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## Assessing the Effectiveness of Date-Based Forward Guidance at the Zero Lower Bound with a Non-Gaussian Affine Term-Structure Model\*

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### Abstract

Using a non-Gaussian affine term-structure model, this paper evaluates the effectiveness of the date-based forward guidance at the zero lower bound. The model extracts the expected dynamics of two state variables (the short-term interest rate and its mean) embedded in the entire Treasury yield curve. Using simulations and an event study, we find that the model's dynamics were significantly altered by the first announcement of date-based forward guidance in August 2011 and speculation about tapering in May 2013. The model offers a probabilistic approach in assessing the market's perception towards the Federal Reserve's projections of the federal funds rate.

Keywords: Forward Guidance, Zero Lower Bound, Non-Gaussian Term-Structure Model JEL Classification: E43, E43, E52, E58

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### 1. Introduction

In the aftermath of the global financial crisis, as further cuts in policy rates become infeasible due to the zero lower bound of nominal interest rates (zero bound), forward guidance about the future path of policy rates was adopted as an unconventional monetary policy tool among a number of central banks. Intuitively, by publicly announcing the commitment to keep policy rates unchanged over a period of time, it is hypothesized that a persistently low short-term rate would depress long-term rates, and hence provide additional monetary stimulus to the real economy. It also prospectively contributes to an eventual exit from quantitative easing (QE).<sup>1</sup>

Through the lens of a three-factor continuous time non-Gaussian affine term-structure model, which allows us to extract the market expectations embedded in the yield curve, the purpose of this paper is twofold. First, we evaluate the effectiveness of the date-based forward guidance used by the Federal Open Market Committee (FOMC). Second, we provide a probabilistic assessment of the market's perception towards the FOMC's projections of the federal funds rate.

In typical affine term-structure models, the instantaneous short-term interest rate (short rate) is modelled as a linear combination of one or more state variables and the short rate is mean reverting around a constant level. Once the state variables are known, the entire yield curve can be determined according to a bond pricing formula.<sup>2</sup> In this paper, we follow Balduzzi et al. (1998) to model the short rate and its long-term mean level as a coupled stochastic process. In particular, we assume that the short rate evolves around a stochastic mean level, which can be identified by long-term yields in the data. As a result, the short rate in our model affects long-term interest rates; because of the interlinkage assumed in the stochastic processes, changes in long-term yields can also have a direct effect on the dynamics of the short rate. Empirically, the estimated short rate and long-term mean level track closely to the federal funds rate and the slope of the yield curve respectively. As a steeper yield curve can reflect better economic prospects, it is not unreasonable to assume that long-term yields could affect the short rate directly since a larger output gap might induce the Federal Reserve (Fed) to raise its policy rate. Indeed, the proposed model is well suited to assess how the Fed can affect market expectations by disentangling its effect on the short-end and the long-end of the yield curve separately. For instance, by observing the dynamics of the estimated state variables, the modelimplied future short rate increased materially in May 2013 amid speculation that the Fed would start an earlier than expected tapered end to QE in 2014, despite the fact that the spot short-term interest

<sup>&</sup>lt;sup>1</sup> Forward guidance and QE (that is, Increasing the size and composition of the central bank's balance sheet) are two popular unconventional monetary policy tools under the zero bound. Eggertsson and Woodford (2003) theoretically show that an explicit commitment to future paths of policy rates is an optimal monetary policy. It is noteworthy that forward guidance and QE stimulates the economy differently. According to Bernanke (2013), forward guidance influences investors' expectations about the future short-term interest rates, while QE aims at lowering the term premium which is the additional return demanded by investors to have an equivalent return between rolling over short-term securities and holding a long-term security of the same maturity. This paper focuses on forward guidance only. For detailed analyses on the effectiveness of QE, see Krishnamurthy and Vissing-Jorgensen (2011) for the US experience and Gambacorta et al. (2014) for a cross-country analysis.

<sup>&</sup>lt;sup>2</sup> See Piazzesi (2010) for a survey of affine term-structure models.

rates were virtually unchanged in the same period.<sup>3</sup>

One of the key challenges in term-structure modelling is to prevent model-implied interest rates from being negative. To fulfil the zero bound constraint, we follow the seminal Cox-Ingersoll-Ross (CIR) model (1985) and specify the short rate as a non-Gaussian square-root process such that it is guaranteed to be non-negative. This represents a significant departure from conventional studies that assume the state variables to follow a Gaussian process.<sup>4</sup> Despite its ability to generate non-negative interest rates, the presence of a persistently low interest rate environment since the global financial crisis presents a challenge to the CIR model. It is well known that the short rate in the CIR model has a tendency to move back towards a higher level quickly, which is at odds with the sticky behaviour of short-term yields near the zero bound in reality. To overcome this problem, we introduce an exogenous factor in the model that helps supress the yield curve and generate the sticky feature of the short rate. The exogenous factor is introduced to capture the surge in investors' demand for the US Treasury bonds, which is not explicitly modelled in the term-structure model. The surge in the demand for US Treasury bonds has been documented extensively in the previous studies. Longstaff (2004) find a flight-to-liquidity premium in the US Treasury bond market suggesting that market participants prefer to hold US Treasury bonds during times of market distress. Krishnamurthy and Vissing-Jorgensen (2012) argue that the liquidity and safety attributes offered by the US Treasury bonds can be interpreted as a convenience yield from holding them. The convenience yield reflects the market's expectations concerning future demand or a surge in the price of the bonds. The greater the possibility that a surge in demand or price will occur in the future, the higher is the convenience yield. We assume that the discount factor in our model is the sum of the short rate and the exogenous factor. Although it is assumed that there is an exogenous component in the discount factor, similar modelling techniques for the discount factor have been used in the previous studies. Duffie and Singleton (1997) introduce an exogenous factor to capture the convenience yield in the pricing of interest rate swaps. Duffie and Singleton (1999) use it to capture the default component in the termstructure of corporate bonds.

In estimating continuous time term-structure models, one common approach is to apply the Euler method to discretize the continuous time process. However, we find that this approach is not suitable when interest rates are near zero. As the probability density function (pdf) in the continuous time CIR model is only defined when interest rates are non-negative, there is no guarantee that the pdf generated by the Euler-discretized process can maintain this desirable property, especially when discretization errors are not negligible. To overcome this, we follow the closed-form maximum likelihood approach developed by Ait-Sahalia and Kimmel (2010) for our estimation to minimise discretization errors and ensure that the desirable property under the continuous time framework is maintained. Specifically, the closed-form maximum likelihood approach naturally precludes the

<sup>&</sup>lt;sup>3</sup> In the standard one-factor term-structure model, long-term interest rates are determined by the short rate only. Hence, the current long-term interest rates can only increase when the spot short rate increases.

<sup>&</sup>lt;sup>4</sup> The Gaussian assumption of the state variables is usually made for computation purposes since the state-space representation of the term-structure models resemble closely a vector autoregressive model and can be estimated easily with the Kalman filter.

occurrence of negative interest rates in the likelihood function. To our knowledge, this paper is the first attempt to use a closed-form maximum likelihood approach to estimating a non-Gaussian affine termstructure model when interest rates are near the zero bound.<sup>5</sup> Using the estimated term-structure model, we simulate the future path of the short rate and its long-term mean to examine their joint behaviour around the window of the Fed's announcements of forward guidance. The simulated interest rates are used as the dependent variables in the event study regressions to examine the effectiveness of forward guidance. Our empirical results show that only the first announcement of date-based forward guidance and the tapering speculation were surprises, as the two events significantly affected market expectations on the future path of both short- and long-term interest rates. The second and third announcement by the FOMC relating to forward guidance did not have a significant impact.

Meanwhile, the simulated interest rates can also be used to probabilistically verify whether the FOMC's projections on the future path of the federal funds rate are consistent with market expectations. Our results show that although the yield curve suggests, with a relatively high probability, that the FOMC will start raising the policy rate in 2015, there remains uncertainty about the pace of interest rate increases and the longer-run path of the policy rate.

Our paper is related to the recent studies in assessing how the Fed might influence investors' expectations through communication. In a seminal work, Bernanke et al. (2004) estimate the market response to central bank communications in both the US and Japan. They provide empirical support in favour of forward guidance and other unconventional policies under the zero bound. By examining intra-day data within a narrow window surrounding policy announcements, Gurkaynak et al. (2005) identify a path factor in the federal funds futures rate which is not only orthogonal to the current policy rate, but also explains a significant portion of longer-tenor Treasury yields. Campbell et al. (2012) distinguish Odyssean forward guidance, which is a public commitment to change policy, and Delphic forward guidance, which merely provides a macroeconomic outlook and likely policy actions. The authors argue that the Fed would not have any difficulty in implementing Odyssean forward guidance. Regarding the evaluation of the date-based forward guidance, Raskin (2013) uses risk neutral pdf derived from interest rate options to solve the identification problem arising from the seemingly nonresponsiveness of short-term interest rates under the zero bound. He finds that date-based forward guidance is successful in changing investors' perception of the Fed's reaction function, and, consistent with our findings, that only the first forward guidance announcement was a surprise to the market. Meanwhile, Moessner (2013) finds that the first and second date-based forward guidance announcements jointly suppressed short- to medium-term interest rates and the term spread in both the US Treasury bond and the Eurodollar futures markets.

Several recent papers employ term-structure modelling techniques to gauge the timing of when the Fed would start to raise its policy rate (the lift-off date). By introducing a zero bound state, Christensen

<sup>&</sup>lt;sup>5</sup> See Egorov et al. (2011) for a recent application of the closed-form maximum likelihood approach in estimating an international affine term-structure model, but their estimations do not cover the zero bound period.

(2013) builds a regime switching term-structure model to estimate the timing and probability of the liftoff date. Following Black (1995), Bauer and Rudebusch (2013) and Wu and Xia (2014) both build shadow rate term-structure models in which actual short-term interest rates are the maximum of zero and the shadow rate, can be negative, zero, or positive.<sup>6</sup> The lift-off date is obtained from the point at which the simulated shadow rate turns from negative to positive. Although it is often argued that the shadow rate term-structure models have superior performance in modelling interest rate behaviour near the zero bound, we demonstrate that a properly-implemented continuous time non-Gaussian affine term-structure model compliments the shadow rate models. More importantly, the estimated results of the model are easy to be interpreted and serve as an analytical tool to study how forward guidance influences investors' expectations in the US Treasury bond market.

The rest of the paper is organized as follows. Section 2 reviews the three announcements of datebased forward guidance. Section 3 presents the non-Gaussian affine term-structure model. Section 4 shows the estimation results. Section 5 illustrates how to use the estimated model to examine the effect of the Fed's communication on market expectations. Section 6 provides a probabilistic assessment of the FOMC's projections of future interest rates. The final section concludes.

## Evolution of Date-Based Forward Guidance

In this section, we review several key FOMC statements about the evolution of date-based forward guidance and its effect on the US Treasury bond market. As further interest rate cuts were not possible due to the zero bound condition, the Fed took the decision to directly inject liquidity to the financial market and made a statement about the duration of the zero interest rate policy (ZIRP). Specifically, in the December 2008 FOMC statement, it was mentioned that, "..., the Committee anticipates that weak economic conditions are likely to warrant exceptionally low levels of federal funds rate for some time". This represented the first time that the FOMC statement attached a time horizon, albeit vaguely, to the duration of its policy action. Subsequently, starting from the March 2009 FOMC statement, the duration of ZIRP was changed from "for some time" to "for an extended period". In the August 2011 FOMC statement, in an attempt to provide a more concrete duration for ZIRP, the FOMC set a terminal date, replacing "for an extended period" with "at least through mid-2013". Then, in the January 2012 FOMC statement, the terminal date was postponed from mid-2013 to late-2014. Finally, in the December 2012 FOMC statement, the FOMC started the threshold-based guidance in which the duration of ZIRP was made contingent on US economic performance.

As argued by Raskin (2013), although the terminal date of ZIRP was explicitly mentioned in three FOMC statements (August 2011, January 2012 and September 2012), the immediate market reaction to these statements differed significantly. Based on a primary dealer survey conducted by the Federal

<sup>&</sup>lt;sup>6</sup> Kim and Singleton (2012) find that the shadow rate term-structure model is successful in explaining the behaviour of the Japanese government bond market in a persistently low interest rate environment.

Reserve Bank of New York, the y-axis in Figure 1 plots the lift-off dates perceived by the median dealers and the three announcements respectively. Before the first announcement, market participants generally thought that the life-off date would be somewhere in 2012Q2-Q3. The announcement that the duration of ZIRP would be at least through to mid-2013 in the August 2011 FOMC meeting had a significant effect on market expectations. The lift-off date was extended by one year from the pre-meeting survey result of December 2012 to the post-meeting result of December 2013. Around the window of the second announcement in January 2012, dealers expected that the lift-off dates both before and after FOMC meeting to be similar and earlier than end-2014, suggesting that the second announcement was not a surprise. Finally, the third announcement also appeared to be anticipated in advance by market participants. For instance, the dealer survey received on 4 September 2012 had already indicated that the lift-off date would be in 2015 Q3, which was very close to the mid-2015 announcement in the subsequent FOMC meeting on 12-13 September 2012. In sum, the relatively stable expectations before and after the second and third announcements suggest that market participants may have already factored-in further strengthening of the forward guidance before the January 2012 and September 2012 FOMC meetings.

The significant effect of the first announcement of date-based guidance can also be observed in longterm US Treasury yields. Figure 2 plots selected US Treasury yields from 2011 to 2013. While shortterm yields remained stagnant near the zero bound throughout the reporting period, there were significant fluctuations in long-term yields.<sup>7</sup> Amid escalation of the European sovereign debt crisis and its spillover risk to other economies in the first half of 2011, long-term US Treasury yields declined steadily due to their safety and liquidity attributes. The first announcement in August 2011 added further downward pressure to declining long-term yields. For example, the 10-year US treasury yield was down from around 3% on July 25 2011 to 2.4% on August 5 2011 prior to the FOMC meeting, and declined a further 30 basis points to 2.1% after the announcement. In anticipation of possible changes in monetary policy, the US Treasury bond market typically moves ahead and prices-in likely action before the FOMC meeting. If there are policy surprises, the yield curve responds and quickly adjusts afterwards. The decline in long-term US Treasury yields after the August 2011 FOMC meeting suggests that explicit mention of a terminal date for ZIRP of mid-2013 was indeed a policy surprise. However, the effect of the second and third announcement is inconclusive since long-term yields remained steady around the respective FOMC meetings in January 2012 and September 2012.

Figure 2 also highlights that the joint dynamics of short-term and long-term interest rates can provide additional insights on expectations embedded in the term-structure. Despite the fact that short-term interest rates are seemingly insensitive near the zero bound, long-term yields fluctuate tangibly as the market evolves. This observation motivates us to develop a term-structure model that allows a two-way interaction between the short- and long-term interest rates.

<sup>7</sup> 

Swanson and Williams (forthcoming) study the responsiveness of yields on US Treasury bonds with a year or more to maturity to news and policy announcements.

### 3. A Three Factor Non-Gaussian Affine Term-Structure Model

We propose a three factor non-Gaussian affine term-structure model in which the long-term mean of the instantaneous short rate varies stochastically over time. Specifically, the instantaneous short rate  $(r_t)$  is described by the following square-root process:

$$dr_t = \kappa(\theta_t - r_t)dt + \sigma\sqrt{r_t}dW_t^P \tag{1}$$

In Eq. (1),  $\kappa$  is is the parameter measuring the speed of reversion to this mean ( $\theta_t$ ),  $\sigma$  is the volatility and  $dW_t^P$  is a standard Brownian motion under the physical measure P. The long-term mean ( $\theta_t$ ) of the short rate in turn follows another square-root process:

$$d\theta_t = \alpha(\beta - \theta_t)dt + \eta\sqrt{\theta_t}dV_t^P \tag{2}$$

where  $\alpha$ ,  $\beta$ ,  $\eta$  and  $dV_t^P$  are the mean reversion parameter, long-term mean, volatility, and the Brownian motion term respectively associated with Eq. (2). Eqs. (1) and (2) together are sometimes referred as the stochastic mean model. Finally, there is an exogenous factor ( $L_t$ ) which evolves as:

$$dL_t = -\xi L_t dt + \gamma dZ_t^P \tag{3}$$

where  $\xi, \gamma$  and  $dZ_t^P$  are the drift term, volatility and the Brownian motion respectively associated with Eq. (3). The functional form for Eq. (3) follows Piazzesi (2005), who assumes a similar exogenous process to capture other information (i.e., information not captured by the state variables in the model) that could affect the yield curve. The Brownian motions in Eqs. (1)-(3) are assumed to be independent to each other (i.e.,  $dW_t^P dV_t^P = 0, dW_t^P dZ_t^P = 0$  and  $dZ_t^P dV_t^P = 0$ )

The proposed term-structure model assumes that the interest rates dynamics in Eqs. (1) and (2) both follow a non-Gaussian square-root process. This assumption ensures the non-negativity of interest rates and is a significant departure from the Gaussian assumption commonly adopted in the previous studies. Those studies have highlighted that Gaussian term-structure models are not appropriate under a zero bound as it cannot preclude the occurrence of negative interest rates. To see this, suppose the short rate follows a Gaussian Ornstein-Uhlenbeck (OU) process and its long-term mean level is assumed to be constant for the sake of illustration:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t^P \tag{4}$$

Conditional on the state variable  $r_t$  and other parameters (i.e.,  $\theta, \kappa, \sigma$ ), Eq. (4) can be integrated to obtain the future short rate at time *T*:

$$r_T = r_t e^{-\kappa(T-t)} + \left[1 - e^{-\kappa(T-t)}\right]\theta + \sigma \int_t^T e^{-\kappa(T-t)} dW_t^P$$
(5)

Due to the Brownian motion term in the integral, the future short rate at time *T* may become negative. This possibility will be more likely when the state variable  $r_t$  is already near the zero bound. On the contrary, Cox et al. (1985) show that non-negativity of interest rates can be ensured under a square-root process.<sup>8</sup> Figure 3 shows the transition densities of  $r_T$  under a square-root (solid line) and OU process (dotted-line) with the same initial conditions and time-to-maturity. We adjust the parameters in Eqs. (1) and (4) such that both models generate similar interest rate volatility and forward curves to facilitate a comparison.<sup>9</sup> It is observed that the OU process yields a symmetric distribution, while the square-root process can generate a skewed and fat-tailed distribution, which is defined only when interest rates are non-negative.

Given Eqs. (1)-(3), it can be shown that the price of a zero-coupon bond with a maturity at time  $\tau = T - t$  is given by:

$$P_t(\tau, r, \theta, L) = E_t^Q \left[ \exp(-\int_t^T (r_t + L_t) dt) \right]$$
(6)

where the expectation is taken under the risk-neutral measure Q. It is noteworthy that we assume that the risk-neutral measure Q has been chosen by the market in such a way that the adjusted discount rate ( $r_t + L_t$ ) is the effective risk-free interest rate. As argued in the introduction, the exogenous factor is introduced in a reduced-form fashion to capture the surge in investor's demand for US Treasury bonds that is not captured in the term-structure model. For instance, a negative  $L_t$  indicates a favouring of US Treasury bonds by investors, which helps to push down the effective interest rate, and hence delivers a higher bond price. Conversely, a positive  $L_t$  implies that investors are seeking extra compensation for holding US Treasury bonds. It is noteworthy that although the short rate is constrained to be non-negative, the effective interest rate in the bond pricing formula in Eq. (6) could be negative.

To preserve analytical tractability, we set the market price of risk as  $(\lambda_r \sqrt{r}, \lambda_\theta \sqrt{\theta}, \lambda_L)$  for the state variables  $(r, \theta, L)$  respectively. With the assumed functional form for risk premium, it is possible to rewrite Eqs. (1)-(3) under the risk-neutral measure Q, and the conditional expectation in Eq. (6) can be calculated by solving a partial differential equation as in Duffie and Kan (1996). In Appendix 1, we show that the solution for Eq. (4) is:

<sup>&</sup>lt;sup>8</sup> It can be shown that in Eq. (1), if  $2\kappa\theta \ge \sigma^2$ , then  $r_t > 0$ , and  $r_t = 0$  if  $2\kappa\theta < \sigma^2$ . Similar condition can be derived to ensure  $\theta$  is always non-negative in Eq. (2). This condition is known as the Feller condition in the literature.

<sup>&</sup>lt;sup>9</sup> Specifically, we calibrate the parameters to ensure that the volatility of the short rate under the square-root process is approximately equal to that under the OU process. Moreover, the limiting forward and spot rates are roughly equal and the shape of the forward curves is similar in both models.

$$P_t(\tau, r, \theta, L) = \exp[A(\tau) - B(\tau)r_t - C(\tau)\theta_t - D(\tau)L_t]$$
(7)

where the functions  $A(\tau)$ ,  $B(\tau)$ ,  $C(\tau)$ ,  $D(\tau)$  can be solved by a system of ordinary differential equations listed in Appendix 1. It is noteworthy that  $A(\tau)$ ,  $B(\tau)$ ,  $C(\tau)$ ,  $D(\tau)$  are determined only by the model parameters ( $\kappa$ ,  $\sigma$ ,  $\alpha$ ,  $\beta$ ,  $\eta$ ,  $\xi$ ,  $\gamma$ ,  $\lambda_{\tau}$ ,  $\lambda_{\theta}$ ,  $\lambda_{L}$ ) and a particular time-to-maturity  $\tau$ .

Before we present the data and estimation results, we discuss why the stochastic mean model can resolve the undesirable tendency of reverting back to a higher mean level as predicted by a standard one-factor CIR model under a zero bound. Specifically, it is well known that the variance of a square-root process would become smaller when the short rate is close to zero, and that the evolution of the short rate would be largely dictated by the drift term. As a result, in the standard CIR model which assumes a constant mean level for the short rate, the constant drift term would tend to pull the short rate back to its higher long-term mean level when the short rate is near zero. In the stochastic mean model, the short rate can remain near the zero bound if the long-term mean level  $\theta$  is also low.

The long-term mean level can be inferred by the cross-section of yields and the exogenous factor. To see this, we let  $\tau_1$  and  $\tau_2$  as the maturities for two zero-coupon bonds, with prices  $P_t(\tau_1, r, \theta, L)$  and  $P_t(\tau_2, r, \theta, L)$  determined by Eq. (7). To simplify notation, we use subscripts to distinguish between  $\tau_1$  and  $\tau_2$ . A straightforward calculation to eliminate *r* in  $P_1$  and  $P_2$  yields:

$$\theta_t \left( \frac{B_2 C_1 - B_1 C_2}{B_1 B_2} \right) = \frac{B_2 A_1 - B_1 A_2}{B_1 B_2} + \left( \frac{ln P_2}{B_2} - \frac{ln P_1}{B_1} \right) + \left( \frac{B_1 D_2 - B_2 D_1}{B_1 B_2} \right) L_t \tag{8}$$

Eq. (8) decomposes  $\theta_t$  into three terms. The first term is a constant. The second term is the difference between the weighted bond prices, which can be interpreted as the slope of the yield curve. The third term is the current level of the exogenous factor times a constant. Holding the exogenous factor constant, changes in  $\theta_t$  can be explained by changes in the slope of the yield curve. It should be noted that Eq. (8) holds for any pairs of zero-coupon bonds of two different maturities. Hence, in an attempt to gauge the time-series of  $\theta_t$ , we use zero-coupon yields data beyond the conventional cutoff at the 10-year maturity.

Meanwhile, the time-series dimension of the yield curve can be used to infer *L*. If we fix a particular maturity of the zero-coupon bond and evaluate its price by Eq. (7). It can be shown that:

$$lnP_t - lnP_{t-1} = B(r_t - r_{t-1}) - C(\theta_t - \theta_{t-1}) - D(L_t - L_{t-1})$$
(9)

where the dependence of  $\tau$  in *P*, *B*, *C* and *D* has been suppressed for presentation purposes. For a bond with maturity  $\tau$ , Eq. (9) shows that the change in its price can be explained by changes in the short rate, the long-term mean level and the exogenous factor. Hence, the dynamics of  $L_t$  are determined residually from the time series dynamics of the yield curve, once the contributions of  $r_t$  and  $\theta_t$  have been accounted for.

### 4. Empirical Analysis of the Term-Structure Model

This section covers the data used in this study and presents the empirical estimates of the termstructure model. We collect daily data of zero-coupon Treasury yields of constant maturities of 3month, 6-month, and 1, 2, 3, 4, 5, 7, 10, 15, 20 and 30-year for the sample period from January 1990 to March 2014.<sup>10</sup> According to the data vendor, the daily yields are stripped from the most recent auctioned on-the-run US Treasury bills and bonds using standard bootstrapping. We compute the weekly average of the daily data for the estimation. It is noteworthy that a long span of data is required to accurately estimate the mean reversion parameters (i.e.,  $\kappa$  and  $\alpha$ ) in Eqs. (1) and (2) respectively.<sup>11</sup> Following Ait-Sahalia and Kimmel (2010), we use the closed-form maximum likelihood method to estimate the term-structure model. As the cross-sectional number of observed bond yields is greater than the number of state variables, we follow previous studies to introduce measurement errors between the observed and model-implied yields. Specifically, we choose the 3-month, 10-year and 30-year maturities as the benchmark maturities (i.e., assuming no measurement errors) and use these bond yields to invert the state variables. It is well known that affine term-structure models with latent factors are invariant upon arbitrary affine transformation which may pose an identification problem (Dai and Singleton, 2000; Cheridito et al., 2010).<sup>12</sup> To this end, we assume that the short rate is observable with measurement error and take the overnight federal fund rate as a proxy. This allows us to pin down the short rate process and hence all the model-implied factors bear the desired economic interpretation.<sup>13</sup> The detailed estimation procedure is discussed in Appendix 2.

Table 1 reports the parameter estimates for the model described in Eqs. (1)-(3) and (6). In comparison with previous studies, four observations are worth mentioning. First, the t-ratios of the volatility estimates for both the short rate and long-term mean processes are well above the conventional significance levels, indicating that a joint characterization of the short- and long-term yield curve dynamics is indeed relevant. Secondly, the short rate exhibits considerably faster mean reversion than the long-term mean process. The intuition behind this result is that interest rates should converge faster towards a time varying mean than a constant mean. Thirdly, the fact that the long-term mean process converges to its average at around 6% probably reflects the fact that inflation expectations remained well anchored after the 1990s.<sup>14</sup> Finally, the exogenous process is almost deterministic as the volatility parameter associated with  $L_t$  is estimated to be not statistically significant, however, we

<sup>&</sup>lt;sup>10</sup> All the data used in this study are obtained from Bloomberg.

<sup>&</sup>lt;sup>11</sup> In a Monte Carlo study, Phillips and Yu (2005) show that the biases in estimating the mean reversion parameters decline with the sample size.

<sup>&</sup>lt;sup>12</sup> Specifically, for a *N*-dimensional affine model with  $r_t = a + b^T X_t$  where  $X_t = (X_{1t}, ..., X_{Nt})$ , for every regular matrix  $\Gamma \in \mathbb{R}^{N \times N}$  and  $z \in \mathbb{R}^N$ , an affine transformation as  $Y_t = \Gamma X_t + z$  can generate the same yield curve as  $X_t$ .

<sup>&</sup>lt;sup>13</sup> The assumption in using the overnight federal fund rate as a proxy for the short rate is made for the sake of maintaining the desired economic meaning of the state variables, as opposed to econometric purposes. We get similar parameter estimates and implied states without imposing this assumption. The details are available upon request.

<sup>&</sup>lt;sup>14</sup> Since long-term yields should contain a premium for expected inflation,  $\beta$  should be substantially higher when inflation expectations are not well anchored. Baldruzzi et al. (1998) provide empirical evidence that the estimated long-term mean was significantly higher during the oil crisis in the 1970s.

can't reject the existence of  $L_t$  due to the significant estimates of its drift term  $\xi$ . Intuitively, the almost deterministic nature of  $L_t$  implies that investors' demand for US Treasury bonds are probably affected by non-random factors that are not explicitly modelled in the term-structure model.

Table 2 reports the pricing errors for the non-benchmark maturities for our model. As a comparison, we also estimate a Gaussian version of our term-structure model and report its pricing errors. We observe that the non-Gaussian term-structure model fits fairly well for yields with short to medium maturities, with the absolute pricing errors ranging from 8 to 29 basis points. The fit for the yields of 15- and 20-year maturities are slightly inferior as the model over predicts the yields by 39 and 50 basis points respectively. On the other hand, the Gaussian version of the model seems to fit the long end of the yield curve relatively better, but at the expense of a slightly inferior fit for the short end. For the second moment, table 2 shows that the two models are able to capture the volatility of the yield curve fairly well. It is noteworthy that although the Gaussian model can provide a good in-sample fit of the yield curve, there is a high probability of having a negative interest rate in the out-of-sample simulation, especially under the current low interest rate environment.

On the contrary, the non-Gaussian model not only precludes negative interest rates naturally, it also provides an adequate fit of the data which is not significantly inferior to the Gaussian counterpart. Indeed, the non-Gaussian model provides a slightly better in-sample fit at the short to medium end of the yield curve. The slightly higher pricing errors at the long end of the yield curve could be explained by the following two arguments. First, it can be difficult to match the longer end of the yield curve using three to four factors, as commonly assumed in the term-structure models Second, there could be measurement errors incurred in the construction of the zero-coupon yields from bootstrapping the raw Treasury yields. Despite the less satisfactory fit in the first moment, the inclusion of yields with maturities beyond 10-year is important to our estimation since they not only help to identify the long-term mean level as in Eq. (8), but they can also affect the short rate dynamics through Eq. (1).

Figure 4 graphs the path of state variables implied by our model. We first discuss the short rate and the exogenous factor. The short rate tracks very closely to the federal funds rate (the dashed line). Meanwhile, the exogenous factor was close to zero during most of the time, with two notable exceptions. The first exception was observed from 1990 to mid-1993 when it was positive, probably reflecting the fact that bond investors were seeking additional compensation for holding Treasury bonds amid a somewhat inflationary environment in the early 1990s. <sup>15</sup> The second notable occurrence of the non-zero exogenous factor began in late 2004 and has remained ever since. In the model, a negative  $L_t$  can be interpreted as stronger than usual demand for the US Treasury bonds, which could be partly due to the global savings gluts phenomenon since 2000<sup>16</sup>, and further amplified

<sup>&</sup>lt;sup>15</sup> The average inflation rate from 1990 to 1993 was around 4%, while the average inflation rate from 1994 onwards was 2.4%.

<sup>&</sup>lt;sup>16</sup> Foreign purchases of the US Treasury bonds, particularly those by other central banks, were substantial during this period. For instance, the ratio of foreign official holdings to the outstanding US Treasury bonds increased from 23.1% in 2004 to 35.2% in 2013. Bernanke et al. (2004), Warnock and Warnock (2009) and Beltran et al. (2013) find that these purchases contributed to lower bond yields.

by the safe heaven feature of US Treasury bonds and the quantitative easing policy after the global financial crisis in 2008.

Figure 5 compares the long-term mean level with a common measure of the slope of the yield curve, which is the spread between the 10-year and 3-month Treasury yields (10-year to 3-month spread). Given the fact that the two series move in tandem, it provides empirical support that the slope of the yield curve helps to determine the long-term mean level as argued in the previous section. Indeed, to further illustrate the importance of incorporating yields with maturities beyond 10-years in our model, we regress the first difference of the long-term mean level on the first difference of various proxies for the slope, as defined in the first column of Table 3. Although the explanatory power of different measures of the slope decline with the chosen maturity at the right endpoint of the yield curve, their effect on the long-term mean level are still significant at the 1% level. Hence, the omission of long-tenor yields likely misses out important information for determining the long-term mean level. Among different measures of the slope, Table 3 shows that the 10-year to 3-month spread is the best candidate in explaining the movement of the long-term mean level, with the adjusted R-squared amounting to 68%.<sup>17</sup>

Meanwhile, the estimated long-term mean level has been persistently trending downward – a phenomenon which also occurs in long-term US Treasury yields. Although the current monetary policy stance undoubtedly has significant influence on the short-end of the yield curve, it is well known that the long-end yield curve contains expectations of future inflation. As a result, the downward trend registered in long-term US interest rates and the estimated long-term mean level of our model could be partly attributed to the demand of the US Treasury bonds as mentioned before or a decline in inflation expectations, or a combination of both.<sup>18</sup>

## 5. Evaluating the Effectiveness of Date-Based Forward Guidance

The estimated term-structure model can be used to examine the effectiveness of the three announcements of date-based forward guidance by extracting expectations embedded in the bond market. Using the realized value of the state variables ( $r_t$ ,  $\theta_t$ ,  $L_t$ ) at time *t* as the initial condition, we simulate the diffusion processes in Eqs. (1)-(3) to obtain future short- and long-term interest rates at the terminal date *T*, where *T* is set according to the dates specified in the announcements (i.e., mid-2013, end-2014, mid-2015).<sup>19</sup> We can then obtain two sequences of interest rates by updating the

<sup>&</sup>lt;sup>17</sup> The superiority of the 10-year and 3-month spread in explaining the long-term mean level is probably due to the fact they are the chosen benchmark yields in the term-structure model.

<sup>&</sup>lt;sup>18</sup> In fact, worldwide long-term real interest rates have declined substantially since the 1980s. The International Monetary Fund's World Economic Outlook (April 2014) identifies three factors which contribute to most of the decline since the 1990s, including: (i) increases in the world total savings brought by higher savings rates in emerging economies;(ii) global demand for safe assets increased; and (iii) a notable decline in investment rates in in advanced economies. For details, see International Monetary Fund (2014).

<sup>&</sup>lt;sup>19</sup> Unless otherwise stated, we use the P measure parameters for the simulation. The simulated short-term interest rates are the effective interest rates (i.e., *r*+*L*).

initial conditions of the simulation. By examining the dynamics of the simulated future interest rates around the windows of the policy announcements, it can shed light on how investors' expectations on short- and long-term interest rates are affected by the Fed's communications.

Figure 6 depicts the median effective interest rate (upper panel) and its long-term mean level (lower panel) at the time of the first announcement of date-based guidance, with the 90<sup>th</sup> and 10<sup>th</sup> percentiles marked as the upper and lower dashed line respectively.<sup>20</sup> Prior to the August 2011 FOMC meeting, both series were already declining but there was a further dip after the announcement. This finding suggests that the first announcement not only lowered expected future short-term rates, but also reduced expectations about future long-term rates, i.e., the slope of the yield curve became flatter. Specifically, within a two-week window, the expected short-term rate at mid-2013 was down from 1.15% in the week of 29 July 2011 to 0.65% in the week of 12 August 2011. The effect on expected long-term rates was more pronounced, which were down from 3.08% to 2.21% in the same period.

Figures 6B and 6C illustrate the evolution of expected interest rates following the second and third announcements of date-based guidance respectively. In contrast to the first announcement in August 2011, there is no apparent change in either the expected short or long-term interest rates, which is consistent with the observation made about the primary dealer survey results in Section 2: that the second and third announcements appeared to be anticipated by market participants before the respective FOMC meetings. Meanwhile, Figure 6C shows that expectations of a rise in interest rates started to take hold in May 2013 when the Fed announced that the tapering could occur sooner than expected. Despite the fact that spot short-term interest rates were still constrained at the zero lower bound at that time, the signal of the Fed's tapering altered market expectations that future short-term rates could rise, as marked by the notable upward trend in expected interest rates after late-May 2013.

To control for the effects of macroeconomic news, we further test the effectiveness of the announcements of forward guidance by employing regressions typically used in previous studies. Specifically, we estimate weekly-frequency regressions of the form:

$$\Delta y_t = a + bDum + cX_t + \varepsilon_t \tag{10}$$

where *y* denotes the expected short or long-term interest rates shown in Figure 5, *Dum* is a dummy variable which takes a value of one following each announcement and  $X_t$  is a vector of surprise components of macroeconomic data releases.<sup>21</sup> We use White heteroskedasticity-consistent standard

<sup>&</sup>lt;sup>20</sup> It is noteworthy that the gap between the interest rates at the 90<sup>th</sup> percentile to its median is wider than the median to 10<sup>th</sup> percentile gap. This is consistent with the theoretical predication that a higher interest rate is more likely in a square-root process.

<sup>&</sup>lt;sup>21</sup> The surprise components are calculated based on the expected and actual values for the selected US macroeconomic data releases. For each data release, we first compute the difference between the actual release and the median forecast value from the Bloomberg survey among the group of professional forecasters, and the surprise is defined as the resulting difference normalized by the standard deviation among the forecasters. The macroeconomic data include surprises on capacity utilization, CPI excluding food and energy, initial claims, ISM manufacturing index, leading index, new home sales, non-farm payroll, retail sales, unemployment rate and personal consumption expenditure.

errors. A significant estimate of *b* would indicate the Fed's communications have altered the market expectations of future interest rates. Table 4 presents the estimation result for Eq. (10) for the three FOMC announcements and speculation about tapering. The empirical results confirm the graphical analysis that market expectations seem to be significantly affected only by the first announcement and the tapering speculation. Specifically, the first announcement leads to a reduction of around 20 and 60 basis points in expected short and long-term interest rates respectively, and these estimates are significant at the 1% level. However, the effect of the second and third announcement on both short and long-term interest rates are not significant at conventional significance levels. Finally, expectations of future interest rates changed during the period when there was speculation about tapering, with estimates of expected short and long-term interest rates edging up tangibly by 6 and 17 basis points respectively.

## Evaluating the FOMC's Projections on Federal Funds Rate

Starting with the January 2012 FOMC statement, the FOMC included its members' assessment on the target federal funds rate at the end of the specified coming calendar year. For instance, in the March 2014 FOMC meeting, the central tendency among the FOMC members was that the the expected federal funds rate at end-2015 and end-2016 would be 1% and 2% respectively. We use the term-structure model to provide a probabilistic assessment on whether market expectations were consistent with the Fed's projections.

Using the realised state variables, we simulate 10,000 paths for the effective interest rate (i.e.,  $r_t + L_t$ ) and calculate its probability of reaching 1% and 2% at end-2015 and end-2016 respectively. Figure 7 presents the time series plot of the two probabilities. Similar to the analysis in the previous section, the probabilities were significantly affected by the first announcement of date-based forward guidance in August 2011 and the tapering speculation in May 2013, while the effect of the second and third announcement was muted. Based on information as of March 2014, it is estimated that the probability of the short-term interest rate reaching 1% at end-2015 and 2% at end-2016 is around 60% and 40% respectively. Although the US Treasury market generally factors in a high likelihood that the FOMC will start raising its policy rate in 2015, the relatively low probability of reaching 2% by end 2016 reveals that longer run policy uncertainty, especially regarding the pace of interest rate increases, remains.

Athough market participants had placed a high probability on the FOMC starting a tapered end to QE around the September 2013 FOMC meeting, the FOMC refrained from doing so on the grounds that the US economic data were not improving quickly enough. In response to this surprise, long-term US Treasury yields declined after the FOMC meeting which results in a decline in the probability for both the end-2015 and end-2016 projections as shown in Figure 7. As a counterfactual exercise, we simulate the probability had the FOMC actually started tapering in September 2013. As a rough approximation, we assume the US Treasury bond market would repeat the experience in May 2013 in which there was an increase in long-term interest rates, but short-term rates remained unchanged. Specifically, we assume that 10-year and 30-year US Treasury yield (the two long-term benchmark

yield of our model) increase by 50 bps permanently, but there is no change in the 3-month US Treasury yield. Figure 8 compares the probability of short-term interest rate reaching 1% or above by end-2015 under the hypothetical scenario with the actual probability.<sup>22</sup> It shows that the probability would shift up tangibly by about 20 percentage points.

### 7. Conclusion

In this paper, we evaluate the effectiveness of the Fed's date-based forward guidance and its projections on future policy rates using a non-Gaussian affine term-structure model. Although previous studies have warned against the use of affine models because of their inferior performance in modelling a persistently low interest rate environment, we show that a carefully implemented non-Gaussian affine model is capable of generating plausible dynamics for both short and long-term interest rates for the US Treasury bond market from 1990 to the present. As the short rate and its long-term mean level are modelled using a coupled stochastic process in the model, their dynamics affect each other.

Consistent with the findings in previous studies, based on the simulations and event study using our estimated model results, we find that the first announcement of date-based forward guidance in the August 2011 FOMC meeting had significant effects on the dynamics of the short rate and its long-term mean. In addition, the tapering speculation in May 2013 also affected market expectations about future interest rate movements. The term-structure model offers a probabilistic approach in assessing investors' perception towards the FOMC's projections of the federal funds rate.

The novel contribution in this paper is that the proposed term-structure model allows us to summarize the information embedded in the entire US Treasury yield curve into two dynamical state variables through which we can assess how the Fed's communications affect investors' expectations about short and long-term interest rate movements. The Fed has recently shifted from the date-based guidance to a threshold-based guidance by linking monetary policy action to the performance of the US economy. We leave a more detailed analysis on forward guidance, which takes account of the dynamic interaction between interest rate expectations and the real economy, to future research.

<sup>&</sup>lt;sup>22</sup> For the sake of illustration, Figure 8 provides the counterfactual exercise for the end-2015 projection only.

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#### Table 1. Maximum Likelihood Estimates

	Estimates	t-ratios
Short rate process		
mean reversion ( $\kappa$ )	0.3522	15.50
volatility ( $\sigma$ )	0.0474	17.01
risk premium $(\lambda_r)$	-0.9628	-2.06
Long-term mean process		
mean reversion $(\alpha)$	0.1012	1.46
long-term mean $(\beta)$	0.0642	6.43
volatility $(\eta)$	0.0505	3.99
risk premium $(\lambda_{\theta})$	-0.4697	-0.38
Exogenous process		
drift ( $\xi$ )	-0.0438	-6.20
volatility ( $\gamma$ )	0.0042	0.73
risk premium $(\lambda_L)$	-0.0486	-0.11

Note: The sample is weekly from January 1990 to March 2014.

Table 2. Mean and Standard	Deviation of the	Absolute Pricing	Errors for	Non-Gaussian	and
Gaussian Models (in	Basis Points)				

	6-mon	1-yr	2-yr	3-yr	4-yr	5-yr	6-yr	7-yr	8-yr	9-yr	15-yr	20-yr
<u>Mean</u>												
NG	0.11	0.18	0.29	0.28	0.25	0.23	0.17	0.14	0.08	0.04	0.39	0.50
G	0.13	0.24	0.31	0.31	0.26	0.20	0.15	0.10	0.07	0.04	0.24	0.29
<u>SD</u>												
NG	0.10	0.15	0.21	0.20	0.17	0.16	0.11	0.09	0.05	0.02	0.13	0.15
G	0.09	0.15	0.20	0.19	0.16	0.12	0.10	0.07	0.05	0.02	0.10	0.12

Note: Absolute pricing errors are defined as absolute difference between the actual yield and the model implied yield. NG and G denote non-Gaussian and Gaussian models respectively, SD is the standard deviation.

Different measures of slope	coefficients	t-ratios	Adj. R <sup>2</sup>
The spread between 10-year yield and			
federal funds rate	0.50	21.86	0.27
3-month yield	1.18	46.10	0.63
1-year yield	1.36	32.30	0.45
The spread between 15-year yield and			
federal funds rate	0.47	19.49	0.23
3-month yield	1.12	37.99	0.53
1-year yield	1.19	25.19	0.33
The spread between 20-year yield and			
federal funds rate	0.42	16.96	0.19
3-month yield	0.99	30.23	0.42
1-year yield	0.91	18.23	0.21
The spread between 30-year yield and			
federal funds rate	0.28	10.86	0.09
3-month yield	0.57	15.73	0.16
1-year yield	0.26	5.51	0.02

# Table 3. The Relationship between Long-Term Mean Level and the Slope of the Yield Curve from January 1990 to March 2014

Note: This table reports the regressions of the first difference of the long-term mean level to the first difference of the slope of the yield curve. The slope is calculated as the spread between a longer-tenor yield and a shorter-tenor yield. All reported coefficients are significant at 1% level.

	1 <sup>st</sup> guidance		2 <sup>nd</sup> guidance		3 <sup>rd</sup> guid	dance	Tapering	
	ST	LT	ST	LT	ST	LT	ST	LT
Dum	-0.24***	-0.61***	0.050	-0.02	0.07	0.07	0.05***	0.15***
t-ratios	-16.70	-19.41	0.91	-0.22	1.37	0.88	3.29	4.86
Adj. R <sup>2</sup>	31.7%	34.5%	11.0%	2.8%	13.0%	6.0%	10.7%	1.4%
Period	12/31/2010-		9/16/2011-		6/15/2012-		12/14/2012-	
	12/23/2011		9/07/2011		6/14/2	2013	12/06/2013	

# Table 4. Reactions of Expected Short-Term (ST) and Long-Term (LT) Interest Rates to Forward Guidance

Note: \*\*\* represents statistical significance at 1% level. This table reports weekly-frequency regressions of the simulated short rate (ST) and its mean (LT) to a dummy variable and surprises in the US macroeconomic variables around the windows of three announcements of date-based forward guidance. The t-ratios are computed based on White heteroskedasticity-consistent standard errors. The estimation results of surprises in US macroeconomic variables are available upon request.









Note: Yields are weekly average of the daily data.

Figure 3. Transition Density for Short Rate under the Gaussian (Dotted Line) and Non-Gaussian (Solid Line) Interest Rate Dynamics



Note: The initial condition is fixed at  $r_0 = 0.01$  and the time-to-maturity is 1 for both pdfs. The parameters for the Gaussian (non-Gaussian) model are  $\kappa = 0.25(0.232)$ ,  $\theta = 0.06(0.06015)$ ,  $\sigma = 0.02(0.082)$ . The parameters are chosen to ensure that both models generate similar interest rate volatility and forward curves.



Figure 4. The Implied State Variables of the Term-Structure Model

Note: The state variables are obtained from inverting Eq. (7) for the three chosen benchmark maturities. For details, see Appendix 2. Data for Fed funds rate are weekly average of daily data.

Figure 5. Long-Term Mean Level and the Spread between 10-Year Yield and 3-Month Yield (Slope)



Note: Data for the slope are weekly average of daily data.



#### Figure 6. Simulated Future Interest Rates around the Three Announcements

Note: In each panel, the grey (dark) line represent the simulated short rate (long-term mean) around the window of each announcement of the date-based guidance, with the 90th to 10th confidence intervals marked in dash-lines. The terminal date in each panel is set to the dates specified in the announcements.





Figure 8. End-2015 Probabilities under the Actual and Counterfactual Scenarios



## Appendix 1. Solution of the Term-Structure Model

Denote the state variable process as  $X_t = (r_t, \theta_t, L_t)$ . Given the initial state variables at time t as  $X_t = X$  such that  $r_t = r$ ,  $\theta_t = \theta$ , and  $L_t = L$ , the price of the zero-coupon bond is given by  $P(X, t; T) = E_t^Q \left[ exp \left( -\int_t^T (r_s + L_s) ds \right) \right]$  and is governed by the partial differential equation (PDE):

$$(r+L)P = \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^{2}r\frac{\partial^{2}P}{\partial r^{2}} + (\kappa\theta - (\kappa + \lambda_{r}\sigma))\frac{\partial P}{\partial r} + \frac{1}{2}\eta^{2}\theta\frac{\partial^{2}P}{\partial \theta^{2}} + (\alpha\beta - (\alpha + \lambda_{\theta}\eta))\frac{\partial P}{\partial \theta} + \frac{1}{2}\gamma^{2}\frac{\partial^{2}P}{\partial L^{2}} - (\xi L + \lambda_{L}\gamma)\frac{\partial P}{\partial L}$$
(A1)

It follows from Duffie and Kan (1996) that Eq. (A1) has the solution of the form

$$P(X, t; T) = \exp[A(\tau) - B(\tau)r_t - C(\tau)\theta_t - D(\tau)L_t]$$
(A2)

where  $\tau = T - t$  is the time-to-maturity, and the coefficient functions  $A(\tau), B(\tau), C(\tau), D(\tau)$  solve a system of ordinary differential equation as

$$\frac{dA(\tau)}{d\tau} = -\alpha\beta C(\tau) + \frac{1}{2}\gamma^2 D^2(\tau) + \lambda_L \gamma D(\tau)$$

$$\frac{dB(\tau)}{d\tau} = 1 - \frac{1}{2}\sigma^2 B^2(\tau) - (\kappa + \lambda_r \sigma)B(\tau)$$

$$\frac{dC(\tau)}{d\tau} = \kappa B(\tau) - \frac{1}{2}\eta^2 C^2(\tau) - (\alpha + \lambda_\theta \eta)C(\tau)$$

$$\frac{dD(\tau)}{d\tau} = 1 - \xi D(\tau)$$
(A3)

for  $\tau \ge 0$  and A(0) = B(0) = C(0) = D(0) = 0.

## Appendix 2. Maximum Likelihood Estimation of the Model

We first illustrate how the state variables at each data point can be inverted from the chosen benchmark bond yields (i.e., assuming no measurement errors). Then, we present the respective likelihood functions implied by benchmark and non-benchmark bond yields.

#### Relationship between state variables and benchmark yields

Let i<sup>th</sup> be the current data point and  $y_i(\tau_m)$  be the model-implied bond yield of maturity  $\tau_m$  such that  $P(\tau_m) = \exp[-y_i(\tau_m)]$ , then Eq. (A2) implies

$$y_i(\tau_m) = f_0(\tau_m) + f_1(\tau_m)^T X_i, \quad i = 1, 2, ..., N$$
 (A3)

where  $f_0(\tau_m) = -A(\tau_m)$ ,  $f_1(\tau_m)^T = [B(\tau_m), C(\tau_m), D(\tau_m)]$  and  $X_i = (r_i, \theta_i, L_i)^T$  is the state vector at the current observation i. Denote the chosen benchmark maturities as  $(\tau_1, \tau_2, \tau_3)$ , we can stack Eq. (A3) as a matrix equation and recover the current value of the state vector  $X_i$  as:

$$Y_i = F_0 + F_1^T X_i \implies X_i = (F_1^T)^{-1} (Y_i - F_0), \quad i = 1, 2, \dots, N$$
(A4)

where  $Y_i = \begin{bmatrix} y_i(\tau_1) \\ y_i(\tau_2) \\ y_i(\tau_3) \end{bmatrix}$  is the benchmark yields, and

$$F_{0} = \begin{bmatrix} f_{0}(\tau_{1}) \\ f_{0}(\tau_{2}) \\ f_{0}(\tau_{3}) \end{bmatrix} \text{ and } F_{1}^{T} = \begin{bmatrix} f_{1}(\tau_{1})^{T} \\ f_{1}(\tau_{2})^{T} \\ f_{1}(\tau_{3})^{T} \end{bmatrix} = \begin{bmatrix} B(\tau_{1}) & C(\tau_{1}) & D(\tau_{1}) \\ B(\tau_{2}) & C(\tau_{2}) & D(\tau_{2}) \\ B(\tau_{3}) & C(\tau_{3}) & D(\tau_{3}) \end{bmatrix} \text{ are coefficients of model parameters}$$

which can be solved from Eq. (A3) and are independent of the state vector.

#### Likelihood functions of the benchmark bond yields

Let  $\Theta = (\kappa, \sigma, \alpha, \beta, \eta, \xi, \delta, \lambda_r, \lambda_{\theta}, \lambda_L)$  denotes the vector of model parameters, the log-likelihood function  $l_X(\Delta t, X_i; X_{i-1})(\Theta)$  of the state variables is

$$l_{X}(\Delta t, X_{i}; X_{i-1})(\Theta) = l_{R}(\Delta t, x_{i}; x_{i-1})(\Theta) + l_{L}(\Delta t, L_{i}; L_{i-1})(\Theta)$$

where  $x_i = (r_i, \theta_i)^T$ ,  $l_R(\cdot)$  is the joint log-likelihood of the short rate and the long-term mean process, and  $l_L(\cdot)$  is log-likelihood of the exogenous process. The latter likelihood has a closed form expression

$$l_{L}(\Delta t, L_{i}; L_{i-1})(\Theta) = -\frac{1}{2}ln(2\pi\tilde{\sigma}_{L}^{2}) - \frac{1}{2\tilde{\sigma}_{L}^{2}}(L_{i} - L_{i-1}e^{-\xi\Delta t})^{2}$$

with  $\tilde{\sigma}_L^2 = \frac{\gamma^2}{2\xi} (1 - e^{-2\xi\Delta t})$ . However, there is no analytical form for  $l_R(\cdot)$  and we follow Ait-Sahalia and Kimmil (2010) to approximate it with a Hermite expansion.<sup>23</sup> Then, the sum of likelihood of benchmark bond yields of all observations is

$$L_B(\Theta) = \sum_{i=2}^{N} det |(F_1^T)^{-1}| \times l_X(\Delta t, X_i; X_{i-1})(\Theta)$$

where  $det|(F_1^T)^{-1}|$  is the Jacobian corresponding to the change of variable as described by Eq. (A4).

#### Likelihood functions of the non-benchmark bond yields

Denote  $\hat{y}_i(\tau_m)$  and  $y_i(\tau_m; \Theta)$  the observed and model-implied yield for the non-benchmark bond with a maturity  $\tau_m$ , then by the assumption that the measurement error  $\varepsilon_i(\tau_m; \Theta)$  is normally distributed, we have :

$$\varepsilon_i(\tau_m; \Theta) = \hat{y}_i(\tau_m) - y_i(\tau_m; \Theta) \sim N(\mu_m, \sigma_m^2), \quad i = 1, 2, \dots, N$$

in which  $\mu_m$  and  $\sigma_m$  are proxied by the sample mean and standard deviation respectively. Hence, the log-likelihood of all the non-benchmark bond yields is:

$$L_M(\Theta) = \sum_m l_i^B(\Theta) = \sum_m \left\{ -\frac{N}{2} ln(2\pi\sigma_m) - \frac{1}{2} \sum_{i=1}^N \left( \frac{\varepsilon_i(\tau_m;\Theta) - \mu_m}{\sigma_m} \right)^2 \right\}$$

for i = 1, 2, ..., N and the summation is taken for all non-benchmark maturities m.

#### Likelihood function of the short rate

Similar to the case of non-benchmark bond yields, we can incorporate the log-likelihood function of the observed short rate  $\hat{r}_i$  as:

$$L_R(\Theta) = -\frac{N}{2}ln(2\pi\sigma_R) - \frac{1}{2}\sum_{i=1}^{N} \left(\frac{r_i - \hat{r}_i}{\sigma_R}\right)^2$$

where  $r_i$  is the model-implied short rate,  $\sigma_R$  is the sample standard deviation, and we ignore the prefactor for simplicity.

The joint log-likelihood is simply the sum of the likelihood of the benchmark and non-benchmark bond yield, i.e.,  $L(\Theta) = L_B(\Theta) + L_M(\Theta) + L_R(\Theta)$ . We then maximize  $L(\Theta)$  using the simplex method.

<sup>&</sup>lt;sup>23</sup> For details, see Ait-Sahalia and Kimmil (2010).