

Hidden Orders and Optimal Submission Strategies in a Dynamic Limit Order Market

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Abstract

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ABSTRACT

Recent empirical evidence on traders' order submission strategies in electronic limit order markets (LOB) shows the growing use of hidden orders. This paper provides a theory of the optimal order submission strategies in an LOB, where traders can choose among limit, market and hidden orders. The dynamic model we propose allows for strategic interaction of traders on the two sides of the LOB, asset volatility and varying order sizes. Hence, it allows traders take a simultaneous three-dimensional strategic choice of price, quantity and exposure. We find that hidden orders increase the liquidity of the LOB. The use of hidden orders increases with volatility, order size, and relative depth on the opposite side of the market, and it decreases with time-to-shock. Agents use order exposure and price aggressiveness as complements.

1 Introduction

Electronic limit order markets have become the dominant market structure for trading financial securities around the world. These are order-driven markets in which traders can either supply liquidity by submitting limit orders or demand liquidity by submitting market orders. Orders posted on the limit order book (LOB) must specify a number of instructions which qualify their sign, size, and eventually their price aggressiveness and degree of disclosure.

Recent empirical evidence about traders' order submission strategies on electronic limit order books shows the growing importance of a special type of order, called "hidden" or "iceberg", that allows traders to limit their exposure by hiding a portion of the quantity they are willing to trade. Hidden orders amount to a striking proportion of trading volume: for example, they correspond to more than 44% of Euronext volume, about 28% of the Australian Stock Exchange volume, and account for more than 15% of total executions on INET and for 16% of executed shares on Xetra. Hidden orders are also extensively used at the NASDAQ and in secondary markets for treasury bonds.¹ Like limit orders, hidden orders contain an instruction about the limit price beyond which submitters are willing to trade; however, unlike limit orders, they contain a further instruction about the fraction of the order that is undisclosed to the market.

By allowing the use of hidden orders, regulators endogenously reduce the degree of pre-trade transparency and hence impact on both liquidity and price informativeness. It is therefore relevant to understand what are the effects on market quality of the widespread adoption of such trading facility. Under which circumstances hidden orders are optimal order submission strategies is the relevant issue for practitioners.

Despite a growing body of empirical research on hidden orders, there is little theoretical guidance on the optimal choice of order exposure. The objective of this paper is indeed to extend the existing literature on dynamic limit order markets with a theory of the optimal order submission strategies, where, beside the standard choice between limit and market orders, traders can also choose their order exposure.

As recent empirical evidence shows that hidden orders are predominantly used by uninformed traders to reduce the cost of being picked-off by front-runners and scalpers,² in our framework hidden orders are modelled as de-

¹See respectively Bessembinder, Panayides and Venkataraman (2007), Aitken, Berkman and Mak (2001), Hasbrouck and Saar (2004 and 2007), Frey and Sandas (2008), Tuttle (2006) and Jiang, Lo and Verdelhan (2007).

²See for example Aitken, Berkman and Mak (2001) , Bessembinder, Panayides and

fensive strategies. The model internalizes three important elements: the strategic dynamic interaction of traders with the two sides of the LOB, asset volatility and different order sizes.

As the choice of hidden orders is related to exposure costs, the model allows for an asset value shock at the end of the trading game, and, to avoid easy detection of hidden orders by market participants, the order size is made endogenous. The model draws from Parlour (1998) the interaction between traders' strategies and the two sides of the LOB, and from Foucault (1999) the impact of front-running costs on traders' order placement strategies.³

Agents have a private evaluation of the asset which determines their degree of impatience; they arrive sequentially at the market and choose the optimal submission strategy contingent on the state of the LOB. A variegated spectrum of trading strategies is available to them: in addition to market and limit orders, they can opt for hidden orders, as well as marketable orders. Furthermore, they can choose their degree of price aggressiveness and hence face a simultaneous three-dimensional strategic choice of price, quantity and exposure.

A group of fast traders who make profits out of the fundamental variations is also active in the market. These scalpers pick-off the mispriced visible liquidity available on the LOB. For this reason, agents run exposure risk when posting visible limit orders.

The model is solved under different specifications. The results of a benchmark model without hidden orders are compared with those of a model which, all else equal, includes undisclosed orders. This comparison allows us to show the effects of hidden orders both on the state of the LOB, and on traders' aggressiveness and willingness to offer and take liquidity.

We find that hidden orders increase the liquidity of the LOB, measured by depth and tightness, and that this effect increases as the asset value shock approaches. This is explained by the fact that hidden orders are submitted to prevent front-running, which takes place right after the shock. Regarding traders' optimal submission strategies, the use of hidden orders increases with order size and decreases with time-to-shock; furthermore, order exposure and price aggressiveness are used as complements by market participants.

The results obtained allow us to further investigate the adverse selection

Venkataraman (2007), De Winne and D'Hondt (2007) and Frey and Sandas (2008).

³Goettler, Parlour and Rajan (2005, 2007) focus as well on the working of an LOB and extend Parlour's framework to model limit order trading as a stochastic sequential game with private and common value; they also introduce endogenous information acquisition. To examine the resiliency and spread dynamic of the LOB, Foucault, Kadan and Kandel (2005) include traders' waiting costs, and Rosu (2006) considers a continuous time model with endogenous undercutting.

component of the spread that in this model arises due to exposure costs rather than inside information. Upon the arrival of information shocks, fast traders can pick-off stale prices; hence limit order submitters who are not quick enough to cancel or update their orders pay adverse selection costs. Clearly the higher the volatility, the greater the exposure costs and the larger the spread as traders react by submitting less aggressive limit orders. This is a testable implication of our model.

This model can be extended in many directions. For example, it is rather straightforward to include a per-period asset value shock, and discuss how time-to-maturity influences the choice of hidden orders; it is also possible to endogenously derive the optimal size of disclosure. In fact, the rules governing hidden orders are heterogeneous around the world, being hidden orders only partially disclosed on Euronext and on most financial markets, and, instead, completely invisible on the limit order book of INET and on the platform of Project Turquoise.⁴

The remainder of the paper is organized as follows. Section 2 discusses the related literature on hidden orders, Section 3 introduces the model's structure and Section 4 analyzes the benchmark model. Section 5 presents the model with hidden orders, and Section 6 concludes. All the proofs are in the Appendix.

2 Literature on Hidden Orders

The existing literature on hidden orders is primarily empirical with a few theoretical works. There seems to be a large consensus that hidden orders are mainly used as defensive strategies by uninformed traders. Aitken, Berkman and Mak (2001) show that in the Australian stock market there is no difference in the stock price reaction between disclosed and undisclosed limit orders and conclude that the latter are not more attractive than disclosed limit orders for informed traders. Pardo and Pascual (2006) show that in the Spanish market hidden volume detection has no relevant impact on returns and volatility. This implies that hidden orders are used to obscure the trading strategies of large uninformed investors. Furthermore, De Winne and D'Hondt (2007) show that traders become significantly more aggressive when there is a signal for hidden depth at the best quotes on the opposite side of the market. They also show that traders tend to hide more when the size of their order is large relative to the prevailing displayed depth, and they conclude that traders use hidden quantity to manage both exposure risk and

⁴Project Turquoise is a Pan-European Multilateral Trading Facility for trading stocks, which will be operative since October 2008.

picking-off risk. Similarly, Frey and Sandas (2008) show that traders bid very aggressively when they suspect hidden liquidity on the opposite side of the market, whereas they become less aggressive when competing with hidden liquidity on the same side.

The empirical evidence also shows that traders use hidden orders more extensively under those market conditions that facilitate quote-matchers and front-runners. More precisely, Harris (1996, 1997) and Aitken et al. (2001) show that traders are more likely to hide their orders when the tick size is small and/or when volatility is high; they conclude that traders use hidden orders to control the option value associated with their limit orders.⁵ Finally, more recently Bessembinder, Panayides and Venkataraman (2007) study the limit order traders' joint decision on price, size and exposure; they find that price aggressiveness and order exposure are used as complements by traders. They also show that this complementarity holds for orders that are not expected to be immediately executed, and suggest that uninformed traders hide their orders to mitigate the option value offered to other traders.

The theoretical works on the use of hidden orders are few. To our knowledge, only two models explicitly include hidden orders among the order types available to market participants. Moinas (2005) extends the model of Kaniel and Liu (2006) to demonstrate how the availability of hidden orders affects informed traders' welfare. The model, however, does not allow uninformed traders to use hidden orders. Furthermore, it does not embody the interaction between the two sides of the LOB. Esser and Mönch (2007) extend the literature on optimal liquidation strategies (e.g. Bertsimas and Lo, 1998; Almgren and Chriss, 2000; Mönch, 2004) to include hidden orders; they determine the optimal limit price and pick size for an iceberg order in a static framework without any strategic interaction among traders.

3 The Framework

We discuss the choice of the optimal order submission strategy in a dynamic LOB by building a model that includes a price grid, an asset value shock and a wide range of order submission strategies. Traders' strategic interaction with the LOB allows us to consider the influence of the depth available at the top of the book on traders' order choice; the price grid defines price priority. Finally, the introduction of an asset value shock allows us to model

⁵ As described in Harris (2003), large uninformed orders run the risk of being front-run by scalpers and quote-matchers, also called parasitic traders. Scalpers' and quote matchers' profits are increasing function of price volatility; quote matchers' profits are also increasing in the minimum tick size.

and discuss the relevance of hidden orders as defensive strategies against front-runners.

3.1 The Working of the LOB

A market for a risky asset takes place over a trading day that is divided into T periods: $t = 1, \dots, T$. At period T , trading finishes and the risky asset pays v_T units. At each period within the trading day, one risk-neutral agent arrives and maximizes his expected profits by choosing an optimal trading strategy that cannot be modified thereafter.

Upon arrival, each trader observes the LOB that is formed by a grid of six prices, three on the ask and three on the bid side of the market. It follows that the prices at which each trader can buy or sell are equal to $A_{1,2,3}$ (ask prices) and $B_{1,2,3}$ (bid prices), with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$; for simplicity we assume that these prices are symmetric around the common value of the asset, v_t . More precisely, traders can demand liquidity over the whole price grid, whereas they can offer liquidity only at the first two levels of the book. This is due to the fact that at A_3 and B_3 a trading crowd absorbs any amount of the risky asset demanded or offered by the incoming trader. As in Parlour (1998) and Seppi (1997), the trading crowd prevents traders from bidding prices that are too far from the inside spread, and is only a theoretical shortcut to limit the price grid. It is also assumed that the difference between the ask and the bid price is equal to the tick size, d , that is the minimum price increment traders are allowed to quote over the existing price.⁶

The state of the book at each period t , $b_t = [A_2, A_1, B_1, B_2]$, is characterized by both the price grid and the number of shares available at each price. It is assumed that the asset value remains constant between $t = 1$ and $t = T - 1$, while between time $T - 1$ and T a shock on the asset value arrives,⁷ so that v_T can either increase, remain constant, or decrease with equal probability:

$$\begin{aligned} v_t &= V + \varepsilon_T & t = T \\ v_t &= V & \forall t = 1, \dots, T - 1 \end{aligned} \tag{1}$$

⁶Hence the fundamental value of the asset is equal to the inside spread mid-quote, i.e. $v = \frac{A_1 + B_1}{2}$, with $A_1 = v + \frac{d}{2}$, $A_2 = v + \frac{3}{2}d$, $A_3 = v + \frac{5}{2}d$ and $B_1 = v - \frac{d}{2}$, $B_2 = v - \frac{3}{2}d$, $B_3 = v - \frac{5}{2}d$.

⁷This assumption greatly simplifies the algebra and allows us to focus only on the last periods of the game. We could include an asset value shock at each trading round, but this would multiply the possible trading strategies and therefore make the computations substantially longer.

with:

$$\varepsilon_T = \begin{cases} +k d & \text{with prob } = \frac{1}{3} \\ 0 & \text{with prob } = \frac{1}{3} \\ -k d & \text{with prob } = \frac{1}{3} \end{cases} \quad (2)$$

where $V > 0$ is constant and is assumed for simplicity equal to one. The parameter k allows us to consider alternative volatility specifications: the higher k , the larger the volatility. As we are interested in potential mispricing on both the first two levels of the LOB, we focus on the case with $k = 2$.⁸ Figures 1 shows the possible evolution of the price grid over time for this asset value shock. We denote the ask and bid prices after a positive (or a negative) price change as A_i^U (A_i^D) and B_i^U (B_i^D) respectively, with $i \in \{1, 2, 3\}$.

[Insert Figure 1 here]

3.2 Market Participants

We model three categories of risk-neutral traders: large institutional traders, who can trade up to ten units, small retail traders, who can only trade one unit, and scalpers. In real markets the latter are agents who trade on their own account and usually do not hold a position for more than a few minutes (Harris, 2003). Scalpers' main profits are due to gone-off prices which they quickly hound down from the book.

3.2.1 Institutional and Retail Traders

Agents arrive sequentially at the market, and at each trading round nature chooses a large or a small trader with equal probability. Each agent is characterized by his personal evaluation of the risky asset, β_t , that is drawn from the following uniform distribution:

$$\beta_t \sim U[\underline{\beta}, \bar{\beta}] \quad \text{where } 0 \leq \underline{\beta} \leq 1 \leq \bar{\beta} \quad (3)$$

Notice that each trader's β_t can be interpreted either as his private evaluation of the asset, or as his degree of impatience: traders with extreme

⁸With a smaller asset value shock ($k = 1$), orders submitted to the second level would never be mispriced and hence would not bear exposure costs. Hence, by construction, traders would not have incentive to submit hidden orders to the second level of the LOB, and, for example, it would not be possible to analyze the complementarity/substitutability relation between aggressiveness and exposure.

value of β_t either value the asset very little, or very much, and in both cases they are respectively the most impatient sellers (low β_t), or buyers (high β_t); traders holding a β_t next to 1 are instead the most patient. We also assume that the distribution of β_t is symmetric around $\beta = 1$.⁹

3.2.2 Scalpers

In our model scalpers are arbitrageurs and hence they are only interested in exploiting the free-option offered by limit order submitters. What distinguishes scalpers from other traders is their speed of reaction: they are much faster than all the other market participants and hence they can pick-off visible stale prices after an asset value shock. These front-runners can alternatively be interpreted as very aggressive informed agents trading ahead of public announcements to exploit short-lived private information.

We assume that scalpers submit marketable limit orders with a limit price equal to the highest stale price, and with size equal to the visible mispriced quantity. Indeed if they submitted orders of larger size, they would run the risk and therefore face the costs of taking a position on the LOB, as the unexecuted quantity would turn into a limit order. Hence we assume that scalpers can pick-off large mispriced limit orders, but they cannot generally hound the invisible quantity of mispriced hidden orders.¹⁰ This implies that a large trader can reduce his losses by submitting hidden orders. He will loose time priority on the hidden part of his order, but he will also reduce the costs of being front-run by scalpers by cancelling the undisclosed depth.

An example of fast traders who search hidden liquidity by submitting aggressive orders is reported in Hasbrouck and Saar (2007). By investigating a sample of stocks listed on INET, where hidden orders have the peculiarity of being completely invisible to market participants, they show that nearly 37% of limit orders are very short-lived (cancelled within the next two seconds) and are posted at aggressive prices above the same-side BBO. The authors suggest that these aggressive limit orders look for hidden depth, as their pattern is similar to that of executions against hidden liquidity. We capture the

⁹This hypothesis makes traders' strategies symmetric around the asset value. Notice however that, with a constant tick size equal to d , when the common value of the asset changes due to the shock, the value of the tick size relative to the asset price changes as well. This effect slightly modifies market orders' execution probability at T , and hence makes the optimal trading strategies at $t \neq T$ not perfectly symmetric.

¹⁰We could relax this hypothesis and allow scalpers to partially hit hidden mispriced liquidity. Indeed, conversations with practitioners informed us that scalpers are cautious when searching hidden liquidity and hence they would not be able to pick-off the whole hidden part as if it were visible.

behavior of these traders in our model, but since hidden orders are partially visible, scalpers use marketable orders to detect them.

3.3 Trader's Order Choice

Orders are instructions to trade: when a trader decides his optimal order submission strategy, he simultaneously chooses the *sign*, the *size*, the aggressiveness and the *degree of exposure* of his order. The following factors drive the choice and the selection of the optimal trading strategy: the costs associated with each strategy, the trader's type and the state of the book.

Trading costs There are three types of costs a trader faces when choosing an order submission strategy: execution costs, price opportunity costs and exposure costs. Execution costs are the waiting costs that traders pay whenever their orders are not immediately executed; they are generally associated with limit orders that are stored on the book and inevitably have a slower execution than market orders. Execution costs are hence minimized by choosing market or marketable orders, that guarantee immediate execution. However, as market orders are generally priced at the top of the book, they bear higher price opportunity costs. Finally, exposure costs arise from the possibility that a fast trader takes advantage of the limit orders submitted to the book. To evaluate exposure costs it is necessary to think of limit orders as free-options offered to market participants by limit order submitters, whose value depend on three interrelated features: time to maturity, volatility, and limit price.

Traders' type As we have assumed that there is no private information, traders only differ by their degree of impatience. Very impatient traders weigh a lot execution costs, whereas patient traders assign more value to price opportunity costs. It follows that facing the trade-off between execution and price opportunity costs, very impatient traders will choose market or marketable orders, whereas patient traders will also consider limit orders. But when opting for limit orders, patient traders will also have to take into account the exposure costs that can arise from scalpers. Very patient traders will then opt for hidden orders.

State of the LOB Before submitting their order, traders also take into account the state of the LOB. Due to time priority, market depth affects execution costs, and hence influences traders' order choice: higher depth on the own side increases execution costs, while the opposite occurs when the

book is deep on the other side of the market, as incoming traders will submit more market than limit orders.

3.4 Order Submission Strategies

Table 1 presents the possible orders that a small trader (Panel A) and a large trader (Panel B) can choose depending on their type (β) and on the state of the LOB. By assumption, large traders can submit orders of size $j = 1, \dots, 10$, whereas small traders can only trade one share ($j = 1$). Moreover, the feasibility and profitability of these strategies depend on the state of the LOB from the time of the order submission (b_t) to the end of the trading game T (b_{t+s} , with $s = 1, \dots, T - t$). It may happen that at time t a trader does not find any profitable strategy and decides not to trade.¹¹

Small Trader An aggressive small trader (Panel A) can demand liquidity by submitting a market sell order that hits one of the three prices on the bid side of the LOB (MO_1B_i); this order will be executed against the limit order with the highest priority available at the corresponding bid price. A more patient small trader can instead opt to act as liquidity supplier and submit a limit sell order either to the first (LO_1A_1), or to the second level of the LOB (LO_1A_2); in this case, his order increases market depth of one share at A_1 or A_2 respectively. Alternatively, he can decide not to trade (NTS). If he submits an LO_1A_2 , his order will be executed when a market buy order arrives which hits the limit price A_2 after that all the other orders on the LOB with lower price and higher time priority have been executed. The small trader's choice between A_2 and a more aggressive limit price, A_1 , depends on both his degree of patience, and on the state of the LOB at the time of his submission. Similar strategies apply to the other side of the LOB.

Large Trader An aggressive large trader (Panel B) can demand liquidity by submitting a market sell order (MO_jB_i) of size j ($1 \leq j \leq 10$) that will hit the limit buy orders with the highest precedence on the bid side. Alternatively, a large trader may opt for a marketable sell order (MRO_jB), which walks down the LOB on the bid side.¹² A more patient large trader

¹¹We do not include order splitting in the model as this strategy is generally dominated by hidden orders or limit orders. Splitting orders could only be optimal when the book is full on the second level; however, this particular case is not relevant to our analysis as here hidden orders are already dominated by limit orders.

¹²Notice that in our setting marketable orders are defined as market orders that walk up or down the book. As a marketable order may hit different prices, we do not use an index for the level of the book as we do for the other order types.

can instead choose to submit a limit sell order of size j to either A_1 or A_2 ($LO_j A_{1,2}$), or he can decide to disclose only 1 unit of his order and submit a hidden sell order to the first or to the second level of the ask grid. By assumption, hidden orders are of size ten with only one visible unit ($HO_{10} A_{1,2}$). This is consistent with the most widespread regulatory requirements on hidden orders that impose a minimum disclosed quantity equal to 10% of the order size.¹³ The undisclosed part of the hidden order loses time priority with respect to incoming limit orders at the same limit price, but has the advantage that it cannot be picked-off by scalpers in case of mispricing. Finally, a large trader can decide not to trade (NTL).

4 The Benchmark Model

We look for the optimal order submission strategies of a three-period model which lasts from time $T - 2$, to time T .¹⁴ We first solve a benchmark model where traders are allowed to choose all the strategies described in Table 1, except hidden orders. We then compare the benchmark's results with those of a model where traders can also choose to hide part of their orders.

4.1 The Trader's Problem

The extensive form of the game is shown in Figure 2. Notice first that at time T the incoming agent is only able to submit market or marketable orders. This is due to the fact that the market closes and the execution probability of a limit order is equal to zero. At time $T - 1$ and $T - 2$, instead, traders can also choose limit orders. Notice that the execution probabilities of these orders are equal to the probabilities that market or marketable orders will hit them before T .

We assume that the market opens at $T - 2$ with an empty book,¹⁵ $b_{T-2} = [0000]$, and from period $T - 2$ onward traders' orders will gradually fill the LOB. Suppose for example that nature selects a small trader at $T - 2$; if he chooses to submit $LO_1 A_1$, then at $T - 1$ the book will open as $b_{T-1} = [0100]$. If we now assume that nature chooses a large trader who selects $LO_9 A_1$, then the opening book at T will be $b_T = [0(10)00]$. This implies that at T

¹³See, for example, Euronext Cash Markets Trading Manual (2007).

¹⁴By starting at time $T - 2$ we can endogenously derive the initial state of the book at time $T - 1$. Furthermore, the three-period framework allows us to analyse the effects of time-to-shock on hidden order submission.

¹⁵This assumption is parsimonious as it will be shown that the probability of hidden orders is the lowest when the book is empty.

large traders' available strategies are MO_jB_3 , NTL and MO_jA_1 , while small traders will choose among MO_1B_3 , NTS and MO_1A_1 .

More generally, the risk-neutral large trader will choose the optimal order submission strategy, o_{L,β_t} , that maximizes his expected profits conditional on the state of the LOB, b_t , and his degree of impatience, β_t . A large seller will submit the order that solves:¹⁶

$$\max_{o_{L,\beta_t} \in [MO_jB_i, MRO_jB, LO_jA_i, NTL]} E[\pi_t(o_{L,\beta_t})] \quad (4)$$

where

$$\begin{aligned} \pi_t(MO_jB_i) &= j(B_i^z - \beta_t v_t) \\ \pi_t(MRO_jB) &= (j - \sum_{i=1,2} f_i)(B_3^z - \beta_t v_t) + \sum_{i=1,2} f_i(B_i^z - \beta_t v_t) \\ E[\pi_t(LO_jA_i)] &= E \left[\sum_{l=t+1}^T (A_l - \beta_t \tilde{v}_l) \sum_{h=0}^{j-1} \Pr \left(\sum_{m=t}^{l-1} w_m = h \mid b_t, v_t \right) \sum_{w_l=1}^{j-h} w_l \times \Pr_{w_l}(A_i | b_l, v_l) \right] \\ \pi_t(NTL) &= 0 \end{aligned}$$

where $i \in [1, 3]$, $B_i^z \in \{B_i, B_i^U, B_i^D\}$, $j \in [1, 10]$, $\sum_i f_i = j$; w_t are the order units executed at t , and $\Pr_{w_l}(A_i | b_l, v_l)$ is the probability that w_l units are executed at price A_i at $t = l$. Notice that because profits from a sell order are decreasing in β_t , agents holding a low evaluation of the asset will more likely sell, whereas those with a high valuation will probably buy.

The small seller will solve a similar problem:

$$\max_{o_{S,\beta_t} \in [MO_1B_i, LO_1A_i, NTS]} E[\pi_t(o_{S,\beta_t})] \quad (5)$$

where

$$\begin{aligned} \pi_t(MO_1B_i) &= (B_i^z - \beta_t v_t) \\ E[\pi_t(LO_1A_i)] &= E \left[\sum_{l=t+1}^T (A_l - \beta_t \tilde{v}_l) \left(\prod_{m=t+1}^{l-1} \Pr_{w_m=0}(A_i | b_m, v_m) \right) \Pr_{w_l=1}(A_i | b_l, v_l) \right] \\ \pi_t(NTS) &= 0 \end{aligned}$$

where $i \in [1, 3]$, $B_i^z \in \{B_i, B_i^U, B_i^D\}$, $\Pr_{w_l}(A_i | b_l, v_l)$ and $\Pr_{w_m}(A_i | b_m, v_m)$ are the probabilities that w units are executed at A_i , respectively at $t = l$ and $t = m$.

¹⁶The optimization program of a buyer is almost symmetric; hence, when possible, we will only discuss the sell side.

Equilibrium definition An equilibrium of the benchmark model is a set of orders o_{L,β_t}^* and o_{S,β_t}^* , with $t = \{T, T-1, T-2\}$, that solves Program (4) and (5), when the execution probabilities $\Pr_w(A_i|b_{T-l}, v_{T-l})$, for $l = \{0, 1\}$, are computed assuming that traders submit orders o_{L,β_t}^* and o_{S,β_t}^* .

[Insert Figure 2 here]

4.2 Equilibrium Order Submission Strategies

We find the solution of this game by backward induction starting from the end-nodes at time T and computing the probabilities of market and marketable orders; these are the execution probabilities of limit orders placed at $T-1$, and hence they allow us to compute the equilibrium order submission strategies at $T-1$. Given the probability of market and marketable orders at $T-1$, we can then compute the equilibrium order submission strategies at $T-2$.

4.2.1 Equilibrium Strategies at T

At time T , a small trader will submit a market sell order if the asset price is higher than his valuation ($B_i^z > \beta_T v_T^z$); he will submit a market buy order in the opposite case ($\beta_T v_T^z > A_i^z$), and he will not trade for intermediate values of β_T . These conditions are satisfied for the values of β_T summarized below:

$$\begin{aligned} \text{submit } MO_1 B_i &\quad \text{if } B_i^z \geq \beta_T v_T^z && \text{i.e. } \beta_T \leq \frac{B_i^z}{v_T^z} \\ \text{submit } MO_1 A_i &\quad \text{if } \beta_T v_T^z \geq A_i^z && \text{i.e. } \beta_T \geq \frac{A_i^z}{v_T^z} \\ NTS &\quad \text{if } \frac{B_i^z}{v_T^z} < \beta_T < \frac{A_i^z}{v_T^z} \end{aligned}$$

where $B_i^z \in \{B_i, B_i^U, B_i^D\}$ and $A_i^z \in \{A_i, A_i^U, A_i^D\}$

Conditional on the state of the book, large traders have the option of submitting either market orders or marketable orders for j shares:

1. When j shares are available either on the first level or on the second level of the book, or when there is no depth on both levels and the agent is forced to trade against the trading crowd on the third level of the book, the large trader's β_T thresholds are the same as those of a retail trader, even if he will be trading j shares rather than one.

2. If there are $f_i < j$ shares on either A_1^z or A_2^z , the large trader will have the option of either submitting a market order of f_i shares on A_i^z , or a marketable order of j shares that will also hit the level q of the book with $q > i$. His β_T thresholds for the ask side are the following:

$$\begin{aligned} \text{submit } MRO_j A &\quad \text{if } \beta_T \geq \frac{A_q^z}{v_T^z} \\ \text{submit } MO_{f_i} A_i &\quad \text{if } \frac{A_i^z}{v_T^z} \leq \beta_T < \frac{A_q^z}{v_T^z} \\ NTL &\quad \text{if } 1 < \beta_T < \frac{A_i^z}{v_T^z} \end{aligned}$$

3. If there are $f_i < j$ shares on the A_i^z level of the book for both $i = 1$ and $i = 2$, with $f_1 + f_2 < j$, a large trader will face the choice of either submitting a market order of f_1 shares, a marketable order of $f_1 + f_2$ shares, or a marketable order of j shares. His β_T thresholds for the ask side are the following:

$$\begin{aligned} \text{submit } MRO_j A &\quad \text{if } \beta_T \geq \frac{A_3^z}{v_T^z} \\ \text{submit } MRO_{f_1+f_2} A &\quad \text{if } \frac{A_2^z}{v_T^z} \leq \beta_T < \frac{A_3^z}{v_T^z} \\ \text{submit } MO_{f_1} A_1 &\quad \text{if } \frac{A_1^z}{v_T^z} \leq \beta_T < \frac{A_2^z}{v_T^z} \\ NTL &\quad \text{if } 1 < \beta_T < \frac{A_1^z}{v_T^z} \end{aligned}$$

Once computed the ranges for β_T at T , to obtain numerical values for these thresholds it is necessary to choose a support for the probability distribution of β . We assume that β is uniformly distributed with support $[0, 2]$. Clearly, the β intervals and the execution probabilities are conditional on both the state of the LOB at time T , b_T , and the realization of v_T . As shown in the Appendix, it is straightforward to derive these probabilities from the thresholds obtained above.

4.2.2 Equilibrium Strategies at T-1 and at T-2

Traders' submission strategies at $T - 1$ depend on both traders' degree of impatience (β_{T-1}) and the state of the LOB (b_{T-1}), that depends in turn on traders' choice at the previous period $T - 2$. For example, if at $T - 2$ the incoming trader submits a 10-unit limit order at A_1 , the book at $T - 1$, $b_{T-1} = [0(10)00]$, will be full on the first level. Indeed, as time priority

applies and a trader at T will submit at the most a market order of size $j = 10$, new limit orders on the ask side will never be executed. Therefore, all the strategies that involve any unit submitted at A_1 and A_2 will not be feasible (i.e. $LO_j A_i$).

To solve the model for traders' equilibrium strategies at $T - 1$, we compare their profits from all the feasible strategies given the state of the LOB. This allows us to obtain the β_{T-1} ranges and to derive both the probabilities of all the feasible order types, and the execution probabilities of the limit orders posted at $T - 2$.¹⁷

Proposition 1 *At $T - 1$ large and small traders' equilibrium strategies depend on both the traders' types, and the state of the LOB.*

- *If the book is empty at $T - 1$, very impatient traders will submit market orders at A_3 and B_3 . More patient traders will choose limit orders at the second level of the book, A_2 and B_2 . No trader will submit limit orders at the first level of the book.*
- *When the book is full on the first level of the ask side, sellers will either submit market orders at B_3 , or they will not trade. Buyers, instead, will submit market orders at A_1 , or limit orders at B_2 . When the book is partially full on the first level (one share on A_1), both small and large traders generally submit limit orders on A_2 , rather than not trading.*
- *When the book is full on the second level of the ask side, very impatient buyers will hit A_2 , and patient sellers will post limit orders at A_1 . Contrary to the case of the empty book, traders may also decide not to trade. When the book is partially full on the second level (one share on A_2), both small and large traders submit limit orders at A_2 .*

At $T - 2$ impatient sellers submit market orders at A_3 . More patient agents submit limit orders at the second level of the book, A_2 .

The equilibrium intervals of β_{T-1} for a small trader are shown in Figures 3.1 to 3.5 and those for a large trader in Figures 3.6 to 3.10. The equilibrium values of β_{T-1} are plotted for $d \in [0, 0.2]$.

This Proposition shows that, whenever possible, traders tend to submit limit orders at the highest level of the LOB. This is due to the fact that the higher exposure costs they bear on the first level of the book more than

¹⁷For analytical convenience we solve the model by considering only sellers' strategies at $T - 2$. Due to the symmetry of the model, specular equilibrium strategies are obtained by including buyers.

outweigh the higher probability of execution. Notice that the larger the tick size, d , the larger the probability that a patient trader chooses a limit order on A_2 , as he would benefit of a 4-tick increments compared to a market order.

When the book is full on the second level and traders cannot submit limit orders at A_2 , they face two options: bear the higher exposure costs associated with a limit order on the first level of the book, or refrain from trading. As very patient traders will indeed decide to exit the market, we can conclude that in the benchmark model exposure costs affect liquidity by reducing agents' willingness to trade.

[Insert Figure 3.1 – 3.10 here]

5 The Model with Hidden Orders

When agents are allowed to submit hidden orders, the depth of the book becomes uncertain. The incoming trader does not know whether there is any undisclosed depth, and hence assigns a probability to each possible state of the LOB. The higher the probability of hidden depth, the greater his aggressiveness and the larger the execution probabilities of the standing limit orders.

Traders rationally compute the probability of hidden depth for orders submitted at $T - 1$. For orders submitted at $T - 2$, we suppose instead that the incoming trader at T holds adaptive expectations, as he assumes that the probability of hidden liquidity is the same as at $T - 1$.¹⁸

5.1 The Trader's Problem with Hidden Liquidity

The extensive form of the game is shown in Figure 4. Notice that now different strategies may imply the same visible LOB for traders arriving the next period. For example, if at $T - 2$ nature selects either a small trader who chooses a LO_1A_2 or a large trader who chooses a $HOS_{10}A_2$, the opening visible book at $T - 1$ will be $b_{T-1} = [1000]$, and the incoming trader will assign a probability to the two possible states: $b_{T-1} = [1000]$ and $b_{T-1} = [(1+9)000]$.

As in the previous case, to optimally choose his trading strategy, a large seller will compare the expected profits associated with all the feasible sell orders (Table 1) and submit the order that solves:

¹⁸This hypothesis reflects the idea that traders carefully monitor the LOB when they are about to trade, but do not pay the same level of attention when they are not planning to trade for some periods.

$$\max_{o_{L,\beta_t} \in [MO_j B_i, MRO_j B, LO_j A_i, HO_{10} A_i, NTL]} E[\pi_t(o_{L,\beta_t})] \quad (6)$$

where

$$\begin{aligned} \pi_t(MO_j B_i) &= j(B_i^z - \beta_t v_t) \\ \pi_t(MRO_j B) &= (j - \sum_{i=1,2} \Pr_{f_i}(B_i|b_t, v_t) f_i)(B_3^z - \beta_t v_t) + \sum_{i=1,2} \Pr_{f_i}(B_i|b_t, v_t) f_i(B_i^z - \beta_t v_t) \\ E[\pi_t(LO_j A_i)] &= E \left[\sum_{l=t+1}^T (A_i - \beta_t \tilde{v}_l) \sum_{h=0}^{j-1} \Pr \left(\sum_{m=t}^{l-1} w_m = h \mid b_t, v_t \right) \sum_{w_l=1}^{j-h} w_l \times \Pr_{w_l}(A_i|b_l, v_l) \right] \\ E[\pi_t(HO_{10} A_i)] &= E \left[\sum_{l=t+1}^T (A_i - \beta_t \tilde{v}_l) \sum_{h=0}^9 \Pr \left(\sum_{m=t}^{l-1} w_m = h \mid b_t, v_t \right) \sum_{w_l=1}^{10-h} w_l \times \Pr_{w_l}(A_i|b_l, v_l) \right] \\ \pi_t(NTL) &= 0 \end{aligned}$$

where $i \in [1, 3]$, $B_i^z \in \{B_i, B_i^U, B_i^D\}$, $j \in [1, 10]$, $\sum_i f_i = j$; w_t are the units of the limit order executed at t , $\Pr_{w_l}(A_i|b_l, v_l)$ is the probability that w_l shares are executed at $t = l$ and $\Pr_{f_i}(B_i|b_t, v_t)$ is the probability that at t there are f_i hidden shares available at B_i .

The small seller will solve a similar problem:

$$\max_{o_{S,\beta_t} \in [MO_1 B_i, LO_1 A_i, NTS]} E[\pi_t(o_{S,\beta_t})] \quad (7)$$

where

$$\begin{aligned} E[\pi_t(MO_1 B_i)] &= (B_i^z - \beta_t v_t) \\ E[\pi_t(LO_1 A_i)] &= E \left[\sum_{l=t+1}^T (A_i - \beta_t \tilde{v}_l) \left(\prod_{m=t+1}^{l-1} \Pr_{w_m=0}(A_i|b_m, v_m) \right) \Pr_{w_l}(A_i|b_l, v_l) \right] \\ E[\pi_t(NTS)] &= 0 \end{aligned}$$

where $i \in [1, 3]$, $B_i^z \in \{B_i, B_i^U, B_i^D\}$, $\Pr_{w_l}(A_i|b_l, v_l)$ and $\Pr_{w_m}(A_i|b_m, v_m)$ are the probabilities that w shares are executed respectively at $t = l$ and $t = m$.

The important difference with the benchmark model is that now traders do not know the exact amount of liquidity available on the LOB. Hence, when computing the execution probabilities of hidden and limit orders, they consider that subsequent traders will face uncertainty on the execution price of their marketable orders, as it is evident from the profit of $MRO_j B$.

Equilibrium definition An equilibrium of the trading game with hidden orders is a set of orders o_{L,β_t}^* and o_{S,β_t}^* and $t = \{T, T-1, T-2\}$, that solves Program (6) and (7), when both the expected execution probabilities,

$\Pr_w(A_i|b_{T-l}, v_{T-l})$, and the probabilities traders assign to the different states of the book, $\Pr_{f_i}(B_i|b_{T-l}, v_{T-l})$, for $l = \{0, 1\}$, are computed assuming that traders submit orders o_{L,β_t}^* and o_{S,β_t}^* .

We solve the model with hidden orders by backward induction, assuming that the tick size is equal to $d = 0.1$.¹⁹

[Insert Figure 4 here]

5.2 Equilibrium Order Submission Strategies

Orders' submission strategies at T are derived as in the benchmark case. Results are summarized in the following Lemma:

Lemma 1 *Small traders choose the same strategies as in the benchmark case. Large traders' strategies differ since now order execution prices for a given visible book are weighted averages of the prices available in all the possible realizations of the LOB.*

Small traders, who submit a one-unit order, are not concerned by undisclosed liquidity and hence their strategies are the same as in the benchmark model. Hidden depth, instead, affects the execution price of marketable orders, and therefore it influences large traders' strategies at T .

The following Proposition summarizes the results from the model.

Proposition 2 *Hidden orders are equilibrium strategies both when the book is empty and when it is partially full on the second level. Hidden orders are only submitted at the second level of the book. The following factors affect the use of partially undisclosed orders:*

- *The use of hidden orders increases with the depth at the opposite side of the LOB*
- *The proportion of hidden orders increases with volatility and decreases with time-to-shock*
- *The use of hidden orders decreases with price aggressiveness, hence order exposure and price aggressiveness are used as complements by market participants*
- *The proportion of hidden orders increases with order size.*

¹⁹We have solved the model for different values of the tick size, d , and found that the results do not change qualitatively. These results are available from the authors on request.

Market Depth As Table 2 (Panel A) shows, the larger the depth on the own side, the lower the execution probability of limit orders and the fewer the hidden orders submitted to the LOB. As an example, $HO_{10}A_2$ is used when the ask side is empty or when there is only one share at A_2 .

Notice also that hidden orders are used more intensively when the book is full or partially full on the opposite side. Consider as an example $HO_{10}B_2$: when the opposite side of the market is deeper, the probability of execution of this hidden buy order increases, as the incoming seller, by observing a long queue on the sell side, will more likely submit a market order. It follows that traders will submit hidden buy orders more extensively.

These results are consistent with the empirical evidence reported by both De Winne and D'Hondt (2007) and Pardo and Pascual (2006), who find that traders are more likely to hide part of their limit orders on the ask (bid) side when the visible bid (ask) depth is larger than the visible ask (bid) depth.

Volatility and Time-to-Shock To capture the effects of volatility on hidden orders, we compare their use across periods.²⁰ In fact, it is straightforward to show that the average volatility, Σ_t , is smaller at $T - 2$ than at $T - 1$:

$$\Sigma_{T-2} = \frac{Var(\tilde{v}_{T-1}) + Var(\tilde{v}_T)}{2} = \frac{4}{3}(.1)^2 < \Sigma_{T-1} = Var(\tilde{v}_T) = \frac{8}{3}(.1)^2$$

We find that, all else equal, hidden orders are used more intensively at $T - 1$ than at $T - 2$ (Table 2, Panel A), hence we can conclude that higher volatility induces traders to use hidden orders more widely. When agents perceive less urgent the need to prevent their orders from being picked-off, they submit hidden orders with a lower probability. Therefore, the shorter the time to the asset value shock, the higher the probability of hidden orders. This result explains the recent empirical evidence from the US bond market: Jiang et al. (2007) show that the relative use of hidden orders significantly increases right before macroeconomic news announcements.

Order Exposure and Price Aggressiveness The model's results show that traders do not submit hidden orders at the first level of the book as the higher execution probability is more than compensated by the higher exposure costs and the lower sell price. Indeed, when the book is empty at $T - 1$, traders submit both limit and hidden orders only on A_2 (Table 2, Panel A). When instead the book is full at A_2 ($b_{T-1} = [(10)000]$) and traders

²⁰Alternatively we could solve the model for different values of k .

are forced to submit more aggressive orders on A_1 , they submit limit orders rather than hidden orders. This suggests that when traders increase price aggressiveness, they also enhance order disclosure. The complementarity between price aggressiveness and disclosure is due to the fact that, when there is a large amount of shares visible at A_2 , hidden orders submitted at A_1 bear the same exposure costs as limit orders, but have lower execution probability. In fact, when front-runners observe a large mispriced order at A_2 , even if only one share is visible at A_1 , they submit a marketable order for the whole visible mispriced quantity. Their order will walk up the book and eventually hit the undisclosed quantity at A_1 . This result is in line with the empirical findings of Bessembinder et al. (2007), who observe complementarity between price aggressiveness and order exposure.

[Insert Table 2 here]

Order Size The most recent empirical evidence shows that the use of hidden orders increases with order size (Aitken et al., 2001; Bessembinder et al., 2007; Pardo and Pascual, 2006). To analyze this effect, we compare the results obtained for $j \leq 10$ with those derived by assuming that, all else equal, large traders are not allowed to submit orders for a size larger than two shares ($j \leq 2$). Table 3 reports the proportion of hidden orders over the total limit orders submitted at A_2 for different states of the LOB. The results show that hidden orders are always used more intensively when the maximum order size is ten shares. For example, when the book is $b_{T-1} = [0000]$ this ratio is .081 for $j \leq 10$, and .075 for $j \leq 2$.

[Insert Table 3 here]

5.3 Market Quality

In light of the growing use of hidden orders in electronic trading platforms, it becomes relevant from a regulatory point of view to investigate whether market participants benefit of hidden orders, and whether such orders improve market quality. The results obtained by comparing the model with hidden orders to the benchmark model are reported in the following Proposition.

Proposition 3 *The use of hidden orders enhances market quality as follows:*

- *it increases the probability that large traders choose to trade*
- *it increases market liquidity*

As previously shown, the use of hidden orders only affects the equilibrium order submission strategies of large traders. We find that with hidden orders the probability that large traders refrain from trading decreases substantially (Table 2, Panel A). For example, in the model with hidden orders when the book is full or partially full at A_2 , no-trading is no longer an equilibrium strategy at $T - 1$.

We then measure the improvement in market quality by comparing the increase in total depth (disclosed and undisclosed) offered by large traders across the two models. For the benchmark model, the expected depth is equal to the probability of observing large limit orders on both sides of the LOB (L_{NH}), while for the model with hidden orders it is equal to the probability of observing both disclosed and hidden limit orders on the two sides of the LOB (L_H). We find that the estimated depth in the model with hidden orders is systematically greater than in the benchmark:

$$L_H = 10 \left(\Pr_H HO_{10}A_2 + \Pr_H HO_{10}B_2 \right) + \sum_j \sum_i j \left(\Pr_H LO_j A_i + \Pr_H LO_j B_i \right) > \\ \sum_j \sum_i j \left(\Pr_{NH} LO_j A_i + \Pr_{NH} LO_j B_i \right) = L_{NH}$$

where \Pr_{NH} and \Pr_H are the order probabilities in the benchmark and in the model with hidden orders respectively.

Our measure of liquidity provision also allows us to infer the effect of hidden orders on the inside spread. In so far as the probability of observing limit orders from large traders increases with hidden orders, it also increases the probability to observe, at $T - 1$, a smaller inside spread. Hence in the hidden model the expected inside spread is tighter.

These findings are consistent with the results obtained by Anand and Weaver (2004) who investigate the effect of the introduction of hidden orders at the Toronto Stock Exchange and show that total depth at the inside increases significantly when traders are allowed to use hidden orders. Analogously, Bessembinder and Venkataraman (2004) find evidence that the use of hidden orders increases market depth on the top of the LOB: they show that at the Paris Bourse the implicit transaction costs of blocks decrease due to the presence of hidden orders.

6 Concluding Remarks

A growing body of empirical evidence shows that hidden orders are widely used in electronic limit order platforms around the world. Hence, it becomes

important to understand whether the intensive use of these orders is beneficial to market participants and/or to market quality. The empirical evidence shows that hidden orders are mostly used by large uninformed traders as defensive strategies against front-runners, but there is no theory to investigate how hidden orders can be used to control exposure costs, and how such orders can affect market liquidity and traders' profits.

In this paper a theory of hidden orders is presented to discuss agents' optimal trading strategies in an LOB where traders are allowed to choose among hidden orders and a wide variety of other order types.

The attractiveness of hidden orders is related to the option that traders offer to market participants by submitting limit orders; the option value depends on price aggressiveness, the state of the LOB, volatility and traders' type. The resulting dynamic model has a price-grid and an asset value shock right before the end of the game, and offers results both on the determinants of hidden orders, as well as on the effects of hidden orders on market liquidity. The results show that hidden orders are positively related to expected volatility, and inversely related to time-to-shock. The use of hidden orders increases both with the order size and with the depth on the opposite side of the LOB. It is also shown that traders use order-exposure and price-aggressiveness as complements, thus confirming the empirical results obtained by Bessembinder et al. (2007).

The use of hidden orders is not only relevant from the point of view of the market participants' optimal trading strategies, but also, and maybe more importantly, it is an instrument that market regulators can use to fine-tune the optimal degree of market transparency. The degree of pre-trade transparency offered to market participants is a timely issue in the design of an LOB, and among the various rules adopted by Exchanges, there is also the authorization to hide limit order. When hidden orders are allowable trading strategies, investors, by observing the screen, are not necessarily informed of the exact depth offered or demanded at the posted quotes. By allowing traders to submit hidden orders, the regulatory authority faces a trade-off between higher liquidity and lower transparency. This important issue is addressed in this paper by comparing a model with hidden orders with a benchmark model without undisclosed depth. The results show that hidden orders increase the total liquidity available on the top of the LOB, and reduce the probability that traders decide not to trade, thus inducing some agents to enter the market. It follows that, even if hidden orders reduce pre-trade transparency, they improve liquidity and hence market quality.

These findings are consistent with the existing empirical evidence from different financial markets (Aitken et al., 2001; Anand and Weaver, 2004; De Winne and D'Hondt, 2007; Frey and Sandas, 2008; Pardo and Pascual, 2006),

and also respond to various issues raised by recent empirical research. For example, Hasbrouck and Saar (2007) show that 37% of limit orders placed on INET are very short-lived and very aggressively priced; the model captures the effects of these fast traders placing aggressive orders in search of liquidity on both agents' trading strategies and market quality.

Finally, an important testable empirical implication of this model is that it suggests to include among the estimated components of the bid-ask spread the option premium due to exposure costs. This component depends on the state of the book, the time of the day and the volatility of the asset value.

This model can be extended in many directions. It is possible to include different values of the asset shock or to make the undisclosed part of the order endogenous. This last extension would be useful to explain why the rules that allow traders to submit hidden orders are not homogeneous across different financial markets. In Euronext, as in most financial trading platforms, traders can only partially hide their limit orders; on INET, instead, traders can submit limit orders that are completely invisible. The optimal regulation of undisclosed depth is, therefore, an interesting topic for future research.

Appendix

Proof of Proposition 1

We solve the model by backward induction, starting from $t = T$. Due to risk neutrality, large traders' profits from market orders are maximized for $j = 10$. For example, if we consider the strategy MO_jB_i , traders' profits will be $j(B_i - \beta_t v_t)$: the larger j , the larger the profits. When computing the expected profits of limit orders, however, the execution probabilities become relevant. It is simple to show that the execution probability of a limit order does not change for $j = 2, \dots, 10$, hence, because of risk neutrality, traders' profits are maximized for $j = 10$. Notice further that a one-unit limit order ($j = 1$) has a higher execution probability, and possibly higher profits, than a 10-unit limit order. Hence, from now onwards we assume that j is equal to either the maximum possible value given the depth of the LOB, or one.

Period T

The thresholds for period T have been derived in Section 4.2; given our assumption that $\beta \sim U[0, 2]$, it is simple to derive the probabilities of the trading strategies. As an example, if $b_{T-1} = [0000]$, agents' trading strategies at T are:

$$\begin{aligned}\Pr_T(MO_1B_3 | b_{T-1}) &= \Pr(S) \left(\frac{2-\beta_{MO_1B_3,NTS}}{2} \right) = (1/2)(\frac{2-(B_3/v_T)}{2}) = (1/8)(2-5d) \\ \Pr_T(MO_{10}B_3 | b_{T-1}) &= \Pr(L) \left(\frac{2-\beta_{MO_{10}B_3,NTL}}{2} \right) = (1/2)(\frac{2-(B_3/v_T)}{2}) = (1/8)(2-5d) \\ \Pr_T(MO_1A_3 | b_{T-1}) &= \Pr(S) \left(\frac{\beta_{MO_1A_3,NTS}}{2} \right) = (1/2)(\frac{A_3/v_T}{2}) = (1/8)(2-5d) \\ \Pr_T(MO_{10}A_3 | b_{T-1}) &= \Pr(MO_1A_3 | b_{T-1}) \\ \Pr_T(NTS | b_{T-1}) &= 2 \left[1 - \Pr_T(MO_1B_3) - \Pr_T(MO_1A_3) \right] = 1 - 1/4(2-5d) \\ \Pr_T(NTL | b_{T-1}) &= \Pr_T(NTS | b_{T-1})\end{aligned}$$

where $\beta_{.,.}$ is the threshold between two trading strategies.

Period $T-1$

To obtain agents' optimal submission strategies at $T-1$, we first consider the possible states of the LOB. To simplify the analysis, we only examine sellers' strategies at $T-2$; hence at $T-1$ the bid side of the LOB is always empty. The possible states of the LOB at $T-1$ are summarized in the following Table:

LOB at $T - 1$					
	0	10	0	1	0
A_2	0	10	0	1	0
A_1	0	0	10	0	1
B_1	0	0	0	0	0
B_2	0	0	0	0	0

Case 1: Small Trader

The small trader solves problem (5), presented in Section 4.1. His available strategies at $T - 1$ depend on the initial state of the book, as shown in Table 4.1. Our focus on the ask strategies at $T - 2$ has some implications for the analysis: $MO_1B_{1,2}$ is never feasible and hence is omitted, while MO_1B_3 , LO_1B_1 and LO_1B_2 are always available. The trader also has the option not to trade, NTS .

[Insert Table 4.1 and Table 4.2 here]

We consider the book $b_{T-1} = [(10)000]$ as an example, the other cases can be derived similarly. In this case, the available strategies are: MO_1B_3 , LO_1A_1 , LO_1B_1 , LO_1B_2 and MO_1A_2 . The profits from these strategies are the following:

$$\begin{aligned}
\pi_{T-1}(MO_1B_3) &= (B_3 - \beta_{T-1}v_{T-1}) \\
\pi_{T-1}(MO_1A_2) &= (\beta_{T-1}v_{T-1} - A_2) \\
E[\pi_{T-1}(LO_1A_1)] &= E \left[(A_1 - \beta_{T-1}\tilde{v}_T) \Pr_{w_T=1}(A_1 | b_T, v_T) \right] \\
&= \frac{1}{3} \left[(A_1 - \beta_{T-1}v_T) \left(\Pr_T(MO_{10}A_1 | b_T) + \Pr_T(MO_1A_1 | b_T) \right) \right] \\
&\quad + \frac{1}{3}(A_1 - \beta_{T-1}v_T^U) \\
\text{where } b_T &= [0100] \\
E[\pi_{T-1}(LO_1B_i)] &= E \left[(\beta_{T-1}\tilde{v}_T - B_i) \Pr_{w_T=1}(B_i | b_T, v_T) \right] \\
&= \frac{1}{3} \left[(\beta_{T-1}v_T - B_i) \left(\Pr_T(MO_{10}B_i | b_T) + \Pr_T(MO_1B_i | b_T) \right) \right] \\
&\quad + \frac{1}{3}(\beta_{T-1}v_T^D - B_i) \\
\text{where } b_T &= [0010] \text{ or } b_T = [0001] \text{ respectively for } LO_1B_1 \text{ and } LO_1B_2
\end{aligned}$$

The equilibrium intervals of β_{T-1} for a small trader are obtained by comparing these profits and by finding the ranges of β_{T-1} associated with his

optimal trading strategies. Results for the five possible states of the book at $T - 1$ are summarized in Figures 3.1 to 3.5, and numerical values for the case $d = 0.1$ are presented in Table 2, Panel B.

Case 2: Large Trader

The large trader solves problem (4), presented in Section 4.1. As before, MRO_jB and $MO_jB_{1,2}$ are not feasible strategies and we omit them, while $MO_{10}B_3$, LO_jB_1 and LO_jB_2 are always feasible strategies. For the remaining strategies, their feasibility depend on the state of the LOB as shown in Table 4.2.

We solve the case $b_{T-1} = [(10)000]$ as an example, the other cases can be derived in a similar way. In this case, the feasible strategies are: $MO_{10}B_3$, LO_jA_1 , LO_jB_1 , LO_jB_2 and $MO_{10}A_2$. The profits from these strategies are the following:²¹

$$\begin{aligned}
\pi_{T-1}(MO_{10}B_3) &= 10(B_3 - \beta_{T-1}v_{T-1}) \\
\pi_{T-1}(MO_{10}A_2) &= 10(\beta_{T-1}v_{T-1} - A_2) \\
E[\pi_{T-1}(LO_{10}A_1)] &= E \left[(A_1 - \beta_{T-1}\tilde{v}_T) \sum_{w_T=1}^{10} w_T \times \Pr(A_1 | b_T, v_T) \right] \\
&= \frac{1}{3} \left[10(A_1 - \beta_{T-1}v_T) \Pr_T(MO_{10}A_1 | b_T) + (A_1 - \beta_{T-1}v_T) \Pr_T(MO_{10}A_2 | b_T) \right] \\
&\quad + \frac{1}{3} 10(A_1 - \beta_{T-1}v_T^U) \\
\text{where } b_T &= [0(10)00] \\
E[\pi_{T-1}(LO_{10}B_i)] &= E \left[(\beta_{T-1}\tilde{v}_T - B_i) \sum_{w_T=1}^{10} w_T \times \Pr(B_i | b_T, v_T) \right] \\
&= \frac{1}{3} \left[10(\beta_{T-1}v_T - B_i) \Pr_T(MO_{10}B_i | b_T) + (\beta_{T-1}v_T - B_i) \Pr_T(MO_{10}B_3 | b_T) \right] \\
&\quad + \frac{1}{3} [10(\beta_{T-1}v_T^D - B_i)] \\
\text{where } b_T &= [00(10)0] \text{ or } b_T = [000(10)] \text{ respectively for } LO_{10}B_1 \text{ and } LO_{10}B_2
\end{aligned}$$

To obtain the equilibrium β_{T-1} intervals for a large trader, we compare these profits and find the ranges of β_{T-1} associated with traders' optimal strategies. Results are summarized in Figures 3.6 to 3.10, and numerical values for the case $d = 0.1$ are presented in Table 2, Panel A.

²¹The profits from a $j = 1$ limit order are derived in Case 1 and hence omitted.

Period $T - 2$

We compute and compare the profits associated to the trader's strategies on the ask side at $T - 2$, and assume that the initial book is empty. Strategies on the bid side will be qualitatively similar, due to the symmetry of our modelization.

Case 1: Small Trader

The small trader solves problem (5). The profits of the feasible strategies on the ask side, MO_1B_3 , LO_1A_1 and LO_1A_2 , are the following:

$$\begin{aligned}
\pi_{T-2}(MO_1B_3) &= (B_3 - \beta_{T-2}v_{T-2}) \\
E[\pi_{T-2}(LO_1A_1)] &= E \left[\sum_{l=T-1}^T (A_1 - \beta_t \tilde{v}_l) \left(\prod_{m=t+1}^{l-1} \Pr_{w_m=0}(A_1 | b_m, v_m) \right) \Pr_{w_l=1}(A_1 | b_l, v_l) \right] \\
&= \frac{1}{2} \left[(A_1 - \beta_{T-2}v_{T-1}) \left[\Pr_{T-1}(MO_1A_1 | b_{T-1}) + \Pr_{T-1}(MRO_{10}A | b_{T-1}) \right] \right. \\
&\quad + \Pr_{T-1}(LO_{10}B_1 | b_{T-1})\gamma_{[01(10)0], A_1} + \Pr_{T-1}(LO_{10}B_2 | b_{T-1})\gamma_{[010(10)], A_1} \\
&\quad + \Pr_{T-1}(LO_1B_2 | b_{T-1})\gamma_{[0101], A_1} + \Pr_{T-1}(LO_1B_1 | b_{T-1})\gamma_{[0110], A_1} \\
&\quad + \left[\Pr_{T-1}(MO_{10}B_3 | b_{T-1}) + \Pr_{T-1}(NTL | b_{T-1}) + \Pr_{T-1}(MO_1B_3 | b_{T-1}) + \Pr_{T-1}(NTS | b_{T-1}) \right] \\
&\quad \times \gamma_{[0100], A_1} \} + \Pr_{T-1}(LO_9A_1 | b_{T-1})\gamma_{[0(10)00], A_1} + \Pr_{T-1}(LO_1A_1 | b_{T-1})\gamma_{[0200], A_1} \\
&\quad \left. + \Pr_{T-1}(LO_1A_2 | b_{T-1})\gamma_{[1100], A_1} + \Pr_{T-1}(LO_9A_2 | b_{T-1})\gamma_{[9100], A_1} \right] \\
E[\pi_{T-2}(LO_1A_2)] &= E \left[\sum_{l=T-1}^T (A_1 - \beta_t \tilde{v}_l) \left(\prod_{m=t+1}^{l-1} \Pr_{w_m=0}(A_1 | b_m, v_m) \right) \Pr_{w_l=1}(A_2 | b_l, v_l) \right] \\
&= \frac{1}{2} \left[(A_2 - \beta_{T-2}v_{T-1}) \left[\Pr_{T-1}(MO_1A_2 | b_{T-1}) + \Pr_{T-1}(MRO_{10}A | b_{T-1}) \right] \right. \\
&\quad + \Pr_{T-1}(LO_{10}B_1 | b_{T-1})\gamma_{[10(10)0], A_2} + \Pr_{T-1}(LO_{10}B_2 | b_{T-1})\gamma_{[100(10)], A_2} \\
&\quad + \Pr_{T-1}(LO_1B_2 | b_{T-1})\gamma_{[1001], A_2} + \Pr_{T-1}(LO_1B_1 | b_{T-1})\gamma_{[1010], A_2} \\
&\quad + \left[\Pr_{T-1}(MO_{10}B_3 | b_{T-1}) + \Pr_{T-1}(NTL | b_{T-1}) + \Pr_{T-1}(MO_1B_3 | b_{T-1}) + \Pr_{T-1}(NTS | b_{T-1}) \right] \\
&\quad \times \gamma_{[1000], A_2} + \Pr_{T-1}(LO_9A_2 | b_{T-1})\gamma_{[(10)000], A_2} + \Pr_{T-1}(LO_1A_2 | b_{T-1})\gamma_{[2000], A_2} \\
&\quad \left. + \Pr_{T-1}(LO_{10}A_1 | b_{T-1}) \frac{1}{3} [(A_2 - \beta_{T-2}v_T^U)] + \Pr_{T-1}(LO_1A_1 | b_{T-1})\gamma_{[1100], A_2} \right]
\end{aligned}$$

where $b_{T-1} = [0100]$ for LO_1A_1 and $b_{T-1} = [1000]$ for LO_1A_2 , and $\gamma_{b_{T,A_i}}$, for $i = \{1, 2\}$, is defined as follows:

$$\begin{aligned}\gamma_{b_{T,A_i}} &= \frac{1}{3}(A_i - \beta_{T-2}v_T) \left[\Pr_T(MRO_{10}A | b_T) + \Pr_T(MO_{10}A_i | b_T) + \Pr_T(MO_1A_i | b_T) \right. \\ &\quad \left. + \Pr_T(MO_2A_i | b_T) + \Pr_T(MRO_2A | b_T) \right] + \frac{1}{3}(A_i - \beta_{T-2}v_T^U)\end{aligned}$$

As for $T-1$, it is straightforward to derive the β_{T-2} intervals associated with a small trader's optimal trading strategies. Results are presented for $d = 0.1$ in Table 2, Panel B. The general solutions are available from the authors on request.

Case 2: Large Trader

The large trader solves again problem (4) and the available strategies on the ask side are $MO_{10}B_3$, LO_1A_1 , $LO_{10}A_1$, LO_1A_2 and $LO_{10}A_2$. The profits from these strategies are the following:²²

$$\pi_{T-2}(MO_{10}B_3) = 10(B_3 - \beta_{T-2}v_{T-2})$$

$$\begin{aligned}E[\pi_{T-2}(LO_{10}A_2)] &= \\ &= E \left[\sum_{l=T-1}^T (A_2 - \beta_l v_l) \sum_{h=0}^9 \left[\Pr \left(\sum_{m=t}^{l-1} w_m = h \mid b_t, v_t \right) \sum_{w_l=1}^{10-h} w_l \times \Pr(w_l | b_l, v_l) \right] \right] \\ &= \frac{1}{2} \left[10(A_2 - \beta_{T-2}v_{T-1}) \Pr_{T-1}(MO_{10}A_2 | b_{T-1}) + \Pr_{T-1}(LO_{10}B_1 | b_{T-1}) \gamma_1 \{[(10)0(10)0], A_2\} \right. \\ &\quad \left. + \Pr_{T-1}(LO_{10}B_2 | b_{T-1}) \gamma_{[(10)00(10)], A_2}^1 + \Pr_{T-1}(LO_1B_1 | b_{T-1}) \gamma_{[(10)010], A_2}^1 \right. \\ &\quad \left. + \Pr_{T-1}(LO_{10}A_1 | b_{T-1}) \frac{1}{3} 10(A_i - \beta_{T-2}v_T^U) + \Pr_{T-1}(LO_1A_1 | b_{T-1}) \gamma_{[(10)100], A_2}^3 \right. \\ &\quad \left. + \left[\Pr_{T-1}(MO_{10}B_3 | b_{T-1}) + \Pr_{T-1}(NTL | b_{T-1}) + \Pr_{T-1}(MO_1B_3 | b_{T-1}) + \Pr_{T-1}(NTS | b_{T-1}) \right] \right. \\ &\quad \left. \times \gamma_{[(10)000], A_2}^1 + \Pr_{T-1}(LO_1B_2 | b_{T-1}) \gamma_{[(10)001], A_2}^1 \right. \\ &\quad \left. + \Pr_{T-1}(MO_1A_2 | b_{T-1}) \left[(A_2 - \beta_{T-2}v_{T-1}) + \gamma_{[9000], A_2}^2 \right] \right]\end{aligned}$$

²²Profits from LO_1A_1 and LO_1A_2 are derived Case 1.

$$\begin{aligned}
E[\pi_{T-2}(LO_{10}A_1)] &= \\
&= E \left[\sum_{l=T-1}^T (A_1 - \beta_t \tilde{v}_l) \sum_{h=0}^9 \left[\Pr \left(\sum_{m=t}^{l-1} w_m = h \mid b_t, v_t \right) \sum_{w_l=1}^{10-h} w_l \times \Pr(A_1 \mid b_l, v_l) \right] \right] \\
&= \frac{1}{2} \left[10(A_1 - \beta_{T-2} v_{T-1}) \Pr_{T-1}(MO_{10}A_1 \mid b_{T-1}) + \Pr_{T-1}(LO_{10}B_1 \mid b_{T-1}) \gamma_{[0(10)(10)0], A_1}^1 \right. \\
&\quad \left. + \Pr_{T-1}(LO_{10}B_2 \mid b_{T-1}) \gamma_{[0(10)0(10)], A_1}^1 + \Pr_{T-1}(LO_1B_1 \mid b_{T-1}) \gamma_{[0(10)10], A_1}^1 \right. \\
&\quad \left. + \left[\Pr_{T-1}(MO_{10}B_3 \mid b_{T-1}) + \Pr_{T-1}(NTL \mid b_{T-1}) + \Pr_{T-1}(MO_1B_3 \mid b_{T-1}) + \Pr_{T-1}(NTS \mid b_{T-1}) \right. \right. \\
&\quad \left. \times \gamma_{[0(10)00], A_1}^1 + \Pr_{T-1}(LO_1B_2 \mid b_{T-1}) \gamma_{[0(10)01], A_1}^1 \right. \\
&\quad \left. + \Pr_{T-1}(MO_1A_1 \mid b_{T-1}) \left[(A_1 - \beta_{T-2} v_{T-1}) + \gamma_{[0900], A_1}^2 \right] \right]
\end{aligned}$$

where $b_{T-1} = [0(10)00]$ for $LO_{10}A_1$ and $b_{T-1} = [(10)000]$ for $LO_{10}A_2$, and γ_{b_{T-1}, A_i}^L , for $L = \{1, 2, 3\}$ and $i = \{1, 2\}$, is defined as follows:

$$\begin{aligned}
\gamma_{b_{T-1}, A_i}^1 &= \frac{1}{3} \left[10(A_i - \beta_{T-2} v_T) \Pr_T(MO_{10}A \mid b_T) + (A_i - \beta_{T-2} v_T) \Pr_T(MO_1A_i \mid b_T) \right] \\
&\quad + \frac{1}{3} 10(A_i - \beta_{T-2} v_T^U) \\
\gamma_{b_{T-1}, A_i}^2 &= \frac{1}{3} \left[9(A_i - \beta_{T-2} v_T) \left[\Pr_T(MRO_{10}A \mid b_T) + \Pr_T(MO_9A_i \mid b_T) \right] + \right. \\
&\quad \left. + (A_i - \beta_{T-2} v_T) \Pr_T(MO_1A_i \mid b_T) \right] + \frac{1}{3} [9(A_i - \beta_{T-2} v_T^U)] \\
\gamma_{b_{T-1}, A_i}^3 &= \frac{1}{3} 9(A_i - \beta_{T-2} v_T) \left[\Pr_T(MRO_{10}A \mid b_T) \right] + \frac{1}{3} 10(A_i - \beta_{T-2} v_T^U)
\end{aligned}$$

Results are presented for $d = 0.1$ in Table 2, Panel A.

Proof of Lemma 1

It is straightforward to show that small traders' trading strategies are unchanged compared to the benchmark case: given that one share is always visible when a hidden order is submitted, small traders face no uncertainty on the execution price of market orders. For large traders, if a shock occurs, traders will cancel their mispriced hidden orders and we are back to the benchmark case. If there is no shock, we have three different cases:

1. When j shares are visible either on A_1 or on A_2 , or when b_T is empty on the ask side, the β_T thresholds are the same as those computed in the benchmark case since there is no uncertainty on the execution price.
2. If $f_i < j$ shares are visible on A_i and $n > j - f_i$ shares on $A_l > A_i$, the large trader's β_T thresholds for the ask side will be the following:

submit $MRO_j A$ if $\beta_T \geq \frac{\Lambda_m}{v_T}$
 submit $MO_{f_i} A_i$ if $\frac{A_i^{U,D}}{v_T^{U,D}} \leq \beta_T < \frac{\Lambda_m}{v_T}$
 no trade if $1 < \beta_T < \frac{A_i}{v_T}$

where $\Lambda_m = \sum \Pr_{j-f_i}(A_m | b_T, b_{T-1}) A_m$, with $m = \{i, l\}$, is a weighted average of the possible prices and $\Pr_{j-f_i}(A_m | b_T, b_{T-1})$ are the probabilities that the remaining $j - f_i$ shares are executed at price A_m . These weights depend on traders' strategies at $T - 1$ and $T - 2$. Consider for example one of the LOB paths described in Figure 4: $b_{T-1} = [0000]$, $b_T = [1000]$ or $b_T = [(1+9)000]$. The incoming trader at T does not know which of the last two books he is facing, as both have one visible unit on A_2 . So, in this case:

$$\Lambda_m = \frac{\Pr_{T-1}(HOS_{10}A_2 | b_T, b_{T-1}) A_2 + \Pr_{T-1}(LO_1A_2 | b_T, b_{T-1}) A_3}{\Pr_{T-1}(HOS_{10}A_2 | b_T, b_{T-1}) + \Pr_{T-1}(LO_1A_2 | b_T, b_{T-1})}$$

3. If there are $f_i < j$ visible shares on A_i for both $i = 1$ and $i = 2$, with $f_1 + f_2 < j$, the large trader's β_T thresholds for the ask side will be the following:

submit $MRO_j A$ if $\beta_T \geq \frac{\Lambda_z}{v_T}$
 submit $MRO_{f_1+f_2} A$ if $\frac{\Lambda_m}{v_T} \leq \beta_T < \frac{\Lambda_z}{v_T}$
 submit $MO_{f_1} A_1$ if $\frac{A_1}{v_T} \leq \beta_T < \frac{\Lambda_m}{v_T}$
 no trade if $1 \leq \beta_T < \frac{A_1}{v_T}$

where $\Lambda_m = \sum \Pr_{f_2}(A_m | b_T, b_{T-1}) A_m$, with $m = \{1, 2\}$, and $\Lambda_z = \sum \Pr_{j-f_1}(A_z | b_T, b_{T-1}) A_z$, with $z \in \{1, 2, 3\}$. The thresholds are computed as in the example for case (2).

The order submission probabilities can be easily derived from the thresholds obtained above.

Proof of Proposition 2

We only provide a sketch of the proof, since there are many similarities with the proof of Proposition 1. The main difference which the benchmark model is that now traders do not always know the depth of the book they are facing, and therefore they have to estimate it to maximize their profits. Clearly, traders adopt the same strategy when they cannot distinguish among two books. Due to risk neutrality, it is possible to show that large traders' profits are maximized for $j = 10$; hence, we assume again that j is equal to its maximum possible value, given the state of the LOB. Notice that in this case one-unit limit orders are never optimal despite their higher execution probability, since they are dominated by hidden orders.

Period T

As shown in Lemma 1 small traders' strategies are identical to the benchmark case. Large traders' strategies are also the same when there is no uncertainty on the LOB, while, if there is uncertainty, they depend on the previous traders' strategies.

Period $T - 1$

The possible states of the LOB at $T - 1$ are summarized in the following Table:

LOB at $T - 1$							
A_2	0	10	0	1	$1 + 9$	0	0
A_1	0	0	10	0	0	1	$1 + 9$
B_1	0	0	0	0	0	0	0
B_2	0	0	0	0	0	0	0

Notice that at $T - 1$ traders can face uncertainty on the state of the book. If for example the visible book is $b_{T-1} = [1000]$, they do not know whether the book is $b_{T-1} = [1000]$ or $b_{T-1} = [(1 + 9)000]$. If instead they face a book with ten shares at A_2 ($b_{T-1} = [(10)000]$), they have no uncertainty.

Case 1: Small Trader

The small trader solves problem (7), presented in Section 5.1. The feasible strategies are unchanged compared to the benchmark case, however there are two other possible books as now $HO_{10}A_1$ and $HO_{10}A_2$ can be submitted at $T - 2$.

As an example we still focus on $b_{T-1} = [(10)000]$; the other cases can be derived in a similar way. Now the available strategies are unchanged (MO_1B_3 , LO_1A_1 , LO_1B_1 , LO_1B_2 and MO_1A_2), and profits for market orders are the same; however, profits for limit orders are different, as at T the incoming trader cannot distinguish between limit and hidden orders at the same level of the book:

$$\begin{aligned}
E[\pi_{T-1}(LO_1A_1)] &= E \left[\Pr_{w_T=1}(A_1|b_T, v_T)(A_1 - \beta_{T-1}\tilde{v}_T) \right] \\
&= \frac{1}{3}(A_1 - \beta_{T-1}v_T) \left\{ \frac{\Pr_{T-1}(HO_{10}A_1|b_{T-1})}{\Pr_{T-1}(HO_{10}A_1|b_{T-1}) + \Pr_{T-1}(LO_1A_1|b_{T-1})} \times \right. \\
&\quad \left(\Pr_T(MRO_{10}A | b_T^a) + \Pr_T(MO_1A_1 | b_T^a) \right) \\
&\quad + \frac{\Pr_{T-1}(LO_1A_1|b_{T-1})}{\Pr_{T-1}(HO_{10}A_1|b_{T-1}) + \Pr_{T-1}(LO_1A_1|b_{T-1})} \times \\
&\quad \left. \left(\Pr_T(MRO_{10}A | b_T^c) + \Pr_T(MO_1A_1 | b_T^c) \right) \right\} + \frac{1}{3}(A_1 - \beta_{T-1}v_T^U) \\
\text{where } b_T^a &= [0(1+9)00] \text{ or } b_T^c = [0100], \quad b_{T-1} = [(10)000]
\end{aligned}$$

$$\begin{aligned}
E[\pi_{T-1}(LO_1B_i)] &= E \left[\Pr_{w_T=1}(B_i|b_T, v_T)(\beta_{T-1}\tilde{v}_T - B_i) \right] \\
&= \frac{1}{3}(\beta_{T-1}\tilde{v}_T - B_i) \left\{ \frac{\Pr_{T-1}(HO_{10}B_i|b_{T-1})}{\Pr_{T-1}(HO_{10}B_i|b_{T-1}) + \Pr_{T-1}(LO_1B_i|b_{T-1})} \times \right. \\
&\quad \left(\Pr_T(MO_{10}B_i | b_T^a) + \Pr_T(MO_1B_i | b_T^a) \right) + \\
&\quad + \frac{\Pr_{T-1}(LO_1B_i|b_{T-1})}{\Pr_{T-1}(HO_{10}B_i|b_{T-1}) + \Pr_{T-1}(LO_1B_i|b_{T-1})} \times \\
&\quad \left. \left(\Pr_T(MO_{10}B_i | b_T^c) + \Pr_T(MO_1B_i | b_T^c) \right) \right\} + \frac{1}{3}(\beta_{T-1}v_T^D - B_i) \\
\text{where } b_T^a &= [(10)0(1+9)0] \text{ or } b_T^c = [(10)010], \quad b_{T-1} = [(10)000] \text{ for } HO_{10}B_1 \\
\text{where } b_T^a &= [(10)00(1+9)] \text{ or } b_T^c = [(10)001], \quad b_{T-1} = [(10)000] \text{ for } HO_{10}B_2
\end{aligned}$$

Results are presented in Table 2, Panel B for $d = 0.1$.

Case 2: Large Trader

The large trader solves problem (6), presented in Section 5.1. Similarly to the benchmark case, MRO_jB and $MO_jB_{1,2}$ are never feasible strategies, while $MO_{10}B_3$, LO_1B_i and HO_1B_i are always feasible strategies, for $i = 1, 2$. Profits are computed as in the benchmark case, however the order execution

probabilities are different as the model is characterized by a further degree of uncertainty.

As an example we solve the case $b_{T-1} = [(10)000]$; the other cases can be derived in a similar way. When ten shares are visible at A_2 , the feasible strategies are: $MO_{10}B_3$, $LO_{10}A_1$, $HO_{10}A_1$, $LO_{10}B_1$, $LO_{10}B_2$, $HO_{10}B_1$, $HO_{10}B_2$, $MO_{10}A_3$. The profits from $MO_{10}B_3$, $LO_{10}A_1$, $LO_{10}B_1$, $LO_{10}B_2$ and $MO_{10}A_3$ are unchanged compared to the benchmark case, as for these strategies there is no uncertainty about the state of the LOB at T . The other profits are the following:

$$\begin{aligned} E[\pi_{T-1}(HO_{10}A_1)] &= E \left[\sum_{w_T=0}^{10} \Pr(A_i|b_m, v_m)(A_1 - \beta_{T-1}\tilde{v}_T)w_T \right] \\ &= \frac{1}{3} \left\{ \frac{\Pr_{T-1}(HO_{10}A_1|b_{T-1})}{\Pr_{T-1}(HO_{10}A_1|b_{T-1}) + \Pr_{T-1}(LO_1A_1|b_{T-1})} \times \right. \\ &\quad \left[10(A_1 - \beta_{T-1}\tilde{v}_T) \Pr_T(MRO_{10}A | b_T^a) + (A_1 - \beta_{T-1}\tilde{v}_T) \Pr_T(MO_1A_1 | b_T^a) \right] \\ &\quad + \frac{\Pr_{T-1}(LO_1A_1|b_{T-1})}{\Pr_{T-1}(HO_{10}A_1|b_{T-1}) + \Pr_{T-1}(LO_1A_1|b_{T-1})} \times \\ &\quad \left. \left[10(A_1 - \beta_{T-1}\tilde{v}_T) \Pr_T(MRO_{10}A | b_T^c) + (A_1 - \beta_{T-1}\tilde{v}_T) \Pr_T(MO_1A_1 | b_T^c) \right] \right\} \\ &\quad + \frac{1}{3} [(A_1 - \beta_{T-1}\tilde{v}_T^U)] \end{aligned}$$

where $b_T^a = [0(1+9)00]$ or $b_T^c = [0100]$, $b_{T-1} = [(10)000]$.

$$\begin{aligned} E[\pi_{T-1}(HO_{10}B_i)] &= E \left[\sum_{w_T=0}^{10} \Pr(B_i|b_m, v_m)(\beta_{T-1}\tilde{v}_T - B_i)w_T \right] \\ &= \frac{1}{3} \left\{ \frac{\Pr_{T-1}(HO_{10}B_i|b_{T-1})}{\Pr_{T-1}(HO_{10}B_i|b_{T-1}) + \Pr_{T-1}(LO_1B_i|b_{T-1})} \times \right. \\ &\quad \left[10(\beta_{T-1}\tilde{v}_T - B_i) \Pr_T(MO_{10}B_i | b_T^a) + (\beta_{T-1}\tilde{v}_T - B_i) \Pr_T(MO_1B_i | b_T^a) \right] \\ &\quad + \frac{\Pr_{T-1}(LO_1B_i|b_{T-1})}{\Pr_{T-1}(HO_{10}B_i|b_{T-1}) + \Pr_{T-1}(LO_1B_i|b_{T-1})} \times \\ &\quad \left. \left[10(\beta_{T-1}\tilde{v}_T - B_i) \Pr_T(MO_{10}B_i | b_T^c) + (\beta_{T-1}\tilde{v}_T - B_i) \Pr_T(MO_1B_i | b_T^c) \right] \right\} \\ &\quad + \frac{1}{3} [(\beta_{T-1}\tilde{v}_T^D - B_i)] \end{aligned}$$

where $b_T^a = [(10)0(1+9)0]$ or $b_T^c = [(10)010]$, $b_{T-1} = [(10)000]$ for $HO_{10}B_1$ and

where $b_T^a = [(10)00(1+9)]$ or $b_T^c = [(10)001]$, $b_{T-1} = [(10)000]$ for $HO_{10}B_2$.

Results for all the possible states of the LOB are presented in Table 2, Panel A for $d = 0.1$.

Period $T - 2$

We compute and compare the profits associated to traders' strategies on the ask side at $T - 2$. Given the similarity with the benchmark case, we omit to present the profits' formulas, that are available from the authors on request. Results are presented for $d = 0.1$ in Table 2, Panel A and B.

Finally, to show how order size affects the use of hidden orders, we solve the model for $j \leq 2$. Results for this case are presented in Table 3.

Proof of Proposition 3

The Proposition is obtained through a straightforward comparison of the results obtained for the benchmark and the model with hidden orders, respectively.

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	$T-2$	$T-1$	T	
$1 + (9/2)d$				A_3^U
$1 + (7/2)d$				A_2^U
$1 + (5/2)d$	A_3	A_3	A_3	A_1^U
$1 + (3/2)d$	A_2	A_2	A_2	B_1^U
$1 + (1/2)d$	A_1	A_1	A_1	B_2^U
$1 - (1/2)d$	B_1	B_1	B_1	B_3^U
$1 - (3/2)d$	B_2	B_2	B_2	A_1^D
$1 - (5/2)d$	B_3	B_3	B_3	B_1^D
$1 - (7/2)d$				B_2^D
$1 - (9/2)d$				B_3^D

$v^U = 1 + 2d \leftarrow$
 $v = 1 \leftarrow$
 $v^D = 1 - 2d \leftarrow$

Figure 1 This figure shows the price grid for $k = 2$. The ask prices are equal to $A_{1,2,3}$ and the bid prices are equal to $B_{1,2,3}$, with $A_1 < A_2 < A_3$ and $B_1 > B_2 > B_3$. These prices are symmetric around the common value of the asset, v that, at time T , can take values v, v^U and v^D respectively.

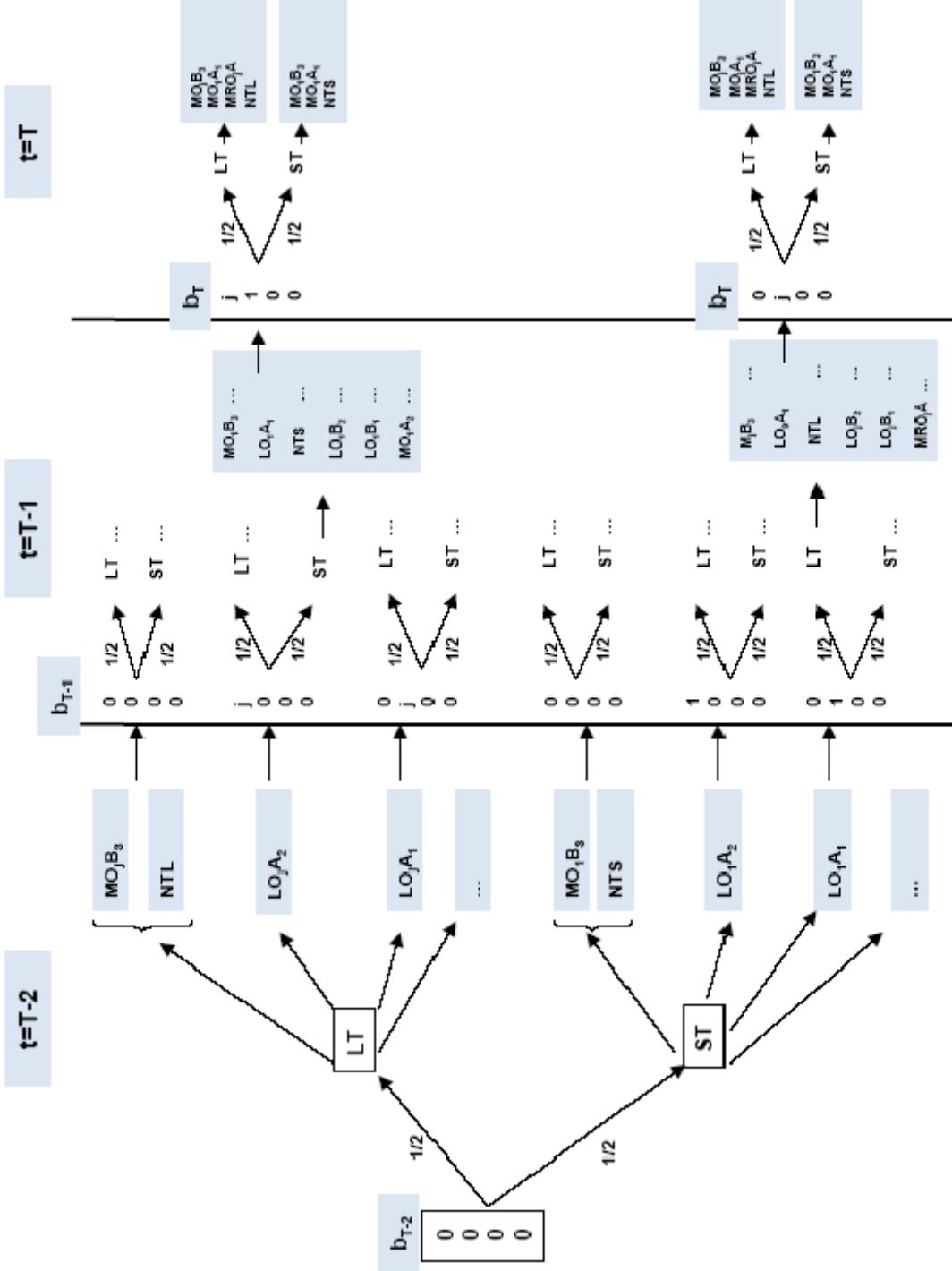


Figure 2 This Figure shows the extensive form of the game. At $T - 2$ the book opens empty, $b_{T-2} = [0000]$; nature chooses with equal probability a large trader (LT) or a small trader (ST) who decides his optimal submission strategy among all the feasible orders (Table 1), except hidden orders. The Figure reports one example for LT and one for ST. If at $T - 2$ a LT chooses LO_jA_2 , at $T - 1$ the book will be $b_{T-1} = [j000]$; if then a ST arrives who, still as an example, chooses LO_1A_1 , then at T the book will open equal to $b_T = [j100]$ so that the incoming LT will submit either MO_jB_3 , MO_1A_1 , MRO_jA , or he will not trade (NTL); the ST instead will choose among MO_1B_2 and MO_1A_1 , or decide not to trade (NTS).

Figure 3.1 - 3.5 Order Submission Strategies at T-1: Small Trader (ST)
 On the vertical axis $\beta_{T-1} \in [0, 2]$; on the horizontal axis the tick size, $d \in [0, 0.2]$.

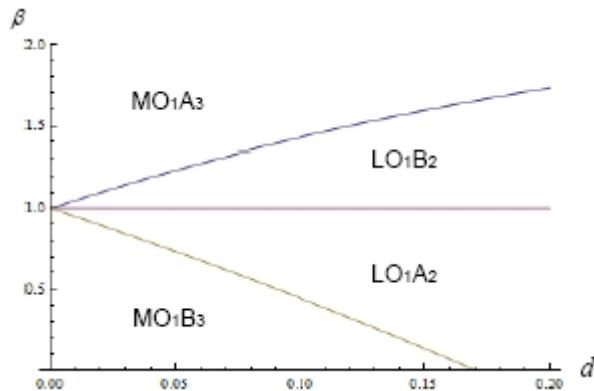


Figure 3.1 $b_{T-1} = [0000]$

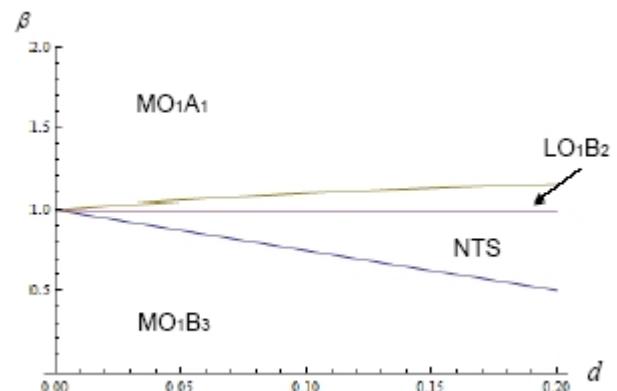


Figure 3.2 $b_{T-1} = [0(10)00]$

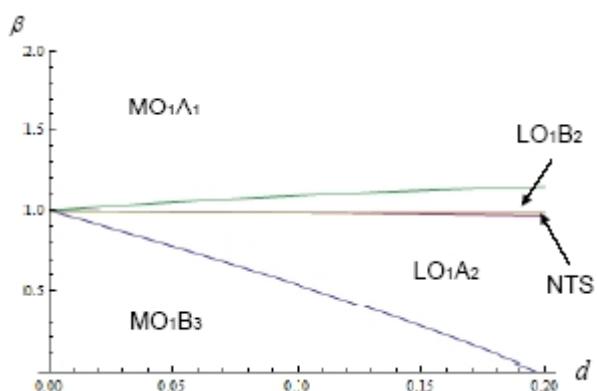


Figure 3.3 $b_{T-1} = [0100]$

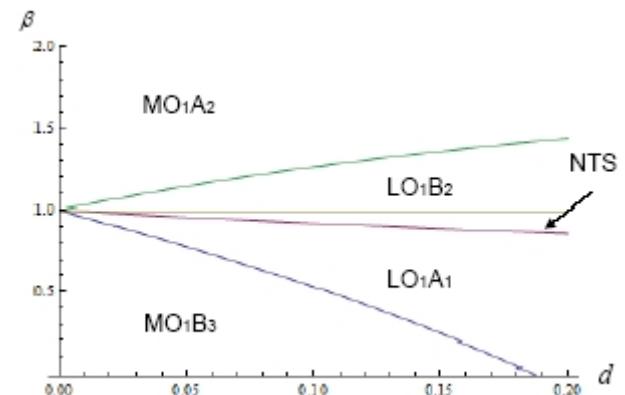


Figure 3.4 $b_{T-1} = [(10)000]$

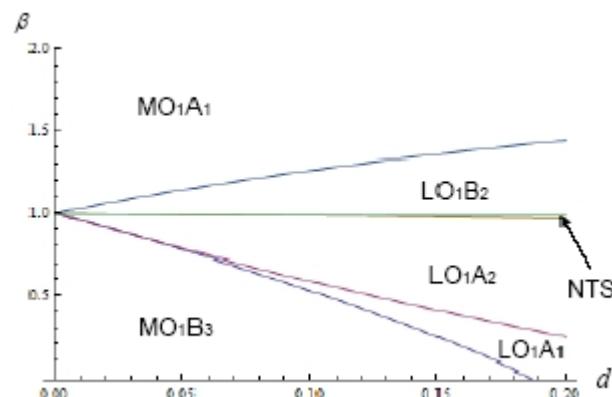


Figure 3.5 $b_{T-1} = [1000]$

Figure 3.6 - 3.10 Order Submission Strategies at T-1: Large Trader (LT)
 On the vertical axis $\beta_{T-1} \in [0, 2]$; on the horizontal axis the tick size, $d \in [0, 0.2]$.

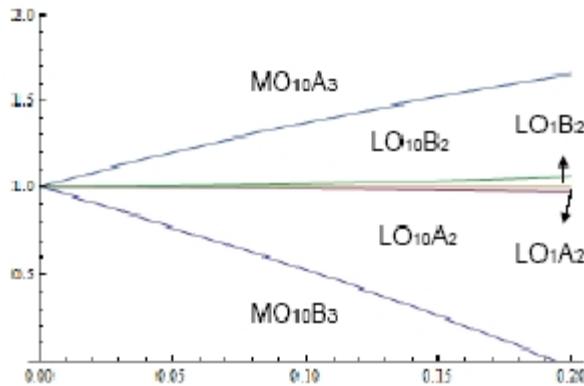


Figure 3.6 $b_{T-1} = [0000]$

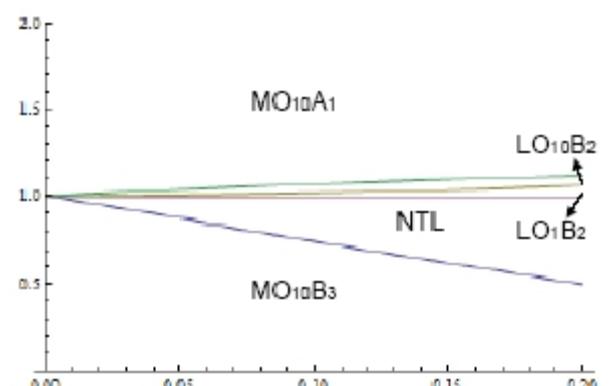


Figure 3.7 $b_{T-1} = [0(10)00]$

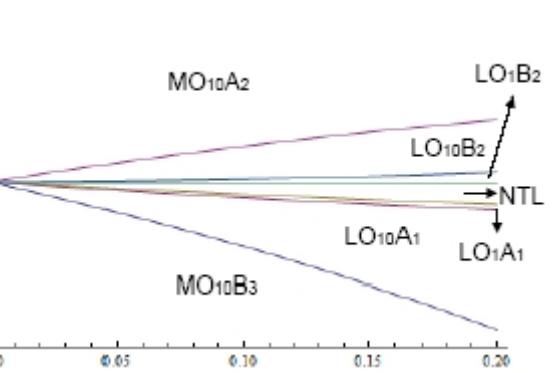
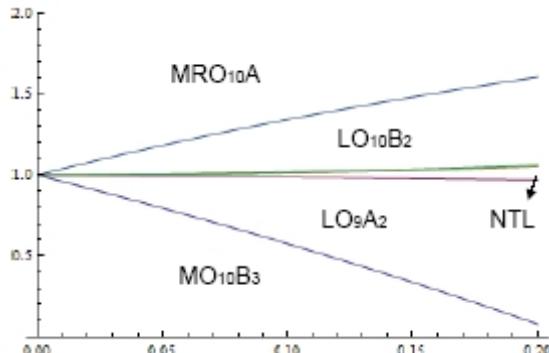
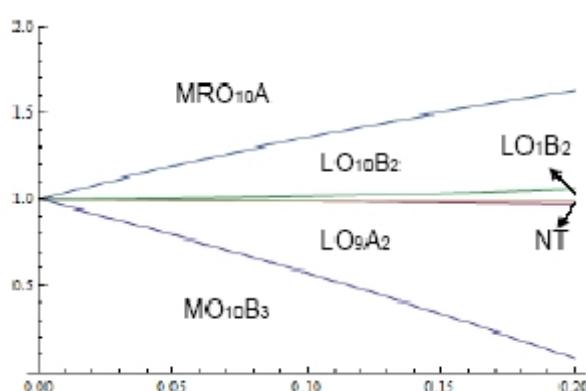


Figure 3.9 $b_{T-1} = [(10)000]$



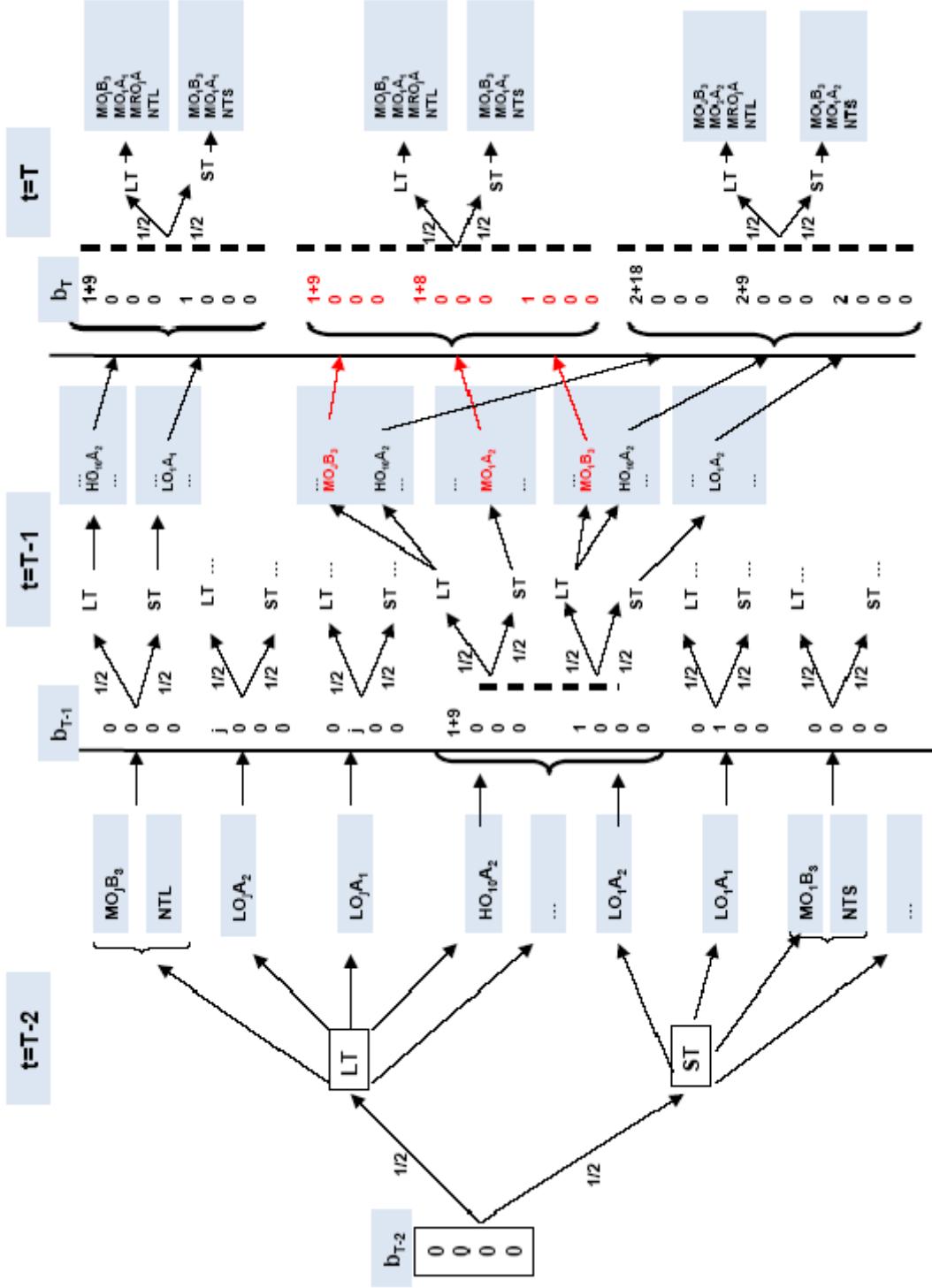


Figure 4 This Figure shows the extensive form of the game with hidden orders. At $T - 2$ the book opens empty $b_{T-2} = [0\ 0\ 0]$, nature chooses with equal probability a large trader (LT) or a small trader (ST) who decides his optimal submission strategy among all the feasible orders (Table 1). The Figure reports eight of the equilibrium game paths, which end into three sets of books at T . Because each of these books may contain hidden depth, before submitting his order, at T the incoming trader has to estimate the probability of observing the hidden liquidity, conditional on the previous states of the LOB. For example, the first two books from above, that appear at T and derive from the same book at $T - 1$ ($b_{T-1} = [0\ 0\ 0]$), differ as the visible unit on A_2 can either come from an $\text{LO}_1 A_2$ submitted at $T - 1$ by a ST, or from a $\text{HO}_{10} A_2$ submitted at $T - 1$ by a LT. Notice that the second sets of states of the LOB is complicated by the fact that they can derive from two different books at $T - 1$ ($b_{T-1} = [1+9\ 0\ 0]$ or $b_{T-1} = [1+9\ 0\ 0]$) and hence are characterized by a further degree of uncertainty. It follows that traders at T have to make inference both on the strategies played at $T - 1$, and on those played at $T - 2$.

TABLE 1: ORDER SUBMISSION STRATEGIES

Panel A: Small Traders ($j = 1, i \in [1, 3]$ for MO , and $i \in [1, 2]$ for LO)		Panel B: Large Trader ($j \in [1, 10]$, $i \in [1, 3]$ for MO , and $i \in [1, 2]$ for LO)	
Market Sell Order	$MO_j B_i$	Market Sell Order	$MO_j B_i$
Limit Sell Order	$LO_j A_i$	Marketable Sell Order Limit Sell Order Hidden Sell Order	$MRO_j B$ $LO_j A_i$ $HO_j A_i$
No Trade	NTS	No Trade Hidden Buy Order Limit Buy Order	NTL $HO_j B_i$ $LO_j B_i$
Limit Buy Order	$LO_j B_i$	Marketable Buy Order Market Buy Order	$MRO_j A$ $MO_j A_i$
Market Buy Order	$MO_j A_i$	Market Buy Order	$MO_j A_i$

Table 1: Order Submission Strategies. This Table presents the possible orders that a small trader (Panel A) and a large trader (Panel B) can choose upon arrival at the market. By definition, small traders can only trade one share ($j = 1$), whereas large traders can submit orders of size one to ten shares. On the sell side (the buy side is symmetrical) small traders can submit a market sell order ($MO_1 B_i$) to one of the three levels of the LOB. Small traders can also opt to submit a limit sell order to the first ($LO_1 A_1$) or to the second level of the LOB ($LO_1 A_2$), and they can also decide not to trade (NTS). A large trader can submit a market sell orders ($MO_j B_i$) of size j to one of the three levels of the LOB, or can he can hit the depth available on the buy side by submitting a marketable sell order ($MRO_j B$). A large trader can also choose to submit a limit sell order of size j to either A_1 or A_2 ($LO_j A_i$), or he can decide to disclose only one unit and submit a hidden sell order to the first or to the second level of the ask grid ($HO_{10} A_{1,2}$). By assumption, hidden orders must be of size 10 with only 1 unit disclosed. Finally, a large trader can decide not to trade (NTL).

Table 2

Panel A	$T - 1$ ($T - 2$)					
	$b_{T-1} = [0000]$			$b_{T-1} = [(10)000]$		
Large Trader	Benchmark	Hidden	Benchmark	Hidden	Benchmark	Hidden
$MO_{10}B_3$.265 (.169)	.265 (.170)	.301	.301	.375	.375
$HO_{10}A_2$.019 (.009)				.288	.288
$LO_{10}A_2$.228 (.331)	.216 (.321)				.013
LO_1A_2	.007				.206	.193
$LO_{10}A_1$.151	.151		
LO_1A_1		.01	.005			
NTL		.032			.119	.078
LO_1B_2	.009	.015		.015		.001
$LO_{10}B_2$.178	.158	.102	.081	.025	.014
$HO_{10}B_2$.029		.073		
$MO_{10}A_1$.466	.466
$MRO_{10}A$.15
$MO_{10}A_2$.034
$MO_{10}A_3$.313	.313				

Table 2 (Panel A) Order Submission Probabilities. This Table reports large traders' submission probabilities for the orders listed in column 1, for both the benchmark and the hidden model. These probabilities are reported for the five states of the book listed in row 2, and for both period $T - 1$ and $T - 2$ (in parenthesis). For example, the second column shows that when the book is empty, $b_{T-1} = [0000]$, large sellers submit market orders at B_3 , $MO_{10}B_3$, with probability .265 at $T - 1$ and .169 at $T - 2$, in the benchmark model.

Table 2

		$T - 1$	$(T - 2) :$	Benchmark \equiv Hidden
Panel B	$b_{T-1} = [0000]$	$b_{T-1} = [(10)000]$	$b_{T-1} = [0(10)00]$	$(b_{T-1} = [1000]) \equiv (b_{T-1} = [(1 + 9)000])$
Small Trader				
MO_1B_3	.222 (0)	.264	.375	.265
LO_1A_2	.278 (.5)			.200
LO_1A_1		.198		.029
NTS		.032	.119	.001
LO_1B_2	.215	.137	.052	.136
MO_1A_1			.454	
MO_1A_2		.369		.369
MO_1A_3	.285			

Table 2 (Panel B) Order Submission Probabilities. This Table reports small traders' submission probabilities for the orders listed in column 1. In this case results for the benchmark and the hidden model are the same. These probabilities are reported for the five states of the book listed in row 2, and for both period $T - 1$ and $T - 2$ (in parenthesis). For example, the second column shows that when the book is empty, $b_{T-1} = [0000]$, small sellers submit limit orders at A_2 , LO_1A_2 , with probability .278 at $T - 1$ and with probability .5 at $T - 2$.

Table 3

		T - 1 (T - 2)				
		[0000]	[2000]	[(10)000]	[0200]	[0(10)00]
Ask Side		j = 2	j = 10	j = 2	j = 10	j = 10
$HO_j A_2$.019 (.011)	.019 (.009)				
$LO_j A_2$.233 (.384)	.216 (.321)				
$\frac{HO_j A_2}{HO_j A_2 + LO_j A_2}$.075 (.027)	.081 (.027)				
Bid Side						
$HO_j B_2$.028	.029	.044	.073	.045	.077
$LO_j B_2$.170	.158	.090	.081	.010	.004
$\frac{HO_j B_2}{HO_j B_2 + LO_j B_2}$.141	.155	.328	.474	.818	.951

Table 3 Hidden Orders and Order Size. This Table reports the order submission probabilities of both hidden orders ($HO_j A_2$ and $HO_j B_2$) and limit orders ($LO_j A_2$ and $LO_j B_2$) posted at A_2 and B_2 ; it also reports the proportion of hidden orders over total liquidity, disclosed and undisclosed, at the second level of the book, for both the model with $j = 10$ and with $j = 2$. Comparisons between the two models are reported for the five states of the book listed in row 2, for periods $T - 1$ and $T - 2$ (in parenthesis). For example, when the book is empty ($b_{T-1} = [0000]$), large hidden orders on A_2 ($HO_j A_2$) for $j = 2$ are submitted at $T - 1$ with probability .019 and at $T - 2$ with probability .011. The same submission probabilities for $j = 10$ are equal to .019 and .009 for $T - 1$ and $T - 2$ respectively.

LOB	Small Trader's Strategies					
	MO_1B_3	LO_1A_2	LO_1A_1	LO_1B_1	LO_1B_2	MO_1A_1
$b_{T-1} = [0000]$	X	X	X	X	X	X
$b_{T-1} = [(10)000]$	X		X	X	X	X
$b_{T-1} = [0(10)00]$	X			X	X	
$b_{T-1} = [1000]$	X	X	X	X	X	X
$b_{T-1} = [0100]$	X	X	X	X	X	X

Table 4.1 This table shows small traders' feasible strategies conditional on the state of the LOB.

LOB	Large Trader's Strategies					
	$MO_{10}B_3$	LO_jA_2	LO_jA_1	LO_jB_1	LO_jB_2	$MO_{10}A_3$
$b_{T-1} = [0000]$	X	$j = 1, 10$	$j = 1, 10$	X	X	X
$b_{T-1} = [(10)000]$	X		$j = 1, 10$	X	X	X
$b_{T-1} = [0(10)00]$	X			X	X	
$b_{T-1} = [1000]$	X	$j = 1, 9$	$j = 1, 10$	X	X	
$b_{T-1} = [0100]$	X	$j = 1, 9$	$j = 1, 9$	X	X	

Table 4.2 This table shows large traders' feasible strategies conditional on the state of the LOB.