$  as the dividend rate the defaultable bond pays before default, even though the bond is actually a zero-coupon bond. This fact highlights the difference between the corporate zero-coupon bond and its treasury counterpart. We would emphasize that such result is a consequence of RMV scheme adopted in this paper, as the appearance of the credit spread  $\delta$  is from this very assumption.

For simplicity, we assume that the write-down rate  $\omega$  is constant. The fact that  $\eta$  is a function of  $\delta$  and  $\omega$  indicates that the  $\eta$  is also a stochastic process and has the same SDE as the credit spread  $\delta$  as we assume  $\omega$  to constant. From the definition of  $\eta$ , it is easy to show that  $\eta = (\frac{1-\omega}{\omega})\delta$ , with  $\omega \in (0, 1]$ . From (8), the dynamics of  $\eta$  is simply

$$d\eta = [\kappa_\delta(\underbrace{\frac{1-\omega}{\omega}\theta_\delta}_{=:\theta_\eta} - \eta(t)) + \frac{1-\omega}{\omega}\sigma_\delta\bar{\lambda}_\delta]dt + \underbrace{\frac{1-\omega}{\omega}\sigma_\delta}_{=:\sigma_\eta}dw_\delta^\bar{P}(t) \quad (13)$$

<sup>17</sup>Correspondingly, the information structure has to be updated. This is shown in Jarrow *et al.* (2005). The essence, however, is the same as in Section 2.1.

<sup>18</sup>In the concurrent work by Walder (2001), he also invokes this conditional diversification argument.

### 3 Optimal Asset Allocation Solution

We assume that an investor in this economy tries to maximize her vNM utility of terminal wealth by dynamically allocating her financial wealth into a money market account, a well-diversified portfolio of corporate zero-coupon bonds in the sense of Jarrow *et al.* (2005), and an equity (index). She has no intermediate consumption and no human-capital income to support her purchase of financial assets.

Her utility function is assumed to be the following standard form:

$$U(W) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma}, & \text{if } W > 0 \\ -\infty, & \text{if } W \leq 0 \end{cases} \quad (14)$$

where  $\gamma > 0$  is the Arrow-Pratt relative risk-aversion coefficient. When  $\gamma = 1$ ,  $U(W) = \log W$  if  $W > 0$ . When  $W$  is below zero, her utility is set at minus infinity, thus effectively restricting her wealth  $W$  from falling below zero. As shown by Dybvig and Huang (1988), this constraint rules out the arbitrage opportunity described by Harrison and Kreps (1979).

The investor is endowed with a positive initial wealth  $W_0$ . Given the financial assets she can invest, she chooses to invest a vector  $\pi$  whose elements are the fraction of her wealth in risky (both market-risky and credit-risky) assets each time  $t \in [0, T]$  so as to maximize her expected utility of terminal wealth. Assume  $T < T_1$ . The percentage of wealth invested in the money market account is given by  $1 - \pi' \mathbf{1}$ , where  $\mathbf{1}$  is a vector of ones with the same dimension as  $\pi$ . The trading strategy is restricted to be self-financing. As a result, the wealth dynamics can be written as

$$\begin{aligned} dW^\pi(t) &= W^\pi(t) \left[ r + \pi(t)' (\mu(t) - r\mathbf{1}) dt + \pi(t)' \Sigma(t) d\mathbf{w}^{\bar{P}}(t) \right] \\ W^\pi(0) &= W_0 > 0 \end{aligned} \quad (15)$$

where  $\mu := (r + \sigma_S \bar{\lambda}_S, r - \eta + \sigma_P \bar{\lambda}_\delta)'$  is the vector of instantaneous expected returns of risky assets under the physical measure  $\bar{P}$ ,  $\Sigma := \begin{pmatrix} \sigma_S & 0 \\ 0 & \sigma_P \end{pmatrix}$  is the volatility matrix of risky assets, and  $\mathbf{w}^{\bar{P}} = (w_S^{\bar{P}}, w_\delta^{\bar{P}})'$ .

Let  $\mathcal{A}(W_0)$  be the set of admissible trading strategies<sup>19</sup> defined by

$$\mathcal{A}(W_0) = \{ \pi(\cdot) \in \mathbf{R}^2 : W^\pi(t) > 0 \text{ } \bar{P} - a.s. \text{ for } t \in [0, T] \}$$

Now the optimization problem the investor faces can be formulated as

$$\max_{\pi(\cdot) \in \mathcal{A}(W_0)} \mathbb{E} U(W^\pi(T)) \quad (16)$$

subject to (15). Where  $\mathbb{E}$  in (16) is the expectation operator under the physical measure  $\bar{P}$  at time 0.

<sup>19</sup>Additional technical conditions have to be satisfied. See Fleming and Rishel (1975) for general exposition. Or see Korn and Kraft (2001) for conditions under similar setup (stochastic interest rates).

To facilitate the exposition, we list the dynamics under the physical measure  $\bar{P}$  of the financial assets in this economy below. In particular, we assume deterministic processes (except  $\sigma_P$ ) in this paper are in fact constant processes in the following analysis.

$$\frac{dS(t)}{S(t)} = (r + \sigma_S \bar{\lambda}_S) dt + \sigma_S dw_S^{\bar{P}}(t)$$

$$\frac{dP(t, T_1)}{P(t, T_1)} = [r - \eta(t) + \sigma_P(t, T_1) \bar{\lambda}_\delta] dt + \sigma_P(t, T_1) dw_\delta^{\bar{P}}(t)$$

and for the money market account

$$\frac{db(t)}{b(t)} = r dt$$

To solve this optimization problem, we use stochastic control method which has been employed extensively in the literature. Assuming  $\omega$ , the write-down rate when default happens, is constant has an implication that the “dividend” rate process  $\eta$  is a state variable in addition to wealth  $W$ . This is also motivated by the fact that credit spreads are observable in markets, which makes  $\eta$  observable in this economy when  $\omega$  is assumed to be constant. Indeed, Collin-Dufresne *et al.* (2002) empirically find that a “market spread factor” probably can proxy for credit market conditions. Following Merton (1971), define the indirect utility function as

$$J(W, \eta, t) = \max_{\{\pi(s) \in \mathcal{A}(W), t \leq s \leq T\}} \mathbb{E} U(W^\pi(T) | \mathcal{F}_t) \quad (17)$$

The Hamilton-Jacobi-Bellman (HJB) equation for this indirect utility function is as follows

$$\max_{\pi(t) \in \mathcal{A}(W), 0 \leq t < T} \mathcal{D}^\pi J(W, \eta, t) = 0 \quad (18)$$

$$\text{with } J(W, \eta, T) = \frac{W^{1-\gamma}}{1-\gamma} \quad (19)$$

where

$$\begin{aligned} \mathcal{D}^\pi J(W, \eta, t) = & J_t + (\kappa_\delta \theta_\eta + \sigma_\eta \bar{\lambda}_\delta - \kappa_\delta \eta) J_\eta \\ & + W[\pi_S \sigma_S \bar{\lambda}_S + \pi_P(-\eta + \sigma_P \bar{\lambda}_\delta) + r] J_W \\ & + W \sigma_\eta (\pi_S \sigma_S \rho + \pi_P \sigma_P) J_{W\eta} + \frac{1}{2} \sigma_\eta^2 J_{\eta\eta} \\ & + \frac{1}{2} W^2 [\pi_S^2 \sigma_S^2 + \pi_P^2 \sigma_P^2 + 2\pi_S \pi_P \sigma_S \sigma_P \rho] J_{WW} \end{aligned} \quad (20)$$

and  $J_t, J_\eta, J_W, J_{W\eta}, J_{\eta\eta}$ , and  $J_{WW}$  are partial derivatives with respect to appropriate variables. For notational brevity, we suppress the parameters' dependence on time and other state variables above. The standard technique used for this problem produces the following result.

**Theorem 1** *In this economy, assume all the deterministic processes (except  $\sigma_P$ ) in the text are in fact constant processes. The indirect utility function  $J$  is given by*

$$J(W, \eta, t) = \begin{cases} g(t) \exp\left(k(t)\eta + \frac{1}{2}l(t)\eta^2\right) \frac{W^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln W & \gamma = 1 \end{cases} \quad (21)$$

where  $g(t)$ ,  $k(t)$  and  $l(t)$  are deterministic functions and given by (35), (33) and (31) respectively in the appendix. The optimal portfolio weights are given by

$$\pi_S^*(t) = \frac{1}{\gamma} \left[ \frac{\bar{\lambda}_S}{\sigma_S(1-\rho^2)} - \frac{\bar{\lambda}_\delta \sigma_P - \eta(t)}{\sigma_P \sigma_S(1-\rho^2)} \rho \right] \quad (22)$$

$$\begin{aligned} \pi_P^*(t) &= \frac{1}{\gamma} \left[ \frac{\bar{\lambda}_\delta \sigma_P - \eta(t)}{\sigma_P^2(1-\rho^2)} - \frac{\bar{\lambda}_S}{\sigma_P(1-\rho^2)} \rho \right] \\ &\quad - \frac{1-\omega}{\omega} \frac{k(t) + l(t) \times \eta(t)}{\gamma} \frac{\kappa_\delta}{1 - e^{-\kappa_\delta(T_1-t)}} \end{aligned} \quad (23)$$

**Proof.** See Appendix.<sup>20</sup> ■

The significance of Theorem 1 is to give a rather clear picture of this optimal asset allocation problem when default risk is an intrinsic risk in the economy. Closed-form solutions, though obtained under some specific assumptions, can be employed to provide clear insights and greatly facilitate numerical exercises. Even though the pricing of defaultable bonds is an uneasy task in general, given the simplified framework laid out here, we can solve the problem in a rather standard way.

When  $\gamma = 1$ , that is, the investor has logarithmic utility, the indirect utility function  $J$  reduces to  $J(W, t) = \ln W$ , which is independent of  $\eta$ . In this case, the investor acts myopically and her derived utility function is just the one-period utility function. The corresponding demands reduce to myopic terms only, as the functions  $k$  and  $l$  become identically zero when  $\gamma = 1$ .

When  $\gamma \neq 1$ , there are two components in the demand for defaultable bond while there is only one in the demand for equity. The second component in  $\pi_P^*$  is indeed a hedging term. Detailed interpretations are provided in the next section.

## 4 Implications for Asset Allocation

In this section, we investigate in detail the implications of Theorem 1 for asset allocations.

### 4.1 Dependence of Optimal Demands on State Variables

Inspection of the optimal portfolio weights given in (22) shows some interesting features. The portfolio weights are not independent of state variables as  $\eta$

<sup>20</sup>Verification of theorem is also discussed in the appendix.

appears in both assets' demand functions. It should also be noted that terms in the square brackets are myopic demands as shown in the proof in the Appendix. This shows that state variable  $\eta$  appears not only in the hedging term in  $\pi_P^*$  but also in myopic demands for both assets provided that there is non-zero correlation between equity market and corporate bond market (Even if  $\rho = 0$ ,  $\eta$  still appears in  $\pi_P^*$ ). In other words, even myopic investors find valuable information in the term structure of credit spreads in their investing decisions. Therefore investors may try to time the market when making their investment decisions. This is equivalent to the investors' timing the market by observing the credit spread because the information content of state variable  $\eta$  is the same as that of credit spread  $\delta$  according to equation (14), provided that the write-down rate is a constant. For example, investors may try to predict  $\eta$  in order to make investment decisions. The shapes of the term structure of credit spreads thus contain valuable market information to investors. This result is from the fact that under RMV scheme adopted in this paper,  $\eta$  is part of the return process of the defaultable bond portfolio as shown in (12). Thus, it directly enters mean-variance considerations of investors. As usual, homogeneity of wealth makes the result independent of  $W$ .

## 4.2 Optimal Equity Demand

The demand for equity contains only the so-called "myopic" term. It is out of the mean-variance considerations investors make. This is because equity is not directly subject to default risk and the interest rate is assumed to be deterministic. In other words, the optimal asset allocation shows some form of separation in that each asset is used to hedge the unique risk it faces. This makes the life of investors much easier in making investment decisions in this setup. Such separation may not hold in general when risk structure becomes more complicated.

As usual, the myopic demand decreases as the risk-aversion coefficient  $\gamma$  increases. There are two terms in this myopic demand for equity. The first term,  $\frac{\bar{\lambda}_S}{\sigma_S(1-\rho^2)}$ , is the pure demand for equity given its market price of equity risk, adjusted by a second term  $\frac{\bar{\lambda}_S \sigma_P - \eta(t)}{\sigma_P \sigma_S(1-\rho^2)} \rho$  involving the correlation between risk sources represented by Brownian motions in this economy. If the correlation coefficient between the two sources of risks is zero, then the first term in the bracket is simply the Sharpe ratio of equity normalized by equity's volatility and the second term vanishes. The presence of the second term is because of non-zero correlation between the two markets as reality suggests. A position in equity market indirectly induces a position in credit market due to common factors in their risk structure.

## 4.3 Optimal Demand for Defaultable Bond

As expected, the optimal portfolio weight of defaultable bonds contains both a myopic term and a hedging term provided that the investor does not behave

myopically and that the recovery rate is not zero. Terms in the myopic demand have the same interpretation as in equity's case. The hedging term is not the usual one which is used to hedge against stochastic changes in the investment opportunity set. In fact, the deterministic volatilities of financial assets and deterministic interest rate in this economy make the investment opportunity set deterministic over time. As a result, there is no hedging demand for it.

As shown in Theorem 1, there are two possibilities that the hedging term can be zero. One is when the investor behaves myopically, that is, when  $\gamma = 1$ , the hedge term vanishes as  $k(t) = l(t) = 0, \forall t$ . The other is that the write-down rate  $\omega$  equals one. In this case, there is no stochastic variability in the risk premium of defaultable bond, eliminating the hedging demand. This fact indicates the relevance of recovery effect of defaultable securities on investors' optimal asset allocation decision. This hedge demand is used by the investor to hedge against or speculate on the stochastic variability of the risk premium of the defaultable bond. Note that the demand for the *level* of  $\eta$  is contained in the myopic term as usual. The hedging demand depends on the planning horizon. As terminal day approaches, the investor has less need to hedge against the default risk *ceteris paribus*, as reflected in the fact that  $k(T) = l(T) = 0$ . This can be called *horizon effect*. Figure 1 shows the optimal hedging demand for a given path of the credit spread. Since credit spread is random, the hedging demand also varies across time randomly. It becomes zero, however, at the terminal date as predicted by the horizon effect.

Since in this paper, default risk is represented by the Brownian driver  $w_\delta$ , and defaultable bond return's volatility  $\sigma_P$  is closely linked with credit spread's volatility  $\sigma_\delta$ ,  $1 - e^{-\kappa_\delta(T_1-t)}$  term in the denominator therefore is proportional to the shares of defaultable bond used to hedge unit default risk at any given time. The effectiveness of the defaultable bond for hedging default risk (defined by  $\left| \frac{\sigma_\delta}{\sigma_P} \right|$ ) is monotonically declining as time to maturity shortens, since the ratio of its return volatility over credit spread volatility decreases over time. This mitigates the decrease in the (absolute) magnitude of hedging demand caused by the horizon effect.

#### 4.4 Correlation between Markets

We also note that the correlation coefficient  $\rho$  plays an important role in this simple setup. To begin with, the hedging demand depends on  $\rho$  through the dependence of  $k$  and  $l$  on  $\rho$ . Further inspection shows that terms in  $k$  and  $l$  that depend on  $\rho$  are symmetric about  $\rho$  except of the term  $d := \frac{(1-\gamma)(\bar{\lambda}_\delta - \rho\bar{\lambda}_\delta)}{\gamma\sigma_P(1-\rho^2)}$ . For an average risk-averse investor who has  $\gamma > 1$ , and a positive equity risk premium,  $d$  decreases as  $\rho$  changes symmetrically from a negative value to a positive one. This leads to an increase in  $k$ . As a result, the hedging term decreases algebraically as the ratio  $\frac{\sigma_\eta}{\sigma_P} < 0$ , keeping other things constant. Given the typical parameter values (see Section 5), this statement can be strengthened further: when  $\rho$ 's change from negative values to positive values in the neighborhood of 0, the hedging demand decreases algebraically but increases



in absolute value. See the top left panel of Figure 2. This is quite intuitive since when the two markets become positively correlated instead of negatively correlated, investors' ability to diversify risks deteriorates, hence their hedging ability. This induces an increase (in absolute terms) in the hedging demand *ceteris paribus*.

Secondly, the change of the correlation coefficient also affects the myopic demands. In this paper, we assume the volatility of credit spread,  $\sigma_\delta$ , is positive, as a result, the volatility of defaultable bond return,  $\sigma_P$ , is negative.<sup>21</sup> If assume the adjusted risk premium of the defaultable bond is positive,<sup>22</sup> and risk premium for equity risk is positive, when  $\rho > 0$ , it can be shown that the myopic demand for defaultable bond has negative relationship with  $\rho$  and the myopic demand for equity has positive relationship with  $\rho$ . The sign of changes of both myopic terms is unclear in general when  $\rho$  changes if  $\rho < 0$ , the case that is very probable as shown by the extant empirical work.

#### 4.5 Default Risk in Asset Allocation

Default risk affects investors' decision both through its market price  $\overline{\lambda_\delta}$ , the "dividend" rate,  $\eta$ , and the stochastic variability of  $\eta$ . By definition,  $\eta = h - \delta = (\frac{1-\omega}{\omega})\delta$ . It is thus clear that default risk affects investors in a rather unique way from other risks such as equity risk which usually affects investors' decision only through market price of risk. It should be noted that even if default recovery *risk* is absent, that is, when there is no uncertainty in the write-down rate  $\omega$  as in this paper, the fact that there is partial recovery of defaultable bond in the event of default still affects agents' investment in defaultable bonds as long as  $\eta \neq 0$ . On top of this recovery *level* effect, uncertainty of recovery may also change investor's behavior. This is not modeled in this paper, however. These features highlight the difference of investment with defaultable securities in the opportunity set of investors from the usual problem when default risk is absent. These insights are not available in Walder (2001) since he assumes zero recovery rate of defaultable bond. As shown above, the correlation between equity market and credit market induces adjustment term in the myopic demand for equity. Consequently, default risk also indirectly affects equity's demand through the presence of  $\eta$ .

#### 4.6 Welfare Effects of Investing in Credit Market

An integral part of the optimal asset allocation problem is to investigate how investors allocate their wealth in such a way as to achieve optimality of their utility. This subsection deals with this welfare issue when agents can invest in credit market.

<sup>21</sup>This is clearly an analogy to Treasury bond's case when interest rate risk is present. See, for example, Musiela and Rutkowski (1998).

<sup>22</sup>This condition must hold, for investors demand positive risk premium for exposures to default risk; otherwise, defaultable bond in this economy would be dominated by the money market account which would have higher rate of return and no risk. Equilibrium argument then eliminates the very presence of defaultable bonds in this economy.

Following Liu and Pan (2003), we define the certainty equivalent wealth  $W_{ce}$  as follows

$$J(W_0, \eta_0, 0) = \begin{cases} \frac{W_{ce}^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln W_{ce} & \gamma = 1 \end{cases} \quad (24)$$

given initial values of the state variables. It can be shown that

$$W_{ce} = \begin{cases} W_0 g(0)^{\frac{1}{1-\gamma}} \exp\left(\frac{1}{1-\gamma} \left[k(0)\eta_0 + \frac{1}{2}l(0)\eta_0^2\right]\right) & \gamma \neq 1 \\ W_0 & \gamma = 1 \end{cases} \quad (25)$$

When credit market is not open to investors or investors do not choose to invest in it for some reason, investors have the classical asset allocation problem of Merton (1971). It is well known that in this case when only equity market and a money market account are available, the indirect utility is

$$J(W, t) = \begin{cases} \frac{W^{1-\gamma}}{1-\gamma} \exp\left(\left[\frac{1-\gamma}{2\gamma}\overline{\lambda}_S^2 + (1-\gamma)r\right](T-t)\right) & \gamma \neq 1 \\ \ln W_0 & \gamma = 1 \end{cases}$$

The certainty equivalent wealth in this case is given by

$$W_{ce}^M = \begin{cases} W_0 \exp\left(\left[\frac{1}{2\gamma}\overline{\lambda}_S^2 + r\right]T\right) & \gamma \neq 1 \\ W_0 & \gamma = 1 \end{cases}$$

We adopt the measure Liu and Pan (2003) have defined:

$$R^W := \frac{\ln W_{ce} - \ln W_{ce}^M}{T}$$

which measures the portfolio improvement in terms of the annualized, continuous compounded return in certainty equivalent wealth of one scenario (credit market available) against another (no credit market available). Thus we have shown the following proposition:

**Proposition 2** *Given the model's setup, the portfolio improvement of an investor in terms of certainty equivalent wealth from investing in credit market is*

$$R^W = \begin{cases} \frac{1}{T(1-\gamma)} \left[\ln g(0) + k(0)\eta_0 + \frac{1}{2}l(0)\eta_0^2\right] - \left(\frac{1}{2\gamma}\overline{\lambda}_S^2 + r\right) & \gamma \neq 1 \\ 0 & \gamma = 1 \end{cases} \quad (26)$$

Note that  $R^W$  does not depend on interest rate  $r$ , though (26) suggests so. In fact,  $g(0)$  contains a term involving  $r$  (see (35) in the appendix). After cancelation, it is easy to verify the preceding statement.

## 5 Numerical Examples

In order to quantify asset allocations and welfare improvement and to investigate parameter sensitivities of interesting variables, we perform numerical analysis in this section. We follow the usual practice in the literature to specify relevant parameter values. For those that we do not have much confidence of their exact values (and which literature is silent on their values), we try to do some robustness examples while changing those parameters in a reasonably interval. All the parameter values below are annual statistics.

### 5.1 Parameter Values

Collin-Dufresne and Solnik (2001) contains maximum likelihood estimates of credit spread parameters using investment grade corporate bonds data. We follow their estimates and set the long-term mean of credit spread  $\theta_\delta = 0.0038$ , the volatility of credit spread  $\sigma_\delta = 0.0131$ , and the reversion speed of credit spread  $\kappa_\delta = 1.4248$ .

For simplicity, the risk-free interest rate  $r$  is set at 5%.

For parameters of equity market, it is rather straightforward to derive them by calibrating to U.S. stock market, say. We set the stock volatility  $\sigma_S = 0.15$ , and the market price of equity risk  $\overline{\lambda}_S = 0.450667$ , which make the equity risk premium equal to 6.76% annually, as shown in Liu and Pan (2003), for instance.

The correlation coefficient  $\rho$  between the two Wiener processes is also an important parameter. As for the sign of  $\rho$ , one might expect that it is negative, since common sense tends to suggest that when the credit of one firm deteriorates (that is, its credit spread  $\delta$  increases) its stock price (if it's listed in the stock market) would decrease. Indeed, Kwan (1996) empirically finds that there exists statistically significant negative correlation between individual stock returns and same firm's bond yields. The relation at the aggregate level, however, is not that clear. For example, Campbell and Ammer (1993) use value-weighted stock index from NYSE and AMEX and US Treasury securities and find the correlation is rather weak. The empirical evidence of the correlation between stock return and *corporate* bond yields at the aggregate level does not seem to avail. We then follow Kwan (1996) to specify the value of  $\rho$ . For investment-grade corporate bonds, we set  $\rho = -0.184$ ; for non-investment-grade bonds, we set  $\rho = -0.423$ . Besides the above baseline values, we also vary  $\rho$  in the interval of  $[-0.5, 0.1]$ .

Duffee (1999) uses Moody's data, which suggests the recovery rate of senior unsecured bonds is 44% on average. This translates to the write-down rate  $\omega = 0.56$ . For junior debt, the write-down rate could be even higher. In our numerical exercise, we set  $\omega = 0.8$  for this debt category.

We set the planning horizon of the investor  $T$  equal to 1 year and assume the maturity of the defaultable bond  $T_1$  to be 10 years. Table 1 summarizes relevant parameters used in the analysis.

## 5.2 Some Results

To illustrate the theoretical results, we adopt the parameter values of the last subsection. We randomly draw a path of  $\eta$  and calculate the optimal asset allocations given this path. To make some comparison, we specify two kinds of corporate bonds, bond  $i$  (investment-grade bond) and bond  $j$  (junk bond), with different initial credit spreads (70 basis points for bond  $i$  and 200 basis points for bond  $j$ ), different write-down rates (.56 for bond  $i$  and .8 for bond  $j$ ) and different  $\bar{\lambda}_\delta$  (-.35 for bond  $i$  and -.4 for bond  $j$ ). Caution has to be taken in interpreting results of bond  $j$  since other parameters of the credit spread process such as  $\theta_\delta$ ,  $\kappa_\delta$  and  $\sigma_\delta$  are obtained from investment-grade bonds. Literature has been rare on their junk-bond counterparts.

The top panel of Figure 3 shows the equity demand over time for two defaultable bond scenarios. Since the equity demand is of myopic nature, it does not vanish at terminal date as the hedging term does. It also depends on  $\eta$  as shown in (22), generating stochastic variation over time. The middle and bottom panels of Figure 3 show how the optimal equity demand changes with parameters at the end of first quarter. As the write-down rate increases, the equity demand decreases. This is because the risk premium of the defaultable bond increases, making equity less attractive. This argument is also shown in the relation between the equity demand and  $\bar{\lambda}_\delta$ . The relationship between equity demand and correlation coefficient  $\rho$  is in general ambiguous as shown in Figure 3, depending on the values of other parameters such as  $\omega$  and  $\bar{\lambda}_\delta$ . With typical parameter values, the equity weight is roughly between 0 and 1.

Figure 4 illustrates the behavior of the myopic bond demand. The magnitude of the myopic demand for corporate bond can be several times of that for equity and it is the dominant part of the whole demand for corporate bond (see the bottom panels of Figure 2). Campbell and Viceira (2001) also find similar results with nominal and index bonds. As the write-down rate increases or  $\bar{\lambda}_\delta$  decreases, the risk premium of the corporate bond increases, making it more appealing to investors.

There are several interesting features of the hedging demand for the corporate bond. Unlike the myopic demand, the hedging demand may not depend on the risk-aversion coefficient  $\gamma$  monotonically, as shown in the top right panel of Figure 2. This is because  $k$  and  $l$  also depend on  $\gamma$ , generating non-linearity of the hedging term in  $\gamma$ . This fact highlights the different natures of myopic demand and hedging demand: myopic demand is mainly out of mean-variance considerations while hedging demand concerns stochasticity in investment opportunity set. The hedging demand decreases as the write-down rate increases, as the increasing write-down rate reduces the stochastic variability of  $\eta$ , providing less incentive to hedge. See Figure 2.

Figure 5 demonstrates the welfare improvement over the case when investors are not allowed to participate in defaultable bond markets. A common message from the upper two panels of Figure 5 is that there exists non-trivial welfare improvement for investors from investing in credit market. This result is quite

robust with respect to different economic scenarios.<sup>23</sup> For example, the lowest level of  $R^W$  (the annualized continuous rate of return in terms of certainty equivalent wealth with credit market against that without credit market) is roughly 2.22% over all parameter values except  $\omega$ . The bottom left panel of Figure 5 shows that the welfare improvement can easily achieve around 3% per year with typical parameter values. Interestingly, comparing the bottom panels of Figure 5 suggests that the welfare improvement from investing in junk bonds may be overshadowed by that from investing in investment-grade bonds. This is quite possible, since investors take into account not only the mean rates of return of the bonds but also their riskiness when making decisions. Junk bonds may appear “trashy” for investors with intermediate risk-aversions.

## 6 Conclusion

In this paper, we study the optimal investment in credit market in addition to equity market. Given the importance of credit market, it is a natural extension of Merton (1971). We adopt the reduced-form approach in pricing the defaultable zero-coupon bond and derive the closed-form solution to the asset allocation problem.

Our analysis provides several important insights. First, state variables (besides time) appears in the optimal weights (both in myopic demand and hedging demand). Therefore, investors in this model try to time the market to maximize welfare. Second, non-zero recovery rate of defaultable bond induces an adjustment term in the drift term under the risk-neutral measure, making the risk premium stochastic. As expected, there is hedging demand for defaultable bond as its risk premium is stochastic for non-zero recovery rate. This insight has been neglected in the literature. As shown in the text, under RMV scheme, the recovery rate (not necessarily recovery *risk*) is an integral part of defaultable bond pricing and has nontrivial effects on investment behavior. Third, market correlation is an important factor in the investment decision. The existence of correlation between markets induces position in one market from a position in the other. Furthermore, the investor’s ability to hedging against or speculate on stochastic risk premium is also affected by correlation.

Finally, our numerical exercises show that investors can achieve substantial welfare improvement by investing in credit market under various market conditions. The rate of return in terms of the certainty equivalent wealth with credit market versus without credit market is about 3% per annum for typical investors. It is also shown that hedging demands may behave quite differently from myopic demands. For example, hedging demand may depend on risk-aversion in a complex way. This highlights the purpose of such demand which is to hedge against stochasticity in investment opportunity set.

This paper is one of the first to study the optimal investment in credit

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<sup>23</sup>Some spikes in the graphs may be due to the non-linearity of  $R^W$  on some parameters and/or to the computation algorithm. Nonetheless, the smooth part of the graphs are clear enough for drawing meaningful conclusions.

market. There is much to be done. For example, we deal with defaultable zero-coupon bond for simplicity and do not study other credit market instruments such as coupon bond, and credit derivatives. We also assume deterministic interest rates. Interest rates, however, are an integral part of fixed-income market and should be formally included in such analysis. Partial equilibrium analysis is adopted in this paper, while equilibrium approach may generate more insights. For instance, two heterogeneous agents may be modeled in this context. Finally, we observe that RMV scheme is pivotal in our analysis. Analysis within other recovery schemes should be conducted. This paper opens several research avenues to be explored in the future.

## 7 Appendix

### 7.1 Proof of Proposition 1

**Proof.**

As in the text, set  $F = 1$ . From (5),  $p(t, T_1) = \tilde{H}(t) v(t, T_1)$ , where  $\tilde{H}(t) := 1 - H(t) = 1_{\{\tau > t\}}$ . Write  $v(t, T_1) = \tilde{b}(t) \phi(t)$ , where  $\tilde{b}(t) := \exp\left(\int_0^t (r(s) + \delta(s)) ds\right)$ ,  $\phi(t) := \mathbb{E}^Q\left(\exp\left(-\int_0^{T_1} (r(s) + \delta(s)) ds\right) \middle| \mathcal{F}_t\right)$  and  $\phi$  is an  $(\mathbb{F}, Q)$ -martingale. Since  $\tilde{b}$  is a finite-variation (FV) process, we have

$$\begin{aligned} dv(t, T_1) &= d\left(\tilde{b}(t) \phi(t)\right) = \phi(t) d\tilde{b}(t) + \tilde{b}(t) d\phi(t) \\ &= (r(t) + \delta(t)) v(t, T_1) dt + \tilde{b}(t) d\phi(t) \end{aligned}$$

An application of Ito's product rule to  $p(t, T_1)$  yields:

$$\begin{aligned} dp(t, T_1) &= \tilde{H}(t-) dv(t, T_1) + v(t-, T_1) d\tilde{H}(t) + \triangle v(t, T_1) \triangle \tilde{H}(t) \\ &= \tilde{H}(t-) \left[ (r(t) + \delta(t)) v(t, T_1) dt + \tilde{b}(t) d\phi(t) \right] + v(t, T_1) d\tilde{H}(t) \end{aligned}$$

where the assumption that the pre-default value  $v$  does not jump at default time is used. Indeed, the pre-default value  $v$  given in (5) is continuous, hence  $v(t-, T_1) = v(t, T_1)$ ,  $\forall t \in [0, T_1]$ . Then

$$dp(t, T_1) = (r(t) + \delta(t)) p(t-, T_1) dt + \tilde{H}(t-) \tilde{b}(t) d\phi(t) - v(t, T_1) dH(t)$$

where  $p(t-, T_1) = \tilde{H}(t-) v(t-, T_1) = \tilde{H}(t-) v(t, T_1)$  and  $dH(t) = -d\tilde{H}(t)$ .

Now we are about to derive  $d\phi(t)$ . Apparently, since  $\phi$  is an  $(\mathbb{F}, Q)$ -martingale, its drift term under  $Q$  has to be zero. We only need to derive its diffusion term.

Write  $\phi(t) = \mathbb{E}^Q\left(\exp\left(-\int_0^{T_1} (r(s) + \delta(s)) ds\right) \middle| \mathcal{F}_t\right) = C_0(t) \phi_\delta(t)$ ,

where  $C_0(t) := \exp\left(-\int_0^t (r(s) + \delta(s)) ds\right) \exp\left(-\int_t^{T_1} r(s) ds\right)$ , and  $\phi_\delta(t) := \mathbb{E}^Q\left(\exp\left(-\int_t^{T_1} \delta(s) ds\right) \middle| \mathcal{F}_t\right)$ . The process of  $\delta$  is Gaussian under the measure

$Q$ . Under this specifications,  $\phi_\delta(t)$  can be easily derived.

$$\begin{aligned}\phi_\delta(t) &= \mathbb{E}^Q \left( \exp \left( - \int_t^{T_1} \delta(s) ds \right) \middle| \mathcal{F}_t \right) \\ &= \exp \left\{ -\mathbb{E}^Q \left( \int_t^{T_1} \delta(s) ds \middle| \mathcal{F}_t \right) + \frac{1}{2} \text{Var}^Q \left( \int_t^{T_1} \delta(s) ds \middle| \mathcal{F}_t \right) \right\} \\ &= C_\delta(t, T_1) \exp(\varsigma_\delta(t, T_1) \delta(t))\end{aligned}$$

where  $\varsigma_\delta(t, T_1) := \frac{\exp(-\kappa_\delta(T_1-t))-1}{\kappa_\delta}$  and  $C_\delta(t, T_1)$  is a deterministic term. We get

$$d\phi(t) = \phi(t) \varsigma_\delta(t, T_1) \sigma_\delta(t) dw_\delta(t)$$

Plug this result into the SDE of  $p(t, T_1)$ , one easily has

$$\begin{aligned}dp(t, T_1) &= (r(t) + \delta(t)) p(t-, T_1) dt + \tilde{H}(t-) \tilde{b}(t) \phi(t) \varsigma_\delta(t, T_1) \sigma_\delta dw_\delta(t) \\ &\quad - v(t, T_1) dH(t) \\ &= p(t-, T_1) [(r(t) + \delta(t)) dt + \varsigma_\delta(t, T_1) \sigma_\delta dw_\delta(t)] - v(t, T_1) dH(t)\end{aligned}$$

■

## 7.2 Proof of Theorem 1

**Proof.**

Assume  $J_{WW} < 0$ . The first order conditions to HJB equation (18) can be used to solve for optimal portfolio weights  $\pi^*$  as a function of the indirect utility function  $J$  and other parameters in this economy. After some straightforward though tedious derivation,  $\pi^*$  can be solved as follows.

$$\begin{aligned}\pi_S^* &= -\frac{J_W}{W J_{WW}} \left[ \frac{\bar{\lambda}_S - \rho \bar{\lambda}_\delta}{\sigma_S (1 - \rho^2)} + \frac{\rho}{\sigma_P \sigma_S (1 - \rho^2)} \eta \right] \\ \pi_P^* &= -\frac{J_W}{W J_{WW}} \left[ \frac{\bar{\lambda}_\delta - \rho \bar{\lambda}_S}{\sigma_P (1 - \rho^2)} - \frac{1}{\sigma_P^2 (1 - \rho^2)} \eta \right] - \frac{J_{W\eta}}{W J_{WW}} \frac{\sigma_\eta}{\sigma_P}\end{aligned}\tag{27}$$

Inserting  $\pi^*$  derived above into the HJB equation yields the following partial differential equation (PDE):

$$\begin{aligned}0 &= J_t J_{WW} + W r J_W J_{WW} + \Gamma_1 J_W^2 + [(\sigma_\eta \bar{\lambda}_\delta + \kappa_\delta \theta_\eta) - \kappa_\delta \eta] J_\eta J_{WW} \\ &\quad + \left( \frac{\eta}{\sigma_P} - \bar{\lambda}_\delta \right) \sigma_\eta J_W J_{W\eta} + \frac{\sigma_\eta^2}{2} J_{\eta\eta} J_{WW} - \frac{\sigma_\eta^2}{2} J_{W\eta}^2\end{aligned}\tag{28}$$

with terminal condition:

$$J(W, \eta, T) = \frac{W^{1-\gamma}}{1-\gamma}$$

where

$$\Gamma_1 := \frac{\bar{\lambda}_\delta - \rho \bar{\lambda}_S}{\sigma_P (1 - \rho^2)} \eta - \frac{1}{2(1 - \rho^2) \sigma_P^2} \eta^2 - \frac{\bar{\lambda}_S^2 - 2\rho \bar{\lambda}_S \bar{\lambda}_\delta + \bar{\lambda}_\delta^2}{2(1 - \rho^2)}$$

Conjecture the solution  $J$  to the PDE (28) is of the form

$$J(W, \eta, t) = f(t, \eta) \frac{W^{1-\gamma}}{1-\gamma} \text{ with } f(T, \eta) = 1 \text{ for } \forall \eta \in \mathbf{R} \quad (29)$$

With this particular functional form, the PDE can be simplified as

$$0 = \gamma f f_t - (1-\gamma) \Gamma_1 f^2 + (1-\gamma) \frac{\sigma_\eta^2}{2} f_\eta^2 + \frac{1}{2} \gamma \sigma_\eta^2 f f_{\eta\eta} - \Gamma_2 f f_\eta + \gamma (1-\gamma) r f^2 - \gamma \kappa_\delta \eta f f_\eta \quad (30)$$

with terminal condition  $f(T, \eta) = 1, \forall \eta \in \mathbf{R}$  and

$$\Gamma_2 := (1-\gamma) \frac{\sigma_\eta}{\sigma_P} \eta - \gamma \kappa_\delta \theta_\eta - \sigma_\eta \overline{\lambda_\delta}$$

Assume

$$f(t, \eta) = g(t) \exp \left( k(t) \eta + \frac{1}{2} l(t) \eta^2 \right)$$

with terminal conditions  $g(T) = 1, k(T) = l(T) = 0$ . Plugging this functional into (30) and simplifying yields

$$\begin{aligned} 0 = & \left\{ \gamma k_t + \sigma_\eta^2 l k - \left[ (1-\gamma) \frac{\sigma_\eta}{\sigma_P} + \gamma \kappa_\delta \right] k + (\gamma \kappa_\delta \theta_\eta + \sigma_\eta \overline{\lambda_\delta}) l - \frac{(1-\gamma) (\overline{\lambda_\delta} - \rho \overline{\lambda_S})}{\sigma_P (1-\rho^2)} \right\} \eta \\ & + \left\{ \frac{\gamma}{2} l_t + \frac{\sigma_\eta^2}{2} l^2 - \left[ (1-\gamma) \frac{\sigma_\eta}{\sigma_P} + \gamma \kappa_\delta \right] l + (1-\gamma) \frac{1}{2(1-\rho^2) \sigma_P^2} \right\} \eta^2 \\ & + \gamma \frac{g_t}{g} + \frac{\sigma_\eta^2}{2} k^2 + \frac{\gamma \sigma_\eta^2}{2} l + (\gamma \kappa_\delta \theta_\eta + \sigma_\eta \overline{\lambda_\delta}) k + \gamma (1-\gamma) r \\ & + \frac{(1-\gamma) (\overline{\lambda_S}^2 - 2\rho \overline{\lambda_S} \overline{\lambda_\delta} + \overline{\lambda_\delta}^2)}{2(1-\rho^2)} \\ \text{or } 0 = & C_1 \eta + C_2 \eta^2 + \left( \gamma \frac{g_t}{g} + C_0 \right) \text{ where definitions of } C_0, C_1, C_2 \text{ are self-evident.} \end{aligned}$$

The coefficients of  $\eta$  and  $\eta^2$  must be identically zero and last term also must be zero. From  $C_2 = 0$ , one can see that it is a Riccati equation which has the following solution:<sup>24</sup>

$$l(t) = \begin{cases} 0 & \text{if } \gamma = 1 \\ \frac{1-\gamma}{\gamma(1-\rho^2)\sigma_P^2} \times \frac{2(e^{\vartheta(T-t)} - 1)}{[\vartheta + 2(\kappa_\delta + \frac{1-\gamma}{\gamma} \frac{\sigma_\eta}{\sigma_P})](e^{\vartheta(T-t)} - 1) + 2\vartheta} & \text{if } \gamma \neq 1 \end{cases} \quad (31)$$

<sup>24</sup>This is not quite mathematically rigorous since  $\sigma_P$  is a deterministic time-varying function, making the coefficient not constant. However, with typical parameters used in this paper, it is virtually constant since the maturity of the defaultable bond  $T_1$  is much larger than investor's planning horizon  $T$ . For example, with  $T_1 = 10, T = 1$ , and  $\kappa_\delta = 1.4248, \sigma_\delta = 0.0131$ , it can be easily seen that  $\sigma_P(0) \approx \sigma_P(1) = -0.00919427$ , which are indistinguishable from each other up to 8 digits after point.



where  $\vartheta := 2\sqrt{(\kappa_\delta + \frac{1-\gamma}{\gamma} \frac{\sigma_\eta}{\sigma_P})^2 - \frac{(1-\gamma)\sigma_\eta^2}{\gamma^2(1-\rho^2)\sigma_P^2}}$ .

From  $C_1 = 0$ , we can solve for  $k(t)$ , given  $l$ . Rewrite the equation as follows:

$$\begin{aligned} k(t)' &= \underbrace{-\frac{\sigma_\eta^2}{\gamma} l(t)}_{=:a} k(t) + \underbrace{[\frac{(1-\gamma)\sigma_\eta}{\gamma\sigma_P} + \kappa_\delta]}_{=:b} k(t) \\ &\quad - \underbrace{\left(\kappa_\delta\theta_\eta + \frac{\sigma_\eta\bar{\lambda}_\delta}{\gamma}\right) l(t)}_{=:c} + \underbrace{\frac{(1-\gamma)(\bar{\lambda}_\delta - \rho\bar{\lambda}_S)}{\gamma\sigma_P(1-\rho^2)}}_{=:d} \end{aligned} \quad (32)$$

Define  $q(t) := a \times l(t) + b(t)$ , if  $\omega \neq 1$ , then the solution to (32) is given by

$$k(t) = \begin{cases} 0 & \text{if } \gamma = 1 \\ \exp\left(-\int_t^T q(s) ds\right) \left[\frac{c}{a} - \int_t^T \left(d(s) - \frac{b(s)c}{a}\right) \exp\left(\int_s^T q(u) du\right) ds\right] - \frac{c}{a} & \text{if } \gamma \neq 1 \end{cases} \quad (33)$$

In the case of  $\omega = 1$ ,  $a = c = 0, b = \kappa_\delta$  since  $\sigma_\eta = \theta_\eta = 0$ . Then equation (32) is reduced to

$$k(t)' = \kappa_\delta k(t) + d$$

The solution is

$$k(t) = \frac{d}{\kappa_\delta} \left( e^{-\kappa_\delta(T-t)} - 1 \right) \quad (34)$$

Note that the apparent difference between (33) and (34) does not mean there is discontinuity of  $k$  with respect to  $\omega$ . In fact, it can be easily shown that when  $\omega = 1$ , (33) indeed becomes (34) after cancelling terms. In sum, function  $k$  is given by

$$k(t) = \begin{cases} 0 & \text{if } \gamma = 1 \\ \frac{d}{\kappa_\delta} (e^{-\kappa_\delta(T-t)} - 1) & \text{if } \gamma \neq 1 \text{ and } \omega = 1 \\ \exp\left(-\int_t^T q(s) ds\right) \left[\frac{c}{a} - \int_t^T \left(d(s) - \frac{b(s)c}{a}\right) \times \exp\left(\int_s^T q(u) du\right) ds\right] - \frac{c}{a} & \text{if } \gamma \neq 1 \text{ and } \omega \neq 1 \end{cases}$$

From  $\gamma \frac{g_t}{g} + C_0 = 0$ , it is easy to derive that

$$g(t) = \begin{cases} 1 & \text{if } \gamma = 1 \\ \exp\left\{\int_t^T \left[\frac{\sigma_\eta^2}{2\gamma} k(s)^2 + \left(\kappa_\delta\theta_\eta + \frac{\sigma_\eta\bar{\lambda}_\delta}{\gamma}\right) k(s) + \frac{1}{2}\sigma_\eta^2 l(s) + (1-\gamma)r + \frac{(1-\gamma)(\bar{\lambda}_S^2 - 2\rho\bar{\lambda}_S\bar{\lambda}_\delta + \bar{\lambda}_\delta^2)}{2\gamma(1-\rho^2)}\right] ds\right\} & \text{if } \gamma \neq 1 \end{cases} \quad (35)$$

Now the indirect utility is

$$J(W, \eta, t) = \begin{cases} g(t) \exp\left(k(t)\eta + \frac{1}{2}l(t)\eta^2\right) \frac{W^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln W & \gamma = 1 \end{cases}$$

with  $g(t)$ ,  $k(t)$  and  $l(t)$  given by (35), (33) and (31) respectively.

Finally, the optimal portfolio weights are given by

$$\begin{aligned}\pi_S^* &= \frac{1}{\gamma} \left[ \frac{\bar{\lambda}_S - \rho \bar{\lambda}_\delta}{\sigma_S (1 - \rho^2)} + \frac{\rho}{\sigma_P \sigma_S (1 - \rho^2)} \eta \right] \\ \pi_P^* &= \frac{1}{\gamma} \left[ \frac{\bar{\lambda}_\delta - \rho \bar{\lambda}_S}{\sigma_P (1 - \rho^2)} - \frac{1}{\sigma_P^2 (1 - \rho^2)} \eta \right] + \frac{k + l \times \eta}{\gamma} \frac{\sigma_\eta}{\sigma_P}\end{aligned}$$

Note that  $\sigma_P := -\frac{1-e^{-\kappa_\delta(T_1-t)}}{\kappa_\delta} \sigma_\delta$  and  $\sigma_\eta := \left(\frac{1-\omega}{\omega}\right) \sigma_\delta$ , these facts give the expressions shown in the text.

An integral part of this “guess-prove” approach to solving this stochastic control problem is *verification*. The standard verification theorems (see, for example, Fleming and Rishel (1975)) need to impose Lipschitz and growth conditions to ensure the existence and uniqueness of the solution to the controlled SDEs of the state variables ( in this case, the SDEs of  $W$  and  $\eta$ ). Because of the stochasticity of  $\eta$ , the standard regularity conditions are not satisfied. However, Korn and Kraft (2001) study the case of stochastic interest rate and have shown that under mild regularity conditions,  $J(W, r, t)$  is indeed the value function. Their argument can be applied to this paper. We refer the interested readers to the original paper for further details and proofs. ■

Table 1: Parameter Definitions and Values

This table summarizes parameters and their values used in this paper. Parameters of credit spread process are taken from Colin-Dufresne and Sohl (2001). Equity market parameters are obtained by calibrating to the U.S. stock market. The correlation coefficient between equity market and corporate bond market is taken from Kwan (1996). Write-down rate for investment-grade bond is from Duffee (1999). Other parameter values are set by the authors. Bond  $i$  refers to investment grade bonds. Bond  $j$  refers to junk bonds.

Symbol	Definition	Value
$r$	risk-free interest rate	.05
$\rho$	correlation coefficient between risk factors	-.184 for Bond $i$ -.423 for Bond $j$
$T$	planning horizon of the investor	1
$T_1$	maturity of bond	10
$\sigma_S$	instantaneous volatility of equity's return	.15
$\lambda_S$	adjusted market price of equity risk	.451
$\theta_\delta$	long-term credit spread level	.0038
$\sigma_\delta$	instantaneous volatility of credit spread	.0131
$\kappa_\delta$	mean-reversion speed of credit spread	1.4248
$\delta_0$	initial credit spread	.007 for Bond $i$ .02 for Bond $j$
$\omega$	write-down rate at default	.56 for Bond $i$ .8 for Bond $j$

Figure 1: Simulated Paths of Optimal Hedging Demand for Defaultable Bond

This figure shows simulated paths of the optimal hedging demand for defaultable bonds of distinct credit quality, once a path of the credit spread is simulated. The optimal hedging demand for a defaultable bond is given in equation (23).

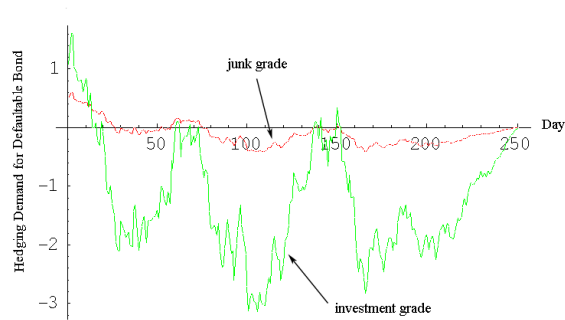


Figure 2: Optimal Hedging Demand for Defaultable Bond

The top left panel shows how the optimal hedging demands for defaultable bonds change with  $\rho$ , the correlation coefficient between the risk factors in the economy, at the end of the first quarter given a simulated credit spread path. The top right panel shows how this demand varies with  $\gamma$ , the relative risk aversion of the investor. The bottom left panel shows how the hedging demand for an investment grade bond varies with  $\omega$ , the writedown rate in the event of default, and with  $\rho$ . The bottom right panel shows how this demand varies with  $\bar{\lambda}_\delta$ , the market price of the default risk, and  $\rho$ . The optimal hedging demand for a defaultable bond is given in equation (23).

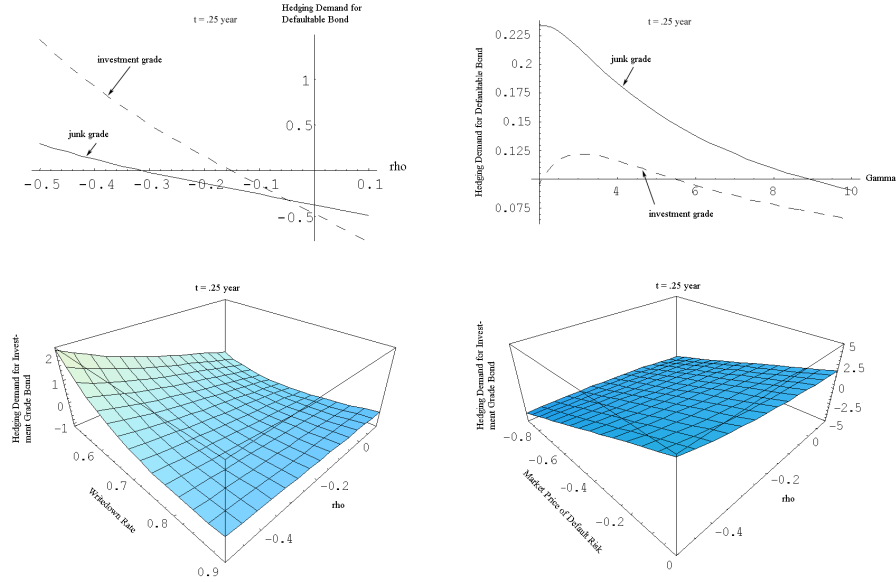


Figure 3: Optimal Demand for Equity

The top panel shows simulated paths of the optimal equity demand when either an investment grade bond or a junk bond is included in the opportunity set, once a path of the credit spread is simulated. The middle panel shows how the equity demand when an investment grade bond is included in the opportunity set varies with  $\omega$ , the writedown rate in the event of default, and with  $\rho$ , the correlation coefficient between the risk factors in the economy, at the end of the first quarter. The bottom panel shows how this demand varies with  $\bar{\lambda}_\delta$ , the market price of the default risk, and  $\rho$ . The optimal equity demand is given in equation (22).

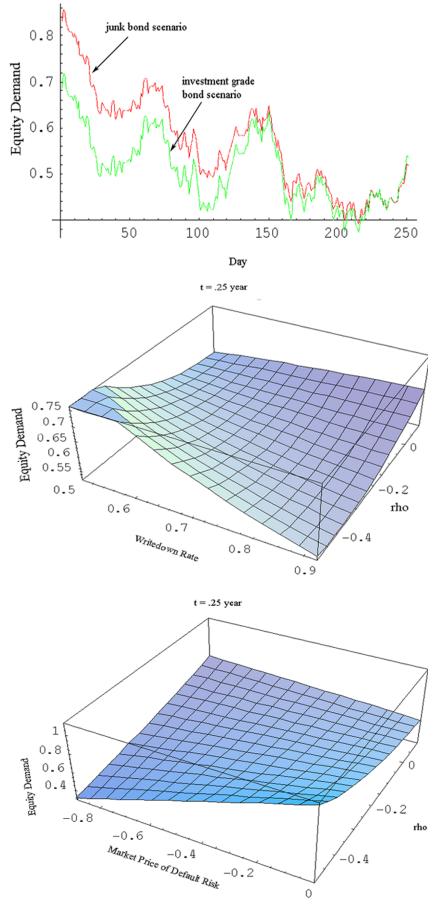


Figure 4: Optimal Myopic Demand for Defaultable Bond

The top panel shows simulated paths of the optimal myopic demand for defaultable bonds demand of distinct credit quality, once a path of the credit spread is simulated. The middle panel shows how the myopic demand for an investment grade bond varies with  $\omega$ , the writedown rate in the event of default, and with  $\rho$ , the correlation coefficient between the risk factors in the economy, at the end of the first quarter. The bottom panel shows how this demand varies with  $\lambda_\delta$ , the market price of the default risk, and  $\rho$ . The optimal myopic demand for a defaultable bond is given in equation (23).

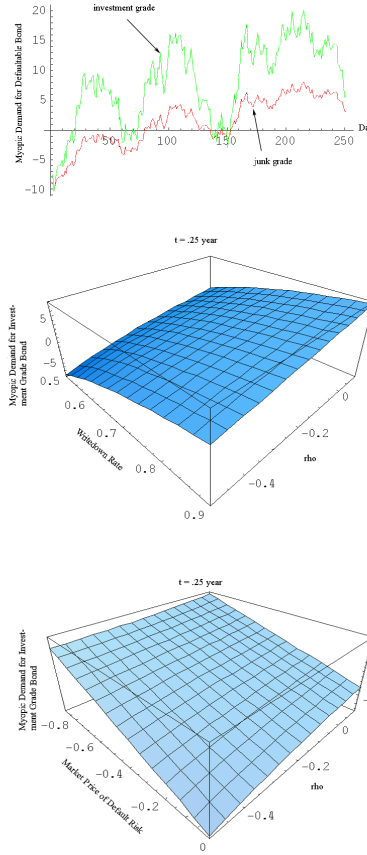
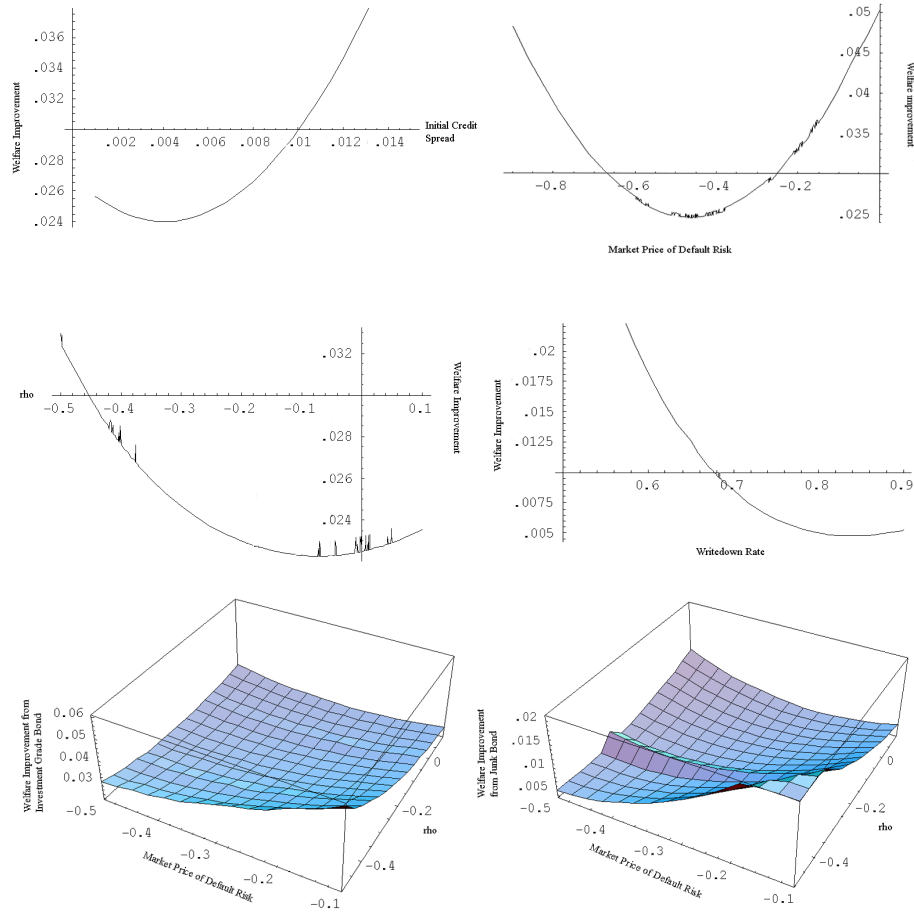


Figure 5: Welfare Improvement with Defaultable Bond

The welfare improvement measures the portfolio improvement in terms of the annualized, continuous compounded return in certainty equivalent wealth when an investor can invest in credit market.  $\delta_0$  is the initial credit spread.  $\bar{\lambda}_\delta$  is the market price of the default risk.  $\rho$  is the correlation coefficient between the risk factors in the economy.  $\omega$  is the writedown rate in the event of default. The bottom panels show how the welfare improvement from investing in an investment grade bond or in a junk bond varies with  $\bar{\lambda}_\delta$  and  $\rho$ . The welfare improvement is given in equation (26).





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