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CURRENCY CRASHES**

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Exchange rate solutions with currency crashes

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Abstract

We present an exchange rate model in which a currency's exchange rate is confined in a wide moving band and a currency crash occurs when the rate breaches the lower boundary. A solution is derived from the standard log exchange rate equation for the model with a smooth-pasting condition at the lower boundary. Using an asymmetric mean-reverting fundamental shock, the solution shows the exchange rate follows a mean-reverting square-root process, which is quasi-bounded at the boundary, and generates left-skewed exchange rate distributions consistent with empirical observations. The probability leakage for the exchange rate across the boundary increases with a weakened mean-reverting force for the exchange rate, suggesting an increase in currency crash risk. The empirical results show the exchange rates of nine major currencies against the US dollar can be calibrated according to the model, where the mean reversion is negatively cointegrated with the risk reversals in currency option markets, as expected by the model, and are consistent with the positive relationship between currency crash risk and risk reversals. The leakage condition for breaching the lower boundaries was met during the 2008 global financial crisis when most of the currencies were under the disaster shock.

Keywords: Exchange rate dynamics, currency crash, stochastic processes, risk reversals

JEL Classification: F31, G13

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1. Introduction

Currency crash risk, which has long been a subject of interest in international finance, is found in both developed and developing economies. We propose an exchange rate model based on the standard log exchange rate equation, in which a currency's exchange rate is confined in a wide-moving band and a currency crash occurs when the exchange rate breaches the lower boundary. A smooth-pasting condition is imposed at the boundary condition, suggesting an optimal boundary condition for the process. If there is no foreseeable jump in the exchange rate and no arbitrage condition at the boundary, the smooth-pasting condition ensures that a currency crash is rare. A solution is derived from the exchange rate equation for the model. By using an asymmetric mean-reverting fundamental shock, the solution shows the log-normalised exchange rate follows a mean-reverting square-root process. This is quasi-bounded at the boundary and can breach the lower boundary, provided the probability leakage condition is met. The asymmetric fundamental shock is consistent with disaster risks, including currency crashes, which are inherently asymmetric, given that crashes are one-sided events. In general, asymmetries in exchange rate dynamics arise when two economies have different risk profiles and/or when investors have different risk preferences. The asymmetric fundamental shock in our model is similar to asymmetric country-specific and global shocks in the context of contributions to violations of uncovered interest rate parity (Backus, et al. (2001)) and exchange rate option prices (Bakshi et al. (2008); Jurek and Xu (2014)).

The mean-reverting square-root exchange rate dynamic derived from the asymmetric mean-reverting fundamental shock in the model is consistent with the observed risk reversals in the currency option market. This is when the price of a currency crash risk is reflected by the risk reversal, which measures the implied volatility difference between an out-of-the-money put on the currency and an out-of-the-money call at the same (absolute) delta. The risk

reversal reflects asymmetric expectations on the directions of exchange rate movement. This suggests that prices of hedging against downside risk (crash risk) of the currency are higher than its up-side risk. The sign and magnitude of risk-reversals are informative about the asymmetry of the exchange rate distribution. A positive risk reversal suggests that the risk-neutral conditional exchange rate distribution is left skewed. In the model, the currency crash risk increases when the mean-reverting force for the exchange rate weakens, indicating that the probability leakage for the rate across the lower boundary increases. The risk reversals are expected to be negatively related to the mean version of the exchange rate.

The model is consistent with the latest proposed theories and empirical evidence about the positive relationship between currency crash risk and risk reversals in major currency option markets.¹ Brunnermeier et al. (2009) show carry trades are subject to crash risk.² Therefore, exchange rate movements between high interest rate and low interest rate currencies are negatively (left) skewed. The price of currency crash risk is reflected by the price of the risk-reversal. Farhi et al. (2015) propose a disaster-based structural model in which investors incorporate a currency crash-risk premium into the value of the exchange rate, and calibrate the crash probability to option prices.³ Farhi and Gabaix (2016) develop a model that makes predictions regarding risk reversals, including: investing in countries with high risk reversals should have high returns on average; and when the risk reversal of a country goes up, its currency contemporaneously depreciates. Regarding currency option pricing, Jurek (2014) derives a measure of crash risk from currency options and finds that exposure to a currency crash can be used to explain at most one-third of the portion of carry

¹ Earlier studies on currency crashes focused more on developing economies in which currency crashes were linked to their ability to defend the currencies reflected by country-specific macro-economic variables, such as output growth, foreign exchange reserves, budget deficit, real effective exchange rate deviation, and foreign direct investment. See Eichengreen et al. (1996), Frankel and Rose (1996), Kaminsky et al. (1998), and Kumar et al. (2003).

² In a carry trade, an investor sells a currency with a relatively low interest rate and uses the funds to buy a different currency yielding a higher interest rate. This strategy attempts to capture the difference between the rates of the two currencies provided that their exchange rate is stable.

³ The quantitative importance of downside risk can be linked to the rare disasters model of Barro (2006).

trade returns. Chernov et al. (2018) find that jump risk related to currency crashes is priced in out-of-the-money options. Husted et al. (2017) demonstrate that an increase in uncertainty in financial markets increases the cost of protection against crash risk reflected in risk reversals. Several studies, including Burnside et al. (2011), Lettau et al. (2014) and Dobrynskaya (2014), explain high returns to carry trades and investigate downside factors related to currency crashes. Lustig et al. (2011) find the effects of broadly defined “global risk aversion” on the profitability of carry trades.

However, the latest literature offers examples of various currency carry trades that are profitable but do not suffer particularly from crashes.⁴ Daniel, Hodrick and Lu (2017) find the exposure of their carry trades to downside market risk is not statistically significantly different from the unconditional exposure. Bekaert and Panayotov (2018) show the distinction between good and bad carry trades significantly alters understanding of currency carry trade returns, and explanations invoking return skewness and crash risk. Related results can also be found in Ready et al. (2017) and Maurer et al. (2016). Verdelhan (2010) presents a model that reproduces the uncovered interest rate parity puzzle without currency crashes. Therefore, the exchange rate model proposed in this paper captures currency crashes, which is consistent with some empirical observations of currency option prices, but does not argue any linkage between carry trades and crashes in the currency market being established.

Using the exchange rates of the major (G10) currencies, including the Australian dollar, Canadian dollar, Swiss franc, euro, British pound, Japanese yen, Norwegian krone, New Zealand dollar and Swedish krona against the US dollar from 2000-2017, the empirical results in this paper demonstrate that their log-normalised exchange rates derived from the model can be calibrated to the mean-reverting square-root process, where the mean reversion is negatively cointegrated with the risk reversals, as expected by the model. The leakage

⁴ The authors gratefully acknowledge the referee to point this out.

condition for breaching the boundaries was met during the 2008 global financial crisis when the exchange rates fell sharply. The exchange rate solution generates left-skewed exchange rate distributions, which shares with empirical observations and some stochastic volatility models, as in Heston (1993) and Bates (2012).

The paper is organised as follows. We develop the exchange rate model in the following section. The exchange rate solution associated with currency crashes is solved from the equation of the model in section 3. The corresponding exchange rate dynamics and probability density function are derived and discussed. The calibrations of the exchange rate dynamics of the nine major currencies against the US dollar and their probability leakage conditions are presented in section 4. The dynamic relationship between the exchange rate dynamics and risk reversals in currency option markets are studied by a cointegration analysis in section 5. The final section of the paper concludes.

2. Exchange rate model

We consider a basic log-linear monetary model of the exchange rate. The log exchange rate s at time t follows the following equation:

$$s(t) = m + \nu + \alpha \frac{E[ds(t)]}{dt}, \quad (1)$$

where m is the logarithm of the constant money supply, ν is a monetary demand shock term (incorporating shifts in real income and velocity, etc.), α is the absolute value of semi-elasticity of the exchange rate with respect to its expected rate of change, and E the expectation operator. The last term captures the expected exchange rate change. Based on a monetary process of exchange rate determination, the “fundamental” (ν) is the source of uncertainty, which follows a stochastic process. The fundamental is assumed to follow a

stochastic process with a drift μ_v which can be a function of v and instantaneous standard deviation σ_v :

$$dv = \mu_v dt + \sigma_v dZ, \quad (2)$$

where dZ is a Wiener process with $E[dZ] = 0$ and $E[dZ^2] = dt$. We apply Ito's lemma to Eqs.(1) and (2), and have

$$\frac{E[ds(t)]}{dt} = \mu_v \frac{ds}{dv} + \frac{1}{2} \sigma_v^2 \frac{d^2 s}{dv^2}. \quad (3)$$

Then substituting Eq.(3) into Eq.(1) yields

$$\frac{1}{2} \alpha \sigma_v^2 \frac{d^2 s}{dv^2} + \alpha \mu_v \frac{ds}{dv} - s = -v - m, \quad (4)$$

which is a second-order linear ordinary differential equation.

Given a constant-drift fundamental, the general solution for Eq.(4) is as follows:

$$s = A_+ e^{\lambda_+ v} + A_- e^{\lambda_- v} + v + m + \alpha \mu_v, \quad (5)$$

where

$$\lambda_{\pm} = \frac{\mu_v}{\sigma_v^2} \left[\pm \sqrt{1 + \frac{2}{\alpha} \left(\frac{\sigma_v}{\mu_v} \right)^2} - 1 \right]. \quad (6)$$

Here A_+ and A_- are free constants to be determined by the prescribed boundary conditions, and need to be linked with the monetary policy and economics of the situation. Their associated terms (the first two terms in Eq.(5)) represent a deviation of the exchange rate from its fundamental value, which is represented by the last three terms in Eq.(5). Monetary authorities may intervene to influence their currencies' exchange rates by altering the stochastic process governing (relative) money-supply growth, which will alter the process driving the fundamental, v . A floating exchange rate regime is in effect when monetary authorities refrain from intervening to offset shocks to fundamentals and is expected to remain passive however the exchange rate moves. It is therefore reasonable to exclude parts

of the solution of the exchange rate in Eq.(5) that deviate far from the fundamental level when ν takes on large positive or negative values. In such a case, A_+ and A_- are zero and the exchange rate equation under a free float is:

$$s = \nu + m + \alpha\mu_\nu. \quad (7)$$

The analysis in this section is based on the driving process of the fundamental, which is relatively simple, i.e., a constant trend. When there is a possibility of a currency crash, fundamentals may not follow a random walk with a constant trend μ_ν in Eq.(2) and the solution in Eq.(7) is therefore no longer valid. In addition, the general solution in Eq.(5) is only applicable to a constant driving process of the fundamental. It is usually impossible to obtain closed-form general solutions similar to Eq.(5) when the stochastic fundamental ν follows a more complicated forcing process (e.g., μ_ν is a function of ν). With currency crash risk, the exchange rate solution depends on the fundamental's dynamic process and the boundary conditions associated with a currency crash.

3. Exchange rate solution and dynamics with currency crashes

This section shows how a currency crash is incorporated into the exchange rate model by imposing a smooth-pasting boundary condition at some lower levels of exchange rates. The exchange rate solution is then derived and analysed accordingly.

3.1 Currency crashes and smooth-pasting boundary condition

We consider the exchange rate S defined as a foreign currency value of a unit of a domestic currency. To derive the exchange rate solution, we identify a free-floating currency that may face a large fall in its value as a crash. To qualify how big a change in the exchange rate, we define a lower boundary as a tolerance limit for a distribution of the exchange rate's statistics (i.e., mean (or moving average) and standard derivation). Without assuming any

distribution of the exchange rate, the lower boundary S_L is taken to be the number (Δ) of standard deviations (Σ) from its mean $\bar{S}$: $S_L = \bar{S} - \Delta\Sigma$. If a normal distribution is assumed, the cumulative normal probabilities when the exchange rate falls below the boundaries are 0.0668 and 0.0227, if Δ is set equal to 1.5 and 2 respectively. The corresponding percentage drops from the mean are 37.5% and 50%. It is noted that even when the exchange rate price is not normally distributed, 1.5 and 2-standard deviations still cover a large area under the distribution of the exchange rate in a given time horizon, suggesting that falling to the lower boundary is a crash. The idea of the definition of a currency crash is similar to value-at-risk (VaR), which is a statistical measure of the riskiness of financial entities or portfolios of assets. VaR is defined as the maximum expected loss at a pre-defined confidence level (say 95%) over a given time horizon. Under a parametric method, also known as variance-covariance method, VaR is calculated as a function of mean and variance of the returns series with assumed normal distributions. Intuitively, a financial institution with a sharp fall in its equity price at the 95th percentile (i.e. a 5% probability of such extreme loss during the sample period) suggests the institution may be in distress. The choice of the level of the lower boundary (provided that it is adequately low) does not affect the process of the exchange rate dynamics. It is not necessary for the exchange rate to breach the lower boundary to capture its dynamics, as shown by the market data in the next section.

Based on qualifying the lower boundary for the exchange rate as a currency crash, historical exchange rates can be used as a guide to set a trading band that the rates are not expected to escape. The historical trend of the exchange rate can be measured by a moving average $S_A(t)$ of the current and past exchange rate. For a domestic currency suffering from depreciation subject to crash risk, the moving average can be scaled by a parameter η_L , with $0 < \eta_L < 1$, such that $\eta_L S_A(t)$ forms a lower boundary for the exchange rate movement. If the exchange rate is assumed to be normally distributed, $\eta_L S_A(t)$ corresponds to the number of

standard deviations from its moving average. The parameter η_L tells how much market participants expect the maximum or extreme downside of holding the currency to be in terms of a fraction of the moving average value $S_A(t)$. A smaller η_L suggests the market expects wider fluctuations over short horizons. In a free-floating exchange rate regime, if no authority has been forced to offer “one-way bets” to short-lived speculative spurts, the exchange rate can fluctuate with a relatively large margin within a band. Such specification of a lower boundary (and of an upper boundary) assumes market participants and monetary authorities care about the behaviour of the exchange rate over a time interval, rather than just its current level. The particular way in which past exchange rates are brought into play does not affect the derivation of the exchange rate solution and the qualitative results of our analysis.

With no loss of generality, the normalised log exchange rate s is defined by:

$$s = \ln \left[\frac{\eta_U S_{At} - S_t}{(\eta_U - \eta_L) S_{At}} \right], \quad (8)$$

where η_L and η_U are adjustable parameters for the lower and upper boundaries of a band respectively, and are not necessary to be symmetric at $S_A(t)$. To solve Eq.(4) with a currency crash when the exchange rate s breaches the lower boundary $s = 0$ (or $S_t = \eta_L S_{At}$), we specify the following boundary conditions at the fundamental of $\nu = 0$:

$$s(0) = 0, \quad (9)$$

$$\left. \frac{ds(\nu)}{d\nu} \right|_{\nu=0} = 0, \quad (10)$$

where the former condition ensures a proper normalisation of the exchange rate and the latter is the smooth-pasting boundary condition suggesting an optimal boundary condition for the process. At the boundary, there is no foreseeable jump in the exchange rate and no expected appreciation or depreciation, i.e., no arbitrage condition. The smooth-pasting condition ensures the exchange rate does not cross the boundary, as shown by Krugman and Rotemberg

(1990). If the condition does not hold, the exchange rate could jump across the boundary, indicating a currency crash, which is a rare event.

Since the boundary at $v \rightarrow \infty$ is inaccessible, the most general form of μ_v is given by

$$\mu_v = \sum_{n=-\infty}^{\infty} A_n v^n, \quad (11)$$

where $A_n < 0$ for $n > 0$. In addition, the assumption of μ_v having no irregular singular point, dictates that $A_n = 0$ for $n < -1$. The simplest possible candidate of this class of μ_v can be obtained by setting $A_n = 0$ for $n > 1$; which is,

$$\mu_v = \frac{A_{-1}}{v} + A_0 + A_1 v \quad (12)$$

for $A_1 < 0$. For the bounded and quasi-bounded condition at $v = 0$, the coefficient A_{-1} must be positive definite. Accordingly, the singular drift component $A_{-1}v^{-1}$ prevents v from breaching the boundary at the origin, while the mean-reverting component $A_1 v$ pulls v away from the boundary at infinity. Likewise, the constant drift term A_0 has a conflicting role – a positive A_0 reinforces the singular barrier at the origin and weakens the mean reversion, whereas a negative A_0 has the opposite effect. It is desirable to have a vanishing A_0 in μ_v . As a result, an asymmetric mean-reverting drift term μ_v for the fundamental v turns out to be the unique choice.

According to Eq.(12), it is reasonable to specify the coefficients of μ_v as:

$$\begin{aligned} A_{-1} &= (\beta - \sigma_v^2)/2 \\ A_0 &= 0 \\ A_1 &= -\kappa/2, \end{aligned} \quad (13)$$

where κ and $\beta > 0$. The parameters κ and β in the specification of Eq.(13) are generic, while the parameter σ_v is incorporated to make the analysis of the exchange rate dynamics at the boundaries convenient with comparison to the exchange rate volatility, given that the coefficient $(\beta - \sigma_v^2)$ remains a constant but has the effective impact on the fundamental dynamics. The fundamental shock therefore follows an asymmetric mean reversion and has the following specification:

$$dv = \frac{1}{2} \left(-\kappa v + \frac{\beta - \sigma_v^2}{v} \right) dt + \sigma_v dZ. \quad (14)$$

When the term $(\beta - \sigma_v^2) > 0$ and v is large (far away from the origin), the first term of the drift will push v down and towards the origin, therefore the currency will depreciate accordingly. Conversely, when v is small (near zero), the second term of the drift in Eq.(14) will push v upward and away from zero. The corresponding domestic currency will appreciate and the exchange rate will move away from its lower boundary. However, the mean-reverting force in Eq.(14) is not symmetric. The restoring force (domestic currency appreciation) given by the second term with v close to zero is stronger than the force (domestic currency depreciation) provided by the first term. This is consistent with the intuition that when a currency has depreciated significantly, the government's actions will push the exchange rate away from the lower boundary.

To further understand the asymmetric mean-reverting fundamental shock, we obtain a “potential well” $U(v)$ by integrating the drift term in Eq.(14), in a negative form, with respect to v :

$$U(v) = - \int \frac{1}{2} \left(-\kappa v + \frac{\beta - \sigma_v^2}{v} \right) dv = -(\beta - \sigma_v^2) \ln v + \frac{\kappa v^2}{4}, \quad (15)$$

in which the fundamental variable v is similar to a ball moving in a well, as shown in Figure 1 by plotting Eq.(15) with different values of κ and β . Decreasing κ will give an extremely

flat potential well covering the whole v , such that the Brownian force in the stochastic term will dominate the motion of the fundamental variable. The fundamental, and therefore the exchange rate, can then move more randomly with a weaker restoring force above the lower boundary, i.e., increasing the crash risk of v breaching the origin. Similarly, decreasing β will allow the fundamentals to approach the origin more easily and increase the crash risk, given the trough of the “well” moving towards the origin – reducing the singular drift component that stops v breaching the lower boundary. This shows that the strength of the mean reversion in the fundamental dynamics determines the crash risk of the currency.

3.2 Deriving exchange rate solution and dynamics

The presence of a regular singular point at $v = 0$ requires that the desired solution s of Eq.(4) takes a power-series form:

$$s = \sum_{n=1}^{\infty} B_n v^n, \quad (16)$$

which vanishes at $v = 0$. The coefficient B_1 can be determined to be zero by the smooth-pasting boundary condition of Eq.(10) at $v = 0$. Applying the two boundary conditions of Eqs.(9) and (10) at $v = 0$ to Eq.(4) yields:

$$B_2 = \frac{1}{2} \frac{d^2 s}{dv^2} \Big|_{v=0} = -\frac{m}{\alpha(\sigma_v^2 + 2A_{-1})} < 0, \quad (17)$$

suggesting s attains its maximum at $v = 0$. In other words, the second-order linear ordinary differential equation uniquely determines the second-order derivative of s with respect to v at $v = 0$ by itself. Substituting the power series of Eq.(16) into Eq.(4) with μ_v as defined in Eqs.(12) and (13), we can easily obtain:

$$B_2 = -\frac{m}{\alpha\beta} \quad (18)$$

$$B_3 = -\frac{1}{\alpha\beta} \left[\frac{2}{3} \left(1 + \frac{\sigma_v^2}{\beta} \right)^{-1} \right] \quad (19)$$

$$B_{n+2} = \frac{1}{\alpha\beta} \left\{ \frac{2+(n+2)\alpha\kappa}{n+4} \left[1 + \frac{(n+2)\sigma_v^2}{\beta} \right]^{-1} \right\} B_n \quad \text{for } n \geq 2. \quad (20)$$

It is noted that B_2 is independent of σ_v and is also obtained in Eq.(17). All the coefficients $\{B_{n+2}\}$ are negative. The series solution can be shown to be a convergent series for all v by means of the ratio test as $\lim_{n \rightarrow \infty} |B_{n+1}/B_n| \rightarrow 0$. The error analysis in Appendix A demonstrates that the convergence of the series solution is fast. Therefore, it is adequate to keep the leading-order term in Eq.(16), and the power series solution is reduced to a simple quadratic relationship between the exchange rate s and the fundamentals v as:

$$\begin{aligned} s(v) &= B_2 v^2 \\ &= -\frac{m v^2}{\alpha\beta}. \end{aligned} \quad (21)$$

The smooth-pasting boundary condition of Eq.(10) determines the relationship between the exchange rate and fundamental expressed in Eq.(21) at the lower boundary where the change of the exchange rate is tangent to the boundary. It is different from the linear relationship of Eq.(7) derived from the model without any crash risk. The boundary condition also suggests an optimal boundary condition for the process, which is shown explicitly by Eq.(17). If there is no foreseeable jump in the exchange rate and no expected appreciation or depreciation (i.e., no arbitrage condition at the boundary), the tangential movement of the exchange rate at the boundary is the only permitted trajectory. The smooth-pasting condition ensures a currency crash, which is defined as breaching the lower boundary, is a rare event, as market forces, including currency hedges modelled by the fundamental dynamics, would stop the exchange rate falling below the boundary. The smooth-pasting condition uniquely determines the form of the fundamental dynamics expressed in Eq.(14). We will then show the smooth-pasting condition could break down when the exchange rate

volatility increases or the mean-reverting force in the exchange rate dynamics decreases, which induces a currency crash with the exchange rate jumping across the lower boundary.

To illustrate the exchange rate dynamics, it is convenient to use the notation $x \equiv -s$ so $0 \leq x < \infty$ with $x=0$ corresponding to the lower boundary. By applying Ito's lemma to Eq.(14) with Eq.(21), x is shown to follow a mean-reverting square-root (MRSR) process:

$$dx = \kappa(\theta - x)dt + \sigma_x \sqrt{x}dZ, \quad (22)$$

where

$$\theta = \frac{\beta}{\kappa} |B_2| = \frac{m}{\alpha\kappa}, \quad (23)$$

$$\sigma_x = \frac{\sigma_v}{2} \sqrt{|B_2|} = \frac{\sigma_v}{2} \sqrt{\frac{m}{\alpha\kappa\theta}}. \quad (24)$$

κ determines the speed of the mean-reverting drift towards the long-term mean θ . When the exchange rate is close to zero, the standard deviation $\sigma_x \sqrt{x}$ also becomes very small. The corresponding exchange rate dynamics become dominated by the mean-reverting drift, which pushes the exchange rate towards the mean. The long-term mean θ associated with the exchange rate dynamic is a time-varying equilibrium level, which can be determined through action by market participants to drive the exchange rate towards its mean level. Such effects increase the mean-reverting force, determined by the size of κ .

Using Feller's classification of boundary points, it can be inferred that there is a non-attractive natural boundary at infinity and the one at the origin is a boundary of no probability leakage for $(\sigma_x^2 / 4\kappa\theta) < 1$ in Eq.(22) [equivalent to $(\sigma_v^2 / \beta) < 1$ in Eq.(14)], and it is not otherwise.⁵ The no-leakage condition ensures the exchange rate will not breach the origin (the lower boundary) and there is no currency crash; otherwise, the exchange rate may pass

⁵ For boundary condition definitions, see Karlin and Taylor (1981).

through the boundary, i.e., the exchange rate is quasi-bounded at the origin.^{6,7} If the no-leakage condition does not hold at the boundary, the smooth-pasting condition of Eq.(10) may break down in the model and a currency crash could occur. Based on the leakage condition in which the volatility and mean reversion of the exchange rate dynamics are the counteracting forces, we expect the mean reversing force to have a long-run negative relationship with the risk reversals. This weakens the mean reversion leading to a higher crash risk anticipated in the currency option market, which is tested empirically in section 5.

One characteristic of the normalised exchange rate in Eq.(8) is that the historical trend of the exchange rate for the normalisation is measured by a moving average $S_A(t)$ of the current and past exchange rate, which captures the drift dynamics of the exchange rate S , i.e., the expected exchange rate. The implication is that the dynamics of the normalised exchange rate x in Eq.(22) describes material part of random fluctuations of S , as shown in Appendix B. This means the mean-reverting drift of the normalised exchange rate stabilises the exchange rate volatility of S . When the mean-reverting drift drops, the effective volatility of S increases. Leakage through the lower boundary only occurs when the fluctuations accompanied with a plunge of the exchange rate (in rare situations) shoot up drastically. This is consistent with the observations in which exchange rate volatility goes up during bad economic periods (see Bates, 2012). The fluctuations of the exchange rate contain the crucial information of crash risk and the leakage condition of the exchange rate dynamics following the MRSR process will signal a possible crash.

⁶ Hui et al. (2016) find empirical evidence that the quasi-bounded process can describe the exchange rate dynamics and interest rate differential of the Swiss franc against the euro during the target zone regimen of September 2011 to January 2015. While the exchange rate was bounded below the strong-side limit during most of the time, as indicated by its dynamics, the condition for breaching the limit was met in November 2014 using only information until that point, i.e., about two months before abandoning the limit. The asymmetric mean-reverting fundamental therefore incorporates the characteristics of intervention and realignment.

⁷ Such a property is similar to the bounded exchange rate dynamics in Ingersoll (1996) and Larsen and Sørensen (2007) in which the variance of the exchange rate vanishes at the weak-side and strong-side limits in a two-sided target zone. In their models the exchange rate is completely bounded under all circumstances determined by the model parameters. However, the exchange rate following the quasi-bounded process can breach the limit under particular conditions.

The probability density function (PDF) of x under the MRSR process is given by:

$$G(x, t; x', t') = \frac{2}{\sigma_x^2 C_1(t-t')} \left(\frac{x}{x'} \right)^{\omega/2} \exp \left[-\frac{\omega+2}{2} C_2(t-t') \right] \times \exp \left\{ -\frac{2x' + 2x \exp[-C_2(t-t')]}{\sigma_x^2 C_1(t-t')} \right\} \times I_\omega \left\{ \frac{4x^{1/2} x'^{1/2} \exp[-C_2(t-t')/2]}{\sigma_x^2 C_1(t-t')} \right\}, \quad (25)$$

where $\omega = 2\kappa\theta/\sigma_x^2 - 1$, $C_1(\tau) = [\exp(\kappa\tau) - 1]/\kappa$, $C_2(\tau) = -\kappa\tau$, I_ω is the modified Bessel function of the first kind of order ω . The associated asymptotic PDF will eventually approach the steady-state exchange rate distribution, which is:

$$K(x, t \rightarrow \infty; x', t') = \frac{2x^{\omega+1/2}}{\Gamma(\omega+1)} \left(\frac{2\kappa}{\sigma_x^2} \right)^{\omega+1} \exp \left(-\frac{2\kappa}{\sigma_x^2} x \right), \quad (26)$$

where Γ is the gamma function. Given the PDF in Eq.(25), the parameters of the MRSR process for the exchange rate dynamics are calibrated in section 4 using market exchange rate data.

Figure 2 shows the steady-state exchange rate distributions in S based on Eq.(26) with two values of the long-term mean θ of 1.0 and 1.5: the former θ is closer to the lower boundary than the latter one. We use the model parameters for $\sigma_x = 0.05, 0.08$ and 0.1 , and $\kappa = 0.01$ and 0.04 , which are consistent with the estimations in section 4. The distributions have their peaks at the right, showing the PDF will decay slower than a Gaussian distribution (the so-called “fat-tails” effect) at the left. This suggests the probability of outlier negative returns. This feature is consistent with the empirical observations of exchange rate returns and the left-skewed distributions in Brunnermeier et al. (2008), Burnside et al. (2011) and Jurek (2014), and the predictions in some models of stochastic volatility, such as Heston (1993) and Bates (2012).

All panels in Figure 2 show fatter tails of the exchange rate distributions with the mean θ further away from the lower boundary, demonstrating that the probability of outlier negative returns becomes more significant for a currency expected to appreciate in the near term. Comparison among Panel A, B and C, where σ_x increases from 0.05 to 0.1, shows the left tails of the distributions become much fatter and hump shaped, and their left-skewness is sensitive to an increase in the exchange rate volatility. The higher exchange rate volatility increases the likelihood of a currency crash, which is reflected by the fat left tails. By keeping $\sigma_x = 0.1$ and increasing κ from 0.01 to 0.04 in Panel D, the exchange rate distributions return to the shapes similar to those in Panel A with less fat left tails, suggesting an increase in the mean reversion in the exchange rate dynamics reduces crash risk. This can be explained by a strong mean reverting force in the normalised exchange rate x reducing the exchange rate volatility in S , given that the dynamics of x describe the material part of random fluctuations of S . The changes in the distributions in Figure 2 with different exchange rate parameters demonstrate that the leakage condition of the MRSR process of the exchange rate dynamics derived from the model is consistent with the left-skewed distributions for exchange rates with crash risk.

4. Calibrations of exchange rate dynamics

In this section, we calibrate the MRSR process for the exchange rate dynamics presented in the previous section. We use nine currencies, including the Australian dollar (AUD), Canadian dollar (CAD), Swiss franc (CHF), euro (EUR), British pound (GBP), Japanese yen (JPY), Norwegian krone (NOK), New Zealand dollar (NZD) and Swedish krona (SEK). Their values are measured as their exchange rates against the US dollar

(USD).⁸ The time series data is from 2 June 1996 (2 January 1999 for the EUR) to July 14, 2017.⁹ The estimations use a three-year rolling window with the initial window covering the period from June 1996 (January 1999) to May 1999 (December 2001). The maximum likelihood estimation (MLE) using daily data is employed to estimate the model parameters in Eq.(22) based on a log-likelihood function constructed by the analytical PDF of Eq.(25). The parameters η_L and η_U for the normalised log exchange rate in Eq.(8) are set to 0.625 and 1.375 respectively, and $S_A(t)$ is defined as a six-month moving average. Therefore, the lower boundary, which is set at 37.5% below $S_A(t)$, can be considered a “large devaluation”. The choice of boundaries only affects the model parameter estimations but not the process of the exchange rate dynamics.

The estimation results are presented in Figure 3 for the AUD, CAD, NOK, NZD and SEK, and Figure 4 for the CHF, EUR, GBP and JPY. Panel D of the two figures show the exchange rates of all currencies do not breach the lower boundaries ($x = 0$). This indicates the smooth-pasting condition at the boundaries determines the exchange rate dynamics with crash risk, while it is not necessary for the exchange rates to breach the boundaries. The estimated volatilities σ_x of the currencies, as shown in Panel A of the two figures, range between 0.01 and 0.05. Their corresponding z-statistics are much higher than 1.96 (i.e., at 5% significance level) except during the short period at the end of 2008 for all currencies and in early 2003 for the CAD, indicating the estimated σ_x is highly significant under the MRSR process. The changes of σ_x are similar among the currencies with substantial increases after the 2008 GFC. The volatility for the CHF jumped in early 2015 after the Swiss National Bank abandoned the target-zone regime for the currency.

⁸ It is noted that the Japanese yen and Swiss franc are considered safe-haven currencies and usually do not suffer from crashes during bad times. To make the comparison in the empirical analysis among currencies consistent, we use the US dollar as the base currency in the analysis.

⁹ The data used in this section is from Bloomberg.

Panels B of Figures 3 and 4 show that the estimates of the drift term κ are significant in terms of the z-statistic (higher than or close to the 5% significance level) for all the currencies except during the period of 2009-2011 and a few other short periods for some currencies. Similar to the volatility estimations, the patterns of changes in κ are similar among the currencies (except the JPY and CHF) with sharp falls at the end of 2008. Panels C of the figures demonstrate the estimated mean θ of all the currencies are also statistically significant at the 5% significance level and steady at the levels of 0.6-0.8 during most of the estimation period. Similar to the estimations of σ_x and κ , the estimated θ becomes insignificant in a short period at the end of 2008. The estimations of κ and θ for the MRSR process show a significant mean-reverting force in the exchange rate dynamics for the currencies.

Panels B and C in Figures 3 and 4 show that κ and θ for all currencies dropped in the last quarter of 2008 when the GFC emerged, which coincided with increases in volatilities σ_x . This suggests currency crash risk building at the lower boundaries with the weakening mean-reverting force and increased volatility. Such pressure, indicated by the critical condition $(\sigma_x^2/4\kappa\theta) > 1$ of probability leakage at the boundaries for the currencies, is presented in Panels D of the figures. A high value of the measure of $(\sigma_x^2/4\kappa\theta)$ reflects a rise in crash risk. The probability leakage measures of the currencies, including the AUD, CAD, NOK, NZD and SEK, in Figure 3 and the EUR and GBP in Figure 4 surged over 1 during the 2008 GFC when the US dollar shortage triggered heavy sell-offs of those currencies and caused their crashes. Conversely, those of the JPY and CHF in Panel D of Figure 3 stayed low due to the fact that they are considered safe-haven currencies.

Based on the model parameter estimations of the EUR/USD exchange rate on 16, 17 and 20 October 2008 before the currency crash, Figure 5 plots their corresponding exchange

rate distributions expressed in Eq.(26). The distributions had fatter left tails when the crash risk intensified in a week during the crisis. Their left skewness was sensitive to the weakened mean reversion (decreasing κ and θ) in the dynamics of the exchange rate x , which represents increases in the volatility of the exchange rate S , as discussed in the previous section. Regarding the euro during the European sovereign debt crisis, its probability leakage measure increased from almost zero to 0.25 in early 2015, when the European Central Bank introduced a quantitative easing program that substantially weakened the euro exchange rate.

In summary, the estimation results shown in Figures 3 and 4 provide evidence that the MRSR process for the exchange rate dynamics expressed as Eq.(22) can be adequately calibrated by the data on the currencies. The surges of the currencies' leakage condition during the GFC indicate that the likelihoods of their exchange rates breaching the lower boundaries increased significantly when the currency crashes occurred.

5. **Dynamic relationship between exchange rates and risk reversals**

The price of currency crash risk is reflected by the price of the risk reversal, which measures the implied volatility difference between an out-of-the-money put and an out-of-the-money call at the same (absolute) delta. It is also interpreted as the market view of the most likely direction of the foreign exchange rate spot movement over the next maturity date. In view of studies, including Brunnermeier et al. (2009), Farhi et al. (2015) and Jurek (2014), on the positive relationship between currency crashes and risk reversals, the estimated model parameters of the exchange rate dynamics in Eq.(22) are expected to be related to the movements of the risk reversal, which reflects the likelihood of a currency crash. In particular, the crash measure in the proposed exchange rate dynamics is determined mainly by the mean reversion, given that the volatility is relatively steady before the currency crash shown in

Figure 3 and 4. We examine the interrelationship between the mean-reversion parameters (κ and θ) and risk reversals through the cointegration method.

We postulate that there is a long-run equilibrium relationship between the mean-reversion parameters and risk reversals. The short-run dynamics represented as a dynamic error-correction model are given by:

$$\Delta y_t = a_{10} + \alpha_y(y_{t-1} - \gamma_1 RR_{t-1}) + \sum_k b_{1k} \Delta y_{t-k} + \sum_k c_{1k} \Delta RR_{t-k} + \varepsilon_{yt} , \quad (27)$$

where y_t is either κ or θ at time t , RR_{t-1} is the risk reversal at time $(t - 1)$ and α_y is less than zero. As specified, the variables will change in response to stochastic shocks and to the previous period's gap from the long-run equilibrium (i.e., $y_{t-1} - \gamma_1 RR_{t-1}$). The parameter α_y is the speed of adjustment. In absolute terms, the larger α_y the greater the response of y_t to the previous period's gap from the long-run equilibrium. If α_y is equal to zero, the long-run equilibrium relationship does not appear and the model is not an error-correction one or cointegrated. Therefore, for a meaningful cointegration and error-correction model, the speed of adjustment α_y must be non-zero.

The estimation is conducted using monthly data for the model parameters (κ and θ) and risk reversals with 10% (RR10) and 25% (RR25) delta from October 2003-June 2017 for the AUD, CAD, EUR GBP, JPY and NZD, and March 2005-June 2017 for the CHF, NOK and SEK.¹⁰ Table 1 provides the Augmented Dickey-Fuller (ADF) and Phillips-Perron test results for RR , κ and θ in levels and changes. It fails to reject at the 10% level the presence of a unit root for the variables in levels. However, the test for the first differences is significant at the 1% level. Therefore, the changes are stationary. This suggests that the variables considered are all $I(1)$ (i.e., integrated of the same order 1), which satisfies the requirement for the variables to be cointegrated.

¹⁰ It is due to availability of market data.

To test the cointegration between the risk reversals and model parameter (κ and θ), we use the Engle–Granger (1987) single-equation test, which is regarded as an easy and super-consistent method of estimation. It determines whether the residuals of the linear combination among the cointegrated variables estimated from the ordinary least squares method are stationary. Table 2 reports the cointegration tests between RR and model parameters κ and θ . The critical values of the tests are based on MacKinnon (1996) and the lag length is determined by the Schwartz criterion. The results are significant at the 10% or less levels. Therefore, we reject the null hypothesis that RR and the model parameters (κ and θ) are not cointegrated in favour of the alternative hypothesis that there is at least one cointegrating vector.

Table 3 reports the estimated cointegrating vectors. The coefficients γ are all negative at the 10% or less significance levels. This shows κ and θ are negatively related to RR , indicating that, when risk reversals increase, the mean-reverting force in the exchange rate dynamics will weaken (i.e., lower κ and θ). Intuitively, the result suggests that crash risk increases when the mean reversion of the exchange rate dynamics weakens. Prices of hedging against downside risk (crash risk) of a currency are therefore higher than its up-side risk. The empirical result is consistent with the interpretation of the crash risk measure, the probability leakage ratio ($\sigma_x^2 / 4\kappa\theta$) derived from the exchange rate model, which is expected to increase with higher risk reversals.

Finally, Table 4 reports the estimates of the short-run dynamics. In all regressions considered, the speeds of adjustment α_y are negative at the 10% or less significance levels and smaller than 1 in absolute value. This suggests that the error correction specification is valid and there is a self-restoring force to close the gap of the link between the model

parameters (κ and θ) and risk reversals, and subsequently adjust to restore the long-run equilibrium.

6. Conclusion

We present an exchange rate model in which a currency's exchange rate is confined in a wide moving band and a currency crash occurs when the exchange rate breaches the lower boundary where a smooth-pasting boundary condition is imposed. A solution is derived from the standard log exchange rate equation for the model. By using an asymmetric mean-reverting fundamental shock, the solution shows the exchange rate follows a mean-reverting square-root process, which is quasi-bounded at the boundary and can breach the boundary with a weakened mean reversion. The boundary condition suggests an optimal boundary condition for the process. If there is no foreseeable jump in the exchange rate and no arbitrage condition at the boundary, the smooth-pasting condition ensures a currency crash is rare. The exchange rate solution generates left-skewed exchange rate distributions, which are a feature it shares with empirical observations and some stochastic volatility models.

The model is consistent with theories and empirical evidence about the positive relationship between currency crash risk and risk reversals in currency option markets. In the model, the probability leakage for the exchange rate across the lower boundary increases with a weakened mean-reverting force in the normalised exchange rate dynamics, suggesting an increase in currency crash risk. Using the exchange rates of the Australian dollar, Canadian dollar, Swiss franc, euro, British pound, Japanese yen, Norwegian krone, New Zealand dollar and Swedish krona against the US dollar from 1996-2017, the empirical results demonstrate that their normalised log exchange rates can be calibrated according to the model, where the mean reversion is negatively cointegrated with the risk reversals, as expected by the model. The leakage condition for breaching the boundaries was met during the 2008 global financial

crisis when most of the currencies were under the disaster shock. Given that the derived exchange rate process is consistent with risk reversals, if this process is used as a basis for pricing currency options and derivatives, its performance in terms of hedging cost should be better (with lower hedging cost) than the conventional option-pricing model, which assumes the lognormal process. Given the empirical finding in relation to risk reversals, information of currency option prices can be incorporated into an exchange rate model to enhance its empirical performance, which will be left for future research.

Appendix A

The relationship between the exchange rate S and the fundamental v is given by

$$S(v) - \eta_L S_{At} = \exp(\sum_{n=2}^{\infty} B_n v^n)(\eta_U - \eta_L) S_{At}. \quad (A1)$$

Since all the B_n s are negative real constants, the leading term provides an upper bound of the exact solution, namely

$$S(v) - \eta_L S_{At} < \exp(B_2 v^2)(\eta_U - \eta_L) S_{At}. \quad (A2)$$

One can also estimate the total error associated with approximating the exact solution by this upper bound as follows:

$$\Delta S \equiv (\eta_U - \eta_L) S_{At} \int_0^{\infty} \exp(B_2 v^2) [1 - \exp(\sum_{n=3}^{\infty} B_n v^n)] dv. \quad (A3)$$

Moreover, a better approximate solution of the form

$$\tilde{S}(v, \varepsilon) - \eta_L S_{At} = \exp(\varepsilon B_2 v^2)(\eta_U - \eta_L) S_{At} \quad (A4)$$

can be determined by minimising the total error

$$\Delta \tilde{S}(\varepsilon) \equiv (\eta_U - \eta_L) S_{At} \sqrt{\int_0^{\infty} [\exp(\varepsilon B_2 v^2) - \exp(\sum_{n=2}^{\infty} B_n v^n)]^2 dv}, \quad (A5)$$

with respect to the positive real parameter ε . It is clear that this optimal approximate solution should be better than the upper-bound solution, which corresponds to the special case of $\varepsilon = 1$.

Figure A1 shows the relationship between the exchange rate S and the fundamental ν using different numbers of terms ($B_2, B_2 + B_3, \dots, B_2 + \dots + B_5$) of the series solution in Eq.(21) and the exchange rate of EUR/USD of 1.34 with parameters $m = 1$, $\alpha = 0.2$, $\kappa = 0.02$, $\beta = 0.02$ and $\sigma = 0.25$, which are broadly consistent with the estimations in section 4. The result demonstrates that the convergence of the series solution is fast and only its leading term contributes significantly. Using higher-order terms does not have material impact on the results, in particular near the lower boundary where $\nu = 0$. It is also noted that the leading term is the upper bound of the series solution. Accordingly, we can conclude that Eq.(21) is a good approximation of the exact relationship between s and ν with extremely small errors.

Appendix B

Let the exchange rate S follow a generic stochastic process with a drift μ and volatility $\sigma_s(S, t)$:

$$dS_t = \mu S_t dt + \sigma_s(S, t) S_t dW, \quad (\text{B1})$$

where dW is a Wiener process. The instantaneous exchange rate expectation value is given by

$$\langle S \rangle_t = S_0 \exp(\mu t). \quad (\text{B2})$$

With no loss of generality, the normalised exchange rate R is defined by:

$$\tilde{R}_t = \frac{\eta_U \langle S \rangle - S_t}{(\eta_U - \eta_L) \langle S \rangle_t}, \quad (\text{B3})$$

where η_U and η_L are adjustable parameters for the lower and upper boundaries of a band. $\tilde{R}_t$ obeys the stochastic differential equation:

$$d\tilde{R}_t = \sigma_s \tilde{R}_t dZ \quad (\text{B4})$$

$$\Rightarrow d(\ln \tilde{R}_t) = -\frac{1}{2} \sigma_s^2 dt + \sigma_s dZ. \quad (\text{B5})$$

It is clear that $\tilde{R}_t$ consists of the purely random part only; in other words, it provides the information about the fluctuations of S . While the exact $\langle S \rangle_t$ is unobservable, it can be estimated as the historical trend of the exchange rate measured by a moving average $S_A(t)$ of the current and past exchange rate. However, unlike $\langle S \rangle_t$, $S_A(t)$ contains the residual effect of the random fluctuations of the exchange rate.

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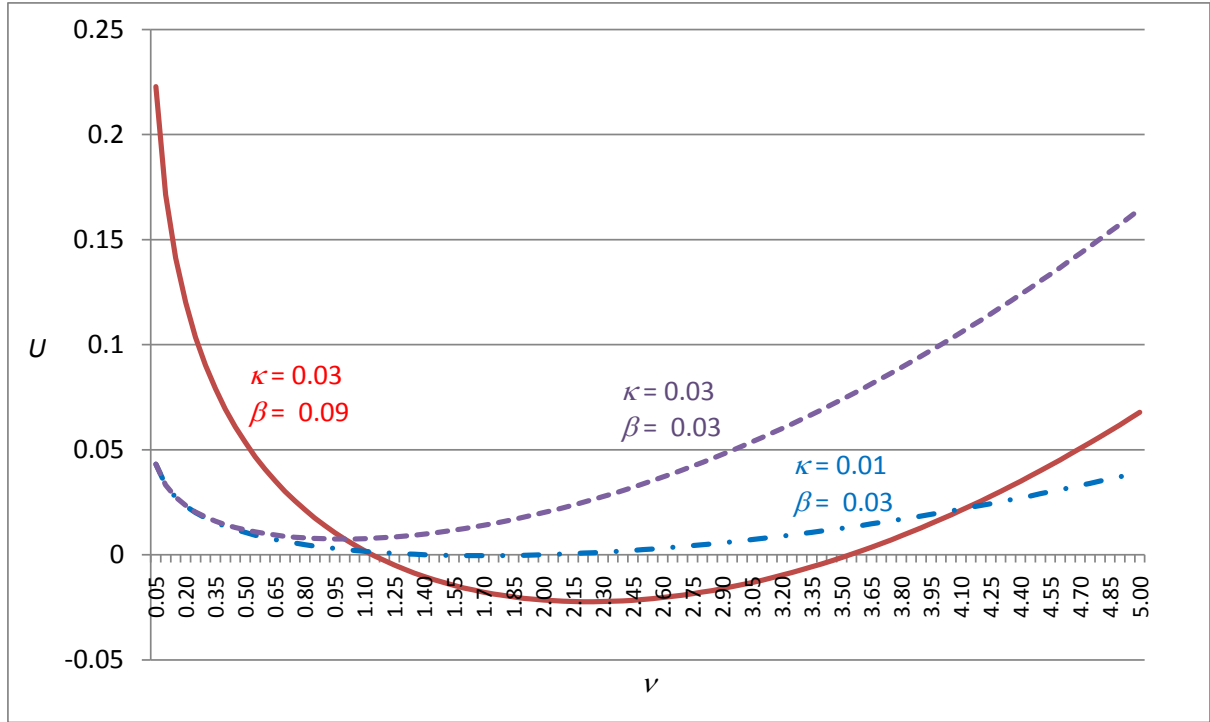


Figure 1: Eq.(15) of $U(\nu)$ by integrating drift term of fundamental dynamics with different model parameters κ and β , and $\sigma_\nu = 0.1$.

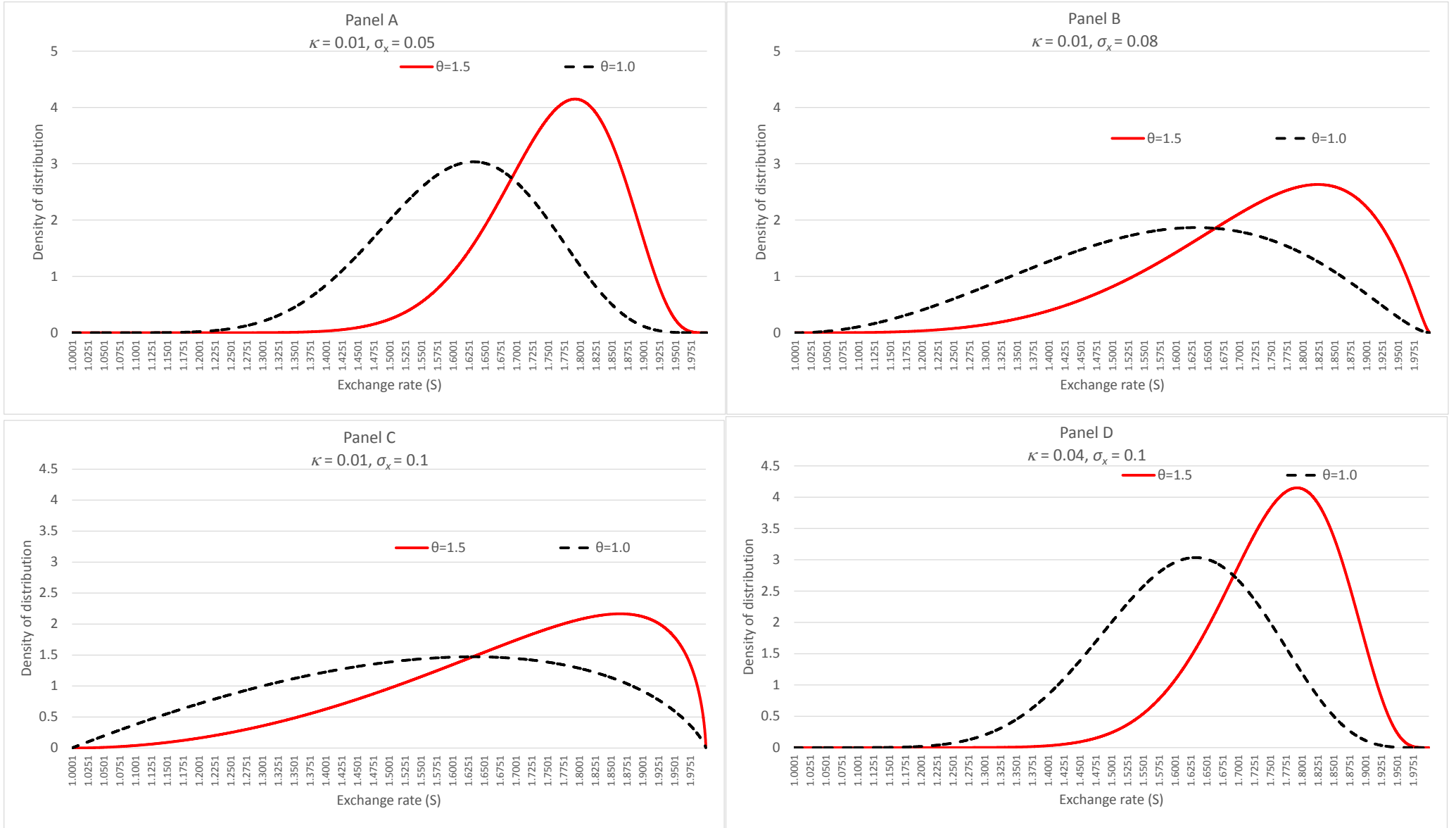


Figure 2: Exchange rate distributions with different values of model parameters σ_x , κ and θ .

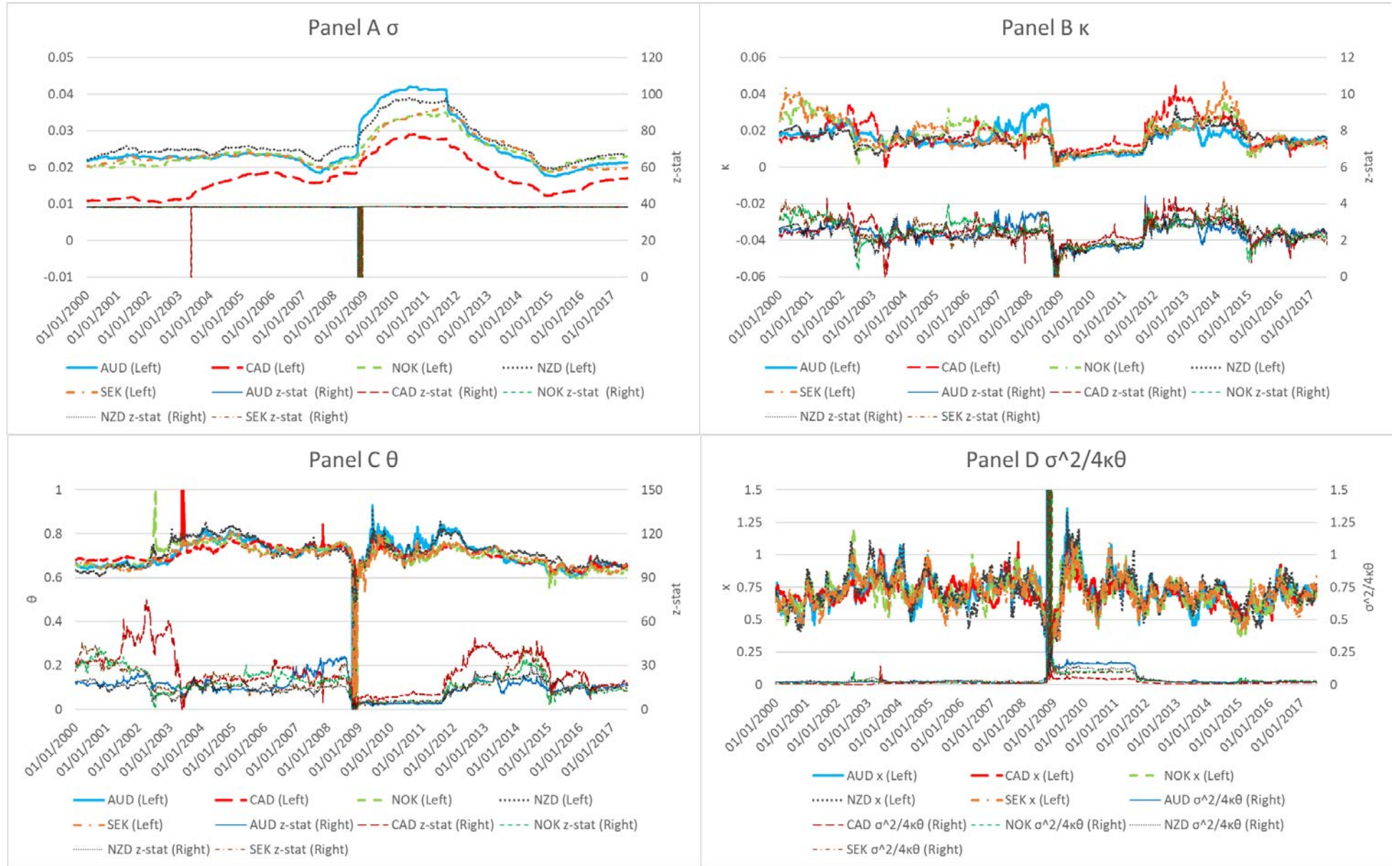


Figure 3: Estimated model parameters of AUD, CAD, NOK, NZD and SEK exchange rates with three-year rolling window: (A) estimates and z-statistic of σ_x , (B) estimates and z-statistic of κ , (C) estimates and z-statistic of θ , (D) x and probability leakage measure $\sigma_x^2/4\kappa\theta$.

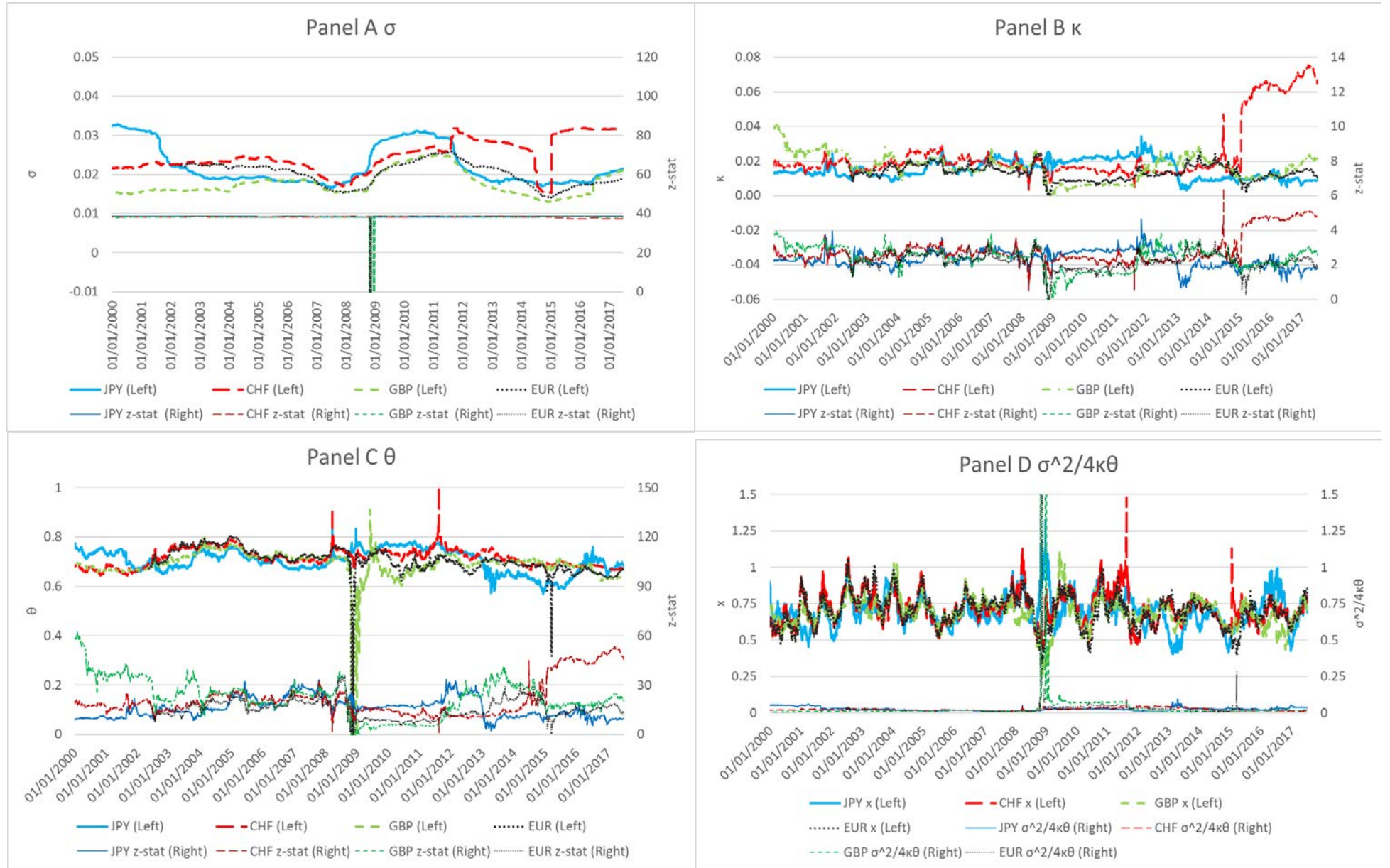


Figure 4: Estimated model parameters of JPY, CHF, GBP and EUR exchanges with three-year rolling window: (A) estimates and z-statistic of σ_x , (B) estimates and z-statistic of κ , (C) estimates and z-statistic of θ , (D) x and probability leakage measure $\sigma_x^2/4\kappa\theta$.

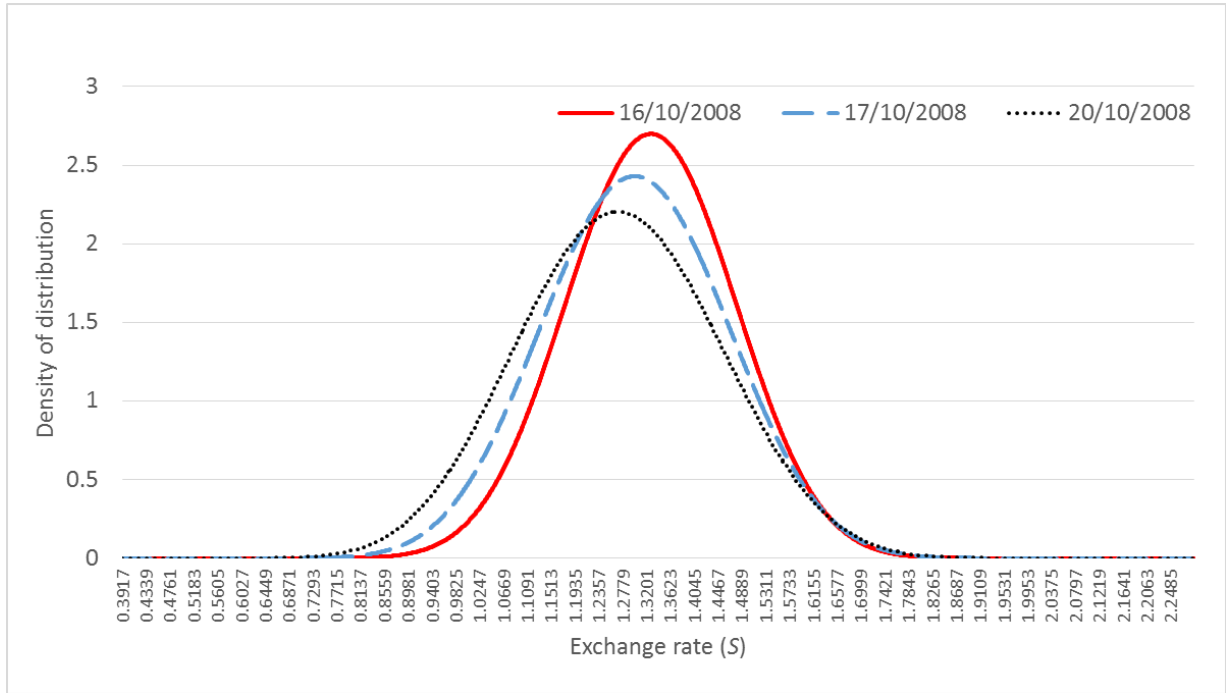


Figure 5: Exchange rate distributions of EUR/USD exchange rate on 16, 17 and 20 October 2008, with $\kappa = 0.0041, 0.0033, 0.0028$, $\theta = 0.69, 0.66, 0.65$, and $\sigma_x = 0.01692, 0.01690, 0.01690$.

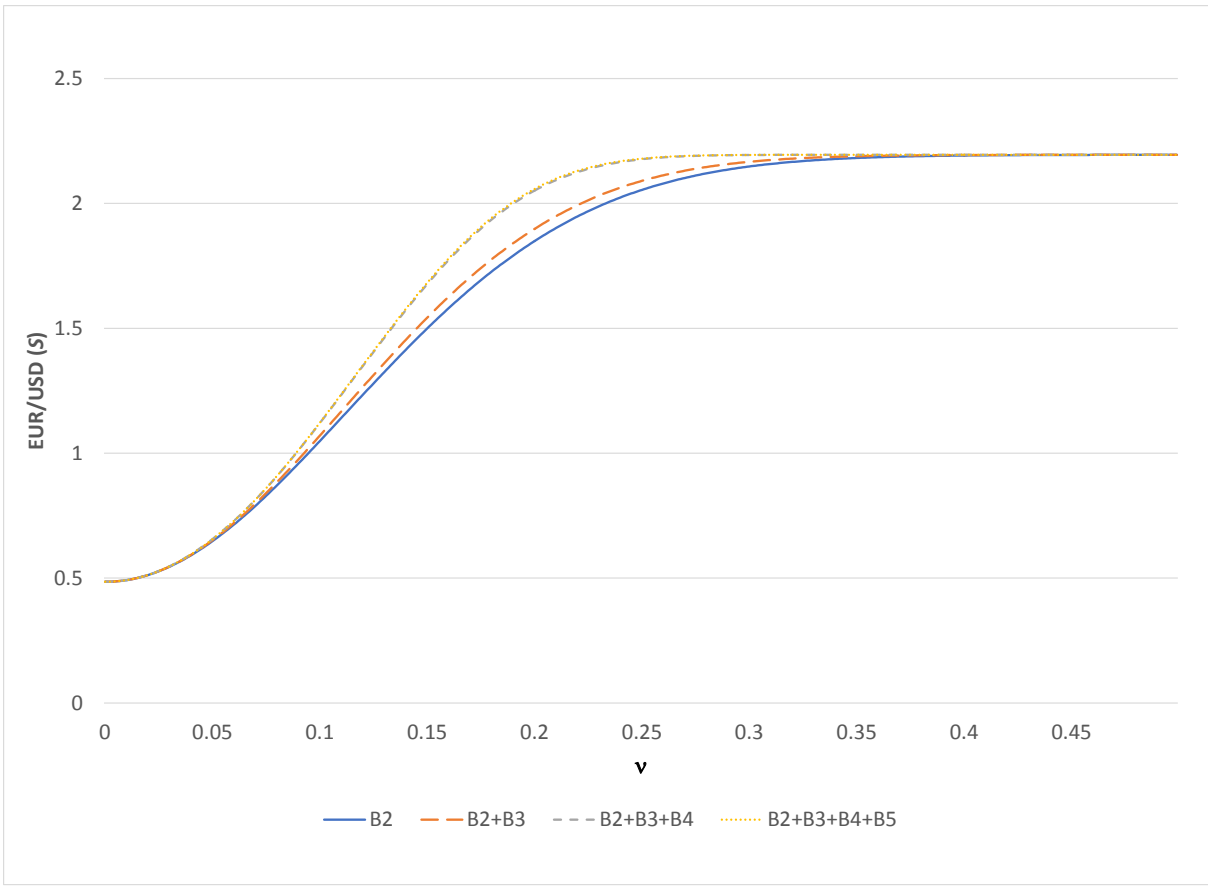


Figure A1: Relationship between exchange rate S and fundamental ν in Eq.(21) using numbers of terms B_n ($n = 2, \dots, 5$) and parameters $m = 1$, $\alpha = 0.2$, $\kappa = 0.02$, $\beta = 0.02$ and $\sigma = 0.25$.

Table 1: ADF and Phillips-Perron tests.

AUD/USD	RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change	
ADF test statistics	-3.02	-7.794	***	-1.23	-8.893	***	-0.527	-7.176	***	-0.626	-10.675	***
Phillips-Perron test statistics	-3.07	-9.831	***	-1.28	-10.310	***	-0.619	-6.167	***	-0.600	-10.623	***
Correlation with RR25/ Δ RR25	-	-		-	-		-0.165	-0.108		0.029	-0.358	
Correlation with RR10/ Δ RR10	-	-		-	-		-0.177	-0.117		0.042	-0.342	
CAD/USD	RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change	
ADF test statistics	-1.24	-8.777	***	-2.98	-9.967	***	-1.549	-10.955	***	-0.546	-9.062	***
Phillips-Perron test statistics	-1.29	-10.749	***	-2.90	-10.870	***	-1.851	-11.048	***	-0.664	-9.132	***
Correlation with RR25/ Δ RR25	-	-		-	-		0.192	0.099		0.042	0.248	
Correlation with RR10/ Δ RR10	-	-		-	-		0.207	0.101		0.030	0.231	
CHF/USD	RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change	
ADF test statistics	-3.02	-11.399	***	-2.87	-10.844	***	-1.394	-9.524	***	-3.082	-11.153	***
Phillips-Perron test statistics	-2.98	-14.289	***	-2.97	-10.844	***	-1.102	-9.580	***	-3.009	-11.153	***
Correlation with RR25/ Δ RR25	-	-		-	-		-0.262	-0.171		0.049	-0.463	
Correlation with RR10/ Δ RR10	-	-		-	-		-0.266	-0.160		0.051	-0.458	
EUR/USD	RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change	
ADF test statistics	-1.24	-8.777	***	-2.69	-8.893	***	-0.496	-8.853	***	-0.724	-10.657	***
Phillips-Perron test statistics	-1.29	-10.749	***	-2.80	-10.310	***	-3.027	-7.723	***	-0.707	-10.518	***
Correlation with RR25/ Δ RR25	-	-		-	-		-0.364	0.013		-0.542	-0.237	
Correlation with RR10/ Δ RR10	-	-		-	-		-0.355	0.021		-0.523	-0.247	
GBP/USD	RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change	
ADF test statistics	-1.49	-10.329	***	-1.42	-9.937	***	-1.109	-7.562	***	0.214	-9.170	***
Phillips-Perron test statistics	-1.59	-10.329	***	-1.58	-9.800	***	-1.111	-7.220	***	0.214	-9.187	***
Correlation with RR25/ Δ RR25	-	-		-	-		-0.297	0.024		-0.611	-0.463	
Correlation with RR10/ Δ RR10	-	-		-	-		-0.290	0.024		-0.601	-0.458	

Cont'd

JPY/USD			RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change		Level	Change
ADF test statistics	-2.66	-11.170 ***		-3.11	-10.992 ***		-2.682	-10.785 ***		-1.914	-10.338 ***			
Phillips-Perron test statistics	-2.82	-11.262 ***		-2.80	-11.031 ***		-2.741	-10.698 ***		-1.918	-10.352 ***			
Correlation with RR25/ Δ RR25	-	-		-	-		-0.212	0.051		-0.363	-0.345			
Correlation with RR10/ Δ RR10	-	-		-	-		-0.153	0.047		-0.314	-0.340			
NOK/USD			RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change		Level	Change
ADF test statistics	-1.24	-8.777 ***		-1.49	-11.544 ***		-0.765	-7.522 ***		-0.704	-9.272 ***			
Phillips-Perron test statistics	-1.29	-10.749 ***		-1.48	-11.584 ***		-0.447	-7.347 ***		-0.525	-9.244 ***			
Correlation with RR25/ Δ RR25	-	-		-	-		-0.379	-0.034		-0.202	-0.147			
Correlation with RR10/ Δ RR10	-	-		-	-		-0.338	-0.046		-0.137	-0.129			
NZD/USD			RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change		Level	Change
ADF test statistics	-1.12	-8.953 ***		-1.12	-9.039 ***		-0.704	-10.489 ***		-0.628	-9.591 ***			
Phillips-Perron test statistics	-1.16	-11.461 ***		-1.18	-10.606 ***		-0.742	-10.394 ***		-0.698	-9.587 ***			
Correlation with RR25/ Δ RR25	-	-		-	-		-0.197	-0.129		-0.088	-0.428			
Correlation with RR10/ Δ RR10	-	-		-	-		-0.194	-0.138		-0.064	-0.388			
SEK/USD			RR25			RR10			κ			θ		
	Level	Change		Level	Change		Level	Change		Level	Change		Level	Change
ADF test statistics	-1.29	-11.739 ***		-1.31	-11.748 ***		-0.925	-8.236 ***		-0.418	-9.853 ***			
Phillips-Perron test statistics	-1.30	-11.740 ***		-1.33	-11.763 ***		-0.798	-8.236 ***		-0.269	-9.853 ***			
Correlation with RR25/ Δ RR25	-	-		-	-		-0.087	-0.182		-0.010	-0.221			
Correlation with RR10/ Δ RR10	-	-		-	-		-0.107	-0.176		-0.012	-0.194			

Notes:

1. *** indicates significance at a levels of 1%.
2. Both tests check the null hypothesis of unit root existence in the time series, assuming non-zero mean in the test equation.
3. The correlations for the level of the variables are the correlations with RR, and those for change are the correlations with the change of RR.

Table 2: Test of cointegration (Euler-Granger).

<u>Engle-Granger single-equation test</u>	2003M10-2017M6				2003M10-2017M6				2006M1-2015M1				2003M10-2017M6				2003M10-2015M5			
	AUD				CAD				CHF				EUR				GBP			
	κ	θ	i		κ	ii	θ		κ	iii	θ	iv	κ	θ			κ	θ		
Dependent variable:																				
ADF test statistic (RR25)	-2.29	**	-2.44	**	-3.62	***	-2.33	**	-2.86	***	-2.89	***	-4.28	***	-2.98	***	-3.03	***	-3.97	***
Phillips-Perron test statistic (RR25)	-2.66	**	-2.33	**	-3.62	***	-1.78	*	-2.63	***	-2.96	***	-3.26	***	-2.84	***	-3.20	***	-4.07	***
ADF test statistic (RR10)	-2.30	**	-2.43	**	-3.64	***	-2.32	**	-2.93	***	-2.90	***	-4.19	***	-2.82	***	-2.93	***	-3.87	***
Phillips-Perron test statistic (RR10)	-2.66	***	-2.34	**	-3.76	***	-1.75	*	-2.46	**	-2.96	***	-3.16	***	-2.66	***	-3.13	***	-3.87	***
-	2003M10-2017M6				2005M3-2017M6				2003M10-2017M6				2006M2-2017M6							
	JPY				NOK				NZD				SEK							
	κ	θ			κ	θ			κ	v	θ		κ	vi	θ	vi				
ADF test statistic (RR25)	-2.30	**	-1.94	*	-2.67	***	-2.24	**	-2.03	**	-2.34	**	-1.99	**	-2.68	***				
Phillips-Perron test statistic (RR25)	-2.33	**	-1.80	*	-2.34	**	-2.64	***	-2.23	**	-2.17	**	-2.04	**	-2.97	***				
ADF test statistic (RR10)	-2.29	**	-1.94	**	-2.56	**	-2.01	**	-2.01	**	-2.38	**	-1.98	**	-2.66	***				
Phillips-Perron test statistic (RR10)	-2.33	**	-1.80	*	-2.05	**	-2.29	**	-2.22	**	-2.20	**	-2.02	**	-2.95	***				

Notes:

1. ***, ** and * indicate significance at a level of 1%, 5% and 10% respectively.
2. The cointegration test uses the Augmented Dickey-Fuller and Phillips-Perron tests to check the null hypothesis that the residuals of the regression of Risk Reversals and the parameters from the MLE calibration with a three-year rolling window, are non-stationary, assuming zero mean in the test equation. The critical value of the test is obtained from MacKinnon (1996).
3. i, ii, iii, iv, v and vi indicate dummies are added during 2011M8-M12, 2011M9-2013M12, 2014M4-M6, 2011M6-M12, 2009M1-M4 and 2011M8-2012M12 respectively.
4. All RR, kappa and theta are I(1) series, being tested by both ADF and Phillips-Perron tests.

Table 3: Estimates of the long-run part of cointegrating vectors.

Dependent variable:	2003M10-2017M6				2003M10-2017M6				2006M1-2015M1				2003M10-2017M6				2003M10-2015M5			
	AUD				CAD				CHF				EUR				GBP			
	κ	θ	i		κ	ii	θ		κ	iii	θ	iv	κ	θ			κ	θ		
RR25 (β)	-0.0010	**	-0.0115	***	-0.0017	***	-0.0196	***	-0.0008	*	-0.0062	**	-0.0014	***	-0.0229	***	-0.0022	***	-0.0322	***
Constant_RR25	0.0179	***	0.7381	***	0.0158	***	0.7214	***	0.0165	***	0.7211	***	0.0148	***	0.7233	***	0.0183	***	0.7289	***
RR10 (β)	-0.0005	**	-0.0053	***	-0.0009	**	-0.0106	***	-0.0004	*	-0.0033	**	-0.0007	***	-0.0121	***	-0.0012	***	-0.0172	***
Constant_RR10	0.0179	***	0.7355	***	0.0158	***	0.7208	***	0.0165	***	0.7213	***	0.0148	***	0.7220	***	0.0183	***	0.7279	***

Dependent variable:	2003M10-2017M6				2005M3-2017M6				2003M10-2017M6				2006M2-2017M6			
	JPY				NOK				NZD				SEK			
	κ	θ			κ	θ			κ	v	θ		κ	vi	θ	vi
RR25 (β)	-0.0008	***	-0.0123	***	-0.0025	***	-0.0121	***	-0.0009	**	-0.0083	**	-0.0017	**	-0.0113	***
Constant_RR25	0.0152	***	0.6858	***	0.0201	***	0.7070	***	0.0173	***	0.7419	***	0.0175	***	0.7026	***
RR10 (β)	-0.0004	**	-0.0064	**	-0.0013	***	-0.0049	**	-0.0004	**	-0.0036	**	-0.0010	**	-0.0054	**
Constant_RR10	0.0152	***	0.6862	***	0.0196	***	0.7016	***	0.0172	***	0.7392	***	0.0177	***	0.7004	***

Notes:

1. ***, ** and * indicate significance at a level of 1%, 5% and 10% respectively.
2. i, ii, iii, iv, v and vi indicate dummies are added during 2011M8-M12, 2011M9-2013M12, 2014M4-M6, 2011M6-M12, 2009M1-M4 and 2011M8-2012M12 respectively.

Table 4: Estimation results of short-run dynamics.

Dependent variable:	2003M10-2017M6 AUD				2003M10-2017M6 CAD				2006M1-2015M1 CHF				2003M10-2017M6 EUR				2003M10-2015M5 GBP			
	κ		θ i		κ ii		θ		κ iii		θ iv		κ		θ		κ		θ	
RR25																				
Speed of adjustment	-0.090	**	-0.055	**	-0.104	***	-0.044	**	-0.266	***	-0.176	***	-0.132	***	-0.096	***	-0.179	***	-0.190	***
Lag length	1		1		1		1		1		1		1		1		1		1	
RR10																				
Speed of adjustment	-0.089	**	-0.055	**	-0.104	***	-0.044	**	-0.267	***	-0.162	***	-0.133	***	-0.094	***	-0.178	***	-0.192	***
Lag length	1		1		1		1		1		1		1		1		1		1	
	2003M10-2017M6 JPY				2005M3-2017M6 NOK				2003M10-2017M6 NZD				2006M2-2017M6 SEK							
	κ		θ		κ		θ		κ v		θ		κ vi		θ vi					
RR25																				
Speed of adjustment	-0.053	**	-0.046	**	-0.056	**	-0.055	**	-0.045	*	-0.052	**	-0.060	*	-0.076	**				
Lag length	1		1		1		1		1		1		7		1					
RR10																				
Speed of adjustment	-0.053	**	-0.045	**	-0.053	**	-0.054	**	-0.046	*	-0.053	**	-0.060	*	-0.079	**				
Lag length	1		1		1		1		1		1		7		1					

Notes:

1. ***, ** and * indicate significance at a level of 1%, 5% and 10% respectively.

2. i, ii, iii, iv, v and vi indicate dummies are added during 2011M8-M12, 2011M9-2013M12, 2014M4-M6, 2011M6-M12, 2009M1-M4 and 2011M8-2012M12 respectively.