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Measuring Contagion-Induced Funding Liquidity Risk in Sovereign Debt Markets

Cho-Hoi Hui*

Hong Kong Monetary Authority

Chi-Fai Lo**

The Chinese University of Hong Kong

Xiao-Fen Zheng**

The Chinese University of Hong Kong

Tom Fong[#]

Hong Kong Monetary Authority

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Abstract

The euro-area sovereign debt crisis demonstrated how liquidity shocks can build up in a sovereign debt market due to contagion. This paper proposes a model based on the probability density associated with the dynamics of sovereign bond spreads to measure contagion-induced systemic funding liquidity risk in the market. The two risk measures with closed-form formulas derived from the model, are (1) the rate of change of the probability of triggering a liquidity shock determined by the joint sovereign bond spread dynamics of the systemically important countries (i.e., Italy and Spain) and small country (i.e., Portugal); and (2) the distress correlation between bond spreads, which can provide forward-looking signals of such risk. A signal of the rate of change of the joint probability appeared in April 2011 before the liquidity shock occurred in November 2011. There exist endogenous critical levels of sovereign spreads, above which the signal materializes. The empirical results show that when funding cost, risk aversion and equity prices pass through certain levels, the rate of change of the joint probability will rise sharply.

Keywords: European Sovereign Debt Crisis, Liquidity Risk, Contagion

JEL Classification: F30, G13

* Correspondence. E-mail: chhui@hkma.gov.hk. Phone: (852) 2878 1485. Fax: (852) 2878 2485. Research Department, Hong Kong Monetary Authority, 55/F, Two International Finance Centre, 8, Finance Street, Central, Hong Kong, China.

** Institute of Theoretical Physics and Department of Physics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong, China. E-mail: cflo@phy.cuhk.edu.hk

Research Department, Hong Kong Monetary Authority, 55/F, Two International Finance Centre, 8, Finance Street, Central, Hong Kong, China.

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1. Introduction

The euro-area sovereign debt crisis emerged after the new Greek government took office in October 2009. The crisis highlighted how contagion risk played out in the sovereign debt market during 2010 and 2011, when the crisis intensified due to deepened concerns about the fiscal sustainability of the debt-ridden countries including Greece, Ireland, Italy, Portugal and Spain.¹ Following a deteriorating sovereign debt situation, the EU and IMF agreed bailout packages to Greece and Ireland in May and November 2010 respectively. Portugal was the third country to request assistance from the EU and IMF in May 2011. Preventing contagion from spreading further to the large and systemically important economies – Italy and Spain – became a pressing task for the euro-area authorities.² Given that European banks were major holders of sovereign government bonds, the decline in the value of the bonds of these two countries weakened banks' capital positions, forcing many out of the funding market.

The instability in the euro-area sovereign debt market intensified during the summer of 2011, when the credit ratings of two of the larger countries, Italy and Spain, were downgraded by the credit-rating agencies. In November 2011, the sovereign bond spreads of Italy and Spain surpassed a threshold of 500 basis points (bp) (see Figure 1). The surge in their bond spreads had a large impact on funding liquidity. Given that Italy is the largest sovereign bond market in the euro area and the third largest in the world after the US and Japan, substantial stress in this market had a severe impact on the credit risk and funding constraints of the market-makers, and banks' capital positions. The Italian sovereign bond market thus faced severe liquidity conditions and other problems, causing bond yields to spike to unsustainable levels. This systemic liquidity shock prompted support from the ECB in the form of sovereign bond purchases to restore sovereign debt market liquidity. The ECB on 8 December 2011 decided on additional support for bank lending and liquidity through two longer-term refinancing operations (LTROs) in December 2011 and February 2012. These policies contained systemic funding liquidity risk by reducing the borrowing costs of Spain and Italy, and easing their sovereign bond market liquidity substantially through strong bond purchases by local banks.

Developments during the European sovereign debt crisis demonstrate how funding liquidity risk and potential total dysfunction of the financial market can emerge when the sovereign credit risk of a small but vulnerable country such as Portugal causes contagion to large and systemically important countries such as Italy and Spain. The spillovers between sovereign credit risks indicate that it is important to have a better understanding of the dynamics of the corresponding bond credit spreads in order to gauge contagion and systemic funding liquidity risk in the sovereign debt market. This could help policy makers in their efforts to improve funding liquidity, and to assess the effectiveness of their

¹ While Greece accounts for only 1.4% of foreign claims in European banks, economies such as Portugal, Ireland, Italy, and Spain, which have had similar fiscal problems as a whole, accounted for 15.4% in September 2009.

² In October 2011 the euro-area ministers approved another bailout loan for Greece, potentially rescuing the country from default. European leaders also agreed on new measures to boost the main bailout fund to 1 trillion euros, and to raise capital requirements of banks to minimize the negative effect under potential stressed scenarios (so-called a "three-pronged" agreement). However, such measures did not improve the situation.

interventions in the market.

This paper proposes a model based on the probability density associated with the dynamics of sovereign bond spreads to measure contagion-induced systemic funding liquidity risk in the euro-area sovereign debt market, and to provide signals of when such risk potentially intensifies. There are two main features of the model relevant to the observations of funding liquidity risk during the crisis. First, there is a threshold of sovereign bond spreads of systemically important countries, above which a systemic liquidity shock will occur. A threshold of 500 bp is often considered as the level which divides the credit spread of investment grade bonds (Standard & Poor's BBB- or better) and speculative grade bonds.³ Banks and market-makers hold those bonds (i.e., the Italian and Spanish sovereign bonds) as liquid assets and collateral for funding liquidity purposes. The 500-bp threshold signals severely low (mark-to-market) bond prices that weaken banks' and market-makers' funding liquidity conditions. This feature is consistent with the theory presented by Brunnermeier and Pedersen (2009) in which traders become reluctant to take positions when funding liquidity is tight due to increases in margin requirements and regulatory capital requirements particularly when their positions are (mark-to-market) capital intensive. Such a constraint causes banks and market-makers to lower market liquidity, and liquidity may suddenly dry up, especially if capital is already low (a nonlinear effect). Consistent with the theory, Pelizzon et al. (2014) find that when Italian sovereign credit default swap (CDS) spreads were above a 500-bp threshold, there was the structural shift in the relationship between changes in credit risk and changes in market liquidity, accompanied by sharply higher illiquidity in the euro-area bond markets due to substantial changes in the clientele of investors who held Italian bonds, margin requirements, accounting treatment, and regulatory capital requirements.

Second, in the proposed model, the contagion-induced systemic funding liquidity risk associated with systemically important countries (such as Italy and Spain) is conditional on the probability of the sovereign bond spread of a small country (e.g., Portugal) breaching a certain level in a given time horizon, suggesting that the small country is vulnerable. The contagion effect is thus contingent on the vulnerability of the small country. Such a characteristic is consistent with the finding in Beirne and Fratzscher (2013) that there was herding contagion in advanced and emerging economies during the European sovereign debt crisis with sharp and simultaneous increases in sovereign yields across countries, but that this contagion was concentrated in time. As the European sovereign debt crisis unfolded from those small countries such as Greece, Portugal and Ireland which received support from the EU/IMF, their sovereign bond spreads signaled risk in the euro-area financial system and the authorities took various measures in order to contain the risk in the system and prevent contagion from spreading further. Conditionality on the vulnerability of a small country in the model is supported by the empirical finding by Kalbaska and Gatkowski (2012). They use impulse response function analysis to study sovereign CDS spreads and find that Portugal was the most vulnerable country in the sample of PIIGS, France, Germany and the UK in the period of 2008–2010. Furthermore, Gorea and Radev (2014) find considerable potential for cascade effects from small to large euro-area

³ This threshold of 500 bp is also used by clearing houses, such as the Depository Trust and Clearing Corporation, to impose more stringent margining.

sovereigns using the corresponding default probabilities estimated from CDS spreads.⁴

The second feature is consistent with the channels of contagion in financial markets discussed in the literature. One channel is the liquidity channel due to the fact that market liquidity has commonality across securities. Brunnermeier and Pedersen (2009) argue that market liquidity and fragility co-moves across assets. Economic agents who experience losses in one market (such as the Portuguese sovereign debt markets) may suffer from funding shortages, which would then result in declines in the liquidity of the other financial assets (such as the Italian and Spanish sovereign debts) in the markets. Through another channel of contagion identified by Vayanos (2004) and Acharya and Pedersen (2005), financial shocks in one market may affect the willingness of market participants to bear risk in any market due to a repricing of equilibrium risk premiums.

Based on the probability density distribution of the sovereign bond spreads, two risk measures are derived from our proposed model to gauge the contagion-induced systemic funding liquidity risk. These are: (1) the rate of change of the joint probabilities above the threshold of the sovereign bond spreads of a small country and systemically important countries; and (2) correlation of the probabilities of the thresholds being breached (i.e., distress correlations). The first measure provides a signal of a potential illiquidity shock in the sovereign debt market when market participants may face a sharp rise in illiquidity in a short period of time due to a substantial deterioration in credit of systemically important countries induced by a small country under stress. We show the existence of critical levels of bond spreads above which the signal of a liquidity shock appears even though the thresholds have not been breached. The second measure identifies the timing of distress spillovers between the small country and systemically important countries when funding liquidity risk in the sovereign debt market may intensify. As the likelihood of triggering the thresholds, based on the probability density, increases with the volatility of the bond spread dynamics, this characteristic is consistent with the empirical findings of Amihud and Mendelson (1989), and Chordia et al. (2005) that liquidity declines as volatility increases.

While joint default probabilities have been used to investigate multiple euro-area sovereign defaults in previous studies (for example, Zhang and Lucas (2012), Zheng (2013), Gorea and Radev (2014), and Pianeti and Giacometti (2015)), the proposed model and measures in this paper are different from them in two aspects. First, thresholds are set for countries' sovereign bond spreads to assess the likelihood of an occurrence of liquidity shocks based on the risk measures calculated from the probability density distributions of the bond spreads. However, the (joint) default probabilities of the

⁴ There are other empirical studies on contagion during the European sovereign debt crisis. For example, Caporin et al. (2012) find that contagion in Europe remained subdued during the period of November 2008 – September 2011. Claey's and Vařićek (2012) demonstrate that using EU sovereign bond spreads relative to the German Bund, spillovers among sovereign yield spreads increased considerably from 2007 but their importance differed across countries. Ludwig (2014) examines the evolution of bond yield spreads in the euro area in 2008–2012 and finds pure contagion of sovereign risk in which the transmission of negative effects after a shock to a country, such as Greece, Ireland and Portugal, are not reflected in the risk pricing of fundamental determinants of sovereign risk of the recipient country like Italy and Spain. Gomez-Puig and Sosvilla-Rivero (2014) demonstrate the existence of possible Granger-causal relationships between the yields of bonds issued by PIIGS during the period 1999-2010. Alter and Beyer (2014) find increasing spillover measures and therefore a high level of potential contagion from Spain and Italy before key financial market events or policy interventions during the sovereign debt crisis.

countries implied from the bond or CDS spreads are not the measures used in our study. Second, the rate of change of the joint probabilities derived from the bond spread dynamics is a new risk measure. We will demonstrate that this measure provides a clearer signal for assessing the likelihood of an occurrence of shocks compared with the joint probabilities.

This paper is structured as follows. Section 2 presents a model for measuring contagion-induced systemic funding liquidity risk using the probability density of the sovereign bond spread dynamics, in which there are single- and multi-systematically important countries. Section 3 shows the numerical results of the model with the model parameters during the European sovereign debt crisis. Section 4 identifies the major determinants of the rate of change of the joint probability. Section 5 concludes.

2. Probability Density Analysis

In theory the CDS spread should be close to the credit spread on a bond issued by the same reference entity with the same maturity. This is because a portfolio consisting of a CDS (buying credit protection) and a par yield bond issued by the reference entity is very similar to a par yield risk-free bond. Hull et al. (2004) and Blanco et al. (2005) find that this theoretical relationship holds fairly well. We therefore assume that sovereign bond spreads follow a lognormal process, which is used for CDS spread dynamics in Hull and White (2003) and Pan and Singleton (2008). Hull and White assume a lognormal distribution of corporate CDS spreads for pricing CDS options. Pan and Singleton use a conditional distribution of sovereign CDS spreads derived from a conditional lognormal distribution to estimate the relevant term structures. Based on this assumption, each individual sovereign bond spread is governed by the following stochastic differential equation:

$$\frac{dR_i}{R_i} = \mu dt + \sigma_i dz_i, \quad \text{for } i = 1, 2, \dots, N \quad (1)$$

where μ is the drift, σ_i is the volatility, and dz_i represents the Wiener process. The Wiener processes dz_i and dz_j are correlated by:

$$dz_i dz_j = \rho_{ij} dt. \quad (2)$$

By Ito's lemma, the multi-dimensional joint probability density function (PDF) $PDF(R_{i0}; R_i; t - t_0)$ satisfies the backward Kolmogorov equation:

$$\left(\frac{\partial}{\partial t_0} + \hat{L} \right) PDF(R_{i0}; R_i; t - t_0) = 0, \quad (3)$$

where

$$\hat{L} = \sum_{i,j=1}^N \frac{1}{2} \rho_{ij} \sigma_i \sigma_j R_i R_j \frac{\partial^2}{\partial R_i \partial R_j} + \mu \sum_{i=1}^N R_i \frac{\partial}{\partial R_i}, \quad (4)$$

R_{i0} is the initial value of the i th bond spread at time t_0 and R_i represents its corresponding bond spread at time t .

2.1 Single-Systemically-Important Country (SIC) Model

We first consider a single-systematically-important country (SIC) model in which country 1 is considered as a country whose sovereign risk has a systemic impact on funding liquidity. When the sovereign bond spread of country 1 breaches a threshold (say 500 bp), illiquidity will rise sharply in the sovereign debt market and a systemic liquidity shock occurs. Country 2 in the model is not systemically important but more vulnerable than country 1. When country 2 is under stress with a surge in its sovereign bond spreads, it could have contagion effects on country 1. Therefore, with $N = 2$ and $\rho = \rho_{12}$ in the model, the two-dimensional PDF associated with Eqs.(3) and (4) is:

$$\begin{aligned} & PDF(R_{10}, R_{20}; R_1, R_2; \tau) \\ &= \frac{1}{R_1 R_2} \frac{1}{\sqrt{2\pi\sigma_1^2\tau}} \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_2^2\tau}} \\ & \times \exp \left[-\frac{\left(\ln \frac{R_{10}}{R_1} + \left(\mu - \frac{1}{2}\sigma_1^2\right)\tau\right)^2}{2(1-\rho^2)\sigma_1^2\tau} - \frac{\left(\ln \frac{R_{20}}{R_2} + \left(\mu - \frac{1}{2}\sigma_2^2\right)\tau\right)^2}{2(1-\rho^2)\sigma_2^2\tau} + \right. \\ & \left. 2\rho \frac{\left(\ln \frac{R_{10}}{R_1} + \left(\mu - \frac{1}{2}\sigma_1^2\right)\tau\right)\left(\ln \frac{R_{20}}{R_2} + \left(\mu - \frac{1}{2}\sigma_2^2\right)\tau\right)}{2(1-\rho^2)\sigma_1\sigma_2\tau} \right], \end{aligned} \quad (5)$$

where $\tau = (t - t_0)$. Based on the two thresholds (H_1 and H_2) of the two countries' sovereign bond spreads, we analyze changes in the PDFs in the four areas according to the level of bond spreads – I, II, III and IV, shown in Figure 2. Both the bond spreads of countries 1 and 2 are below their thresholds within area I. In area II, the bond spread of country 2 breaches its threshold while the bond spread of country 1 remains below the threshold. In area III, both the bond spreads of the two countries breach their thresholds, i.e., country 1 being in danger conditional on country 2 being under stress. We thus consider that contagion-induced systemic funding liquidity risk occurs in area III. In area IV, the bond spread of country 1 breaches its threshold while the bond spread of country 2 is below its threshold. Systemic funding liquidity risk occurs in area IV but it is not contagion induced. The corresponding probabilities in areas I, II, III, IV are given as:

$$\begin{aligned} P_I &= \int_0^{H_1} \int_0^{H_2} PDF(R_{10}, R_{20}; R_1, R_2; \tau) dR_1 dR_2 \\ &= N_2(A_1, A_2, \rho) \end{aligned} \quad (6)$$

$$\begin{aligned}
P_{II} &= \int_0^{H_1} \int_{H_2}^{\infty} PDF(R_{10}, R_{20}; R_1, R_2; \tau) dR_1 dR_2 \\
&= N(A_1) - N_2(A_1, A_2, \rho)
\end{aligned} \tag{7}$$

$$\begin{aligned}
P_{III} &= \int_{H_1}^{\infty} \int_{H_2}^{\infty} PDF(R_{10}, R_{20}; R_1, R_2; \tau) dR_1 dR_2 \\
&= 1 - N(A_1) - N(A_2) + N_2(A_1, A_2, \rho)
\end{aligned} \tag{8}$$

$$\begin{aligned}
P_{IV} &= \int_{H_1}^{\infty} \int_0^{H_2} PDF(R_{10}, R_{20}; R_1, R_2; \tau) dR_1 dR_2 \\
&= N(A_2) - N_2(A_1, A_2, \rho)
\end{aligned} \tag{9}$$

where $N(\cdot)$ is the cumulative normal distribution function, $N_2(\cdot)$ is the bivariate cumulative normal distribution function, and

$$A_1 = \frac{\ln \frac{H_1}{R_{10}} - (\mu - \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}}, \quad A_2 = \frac{\ln \frac{H_2}{R_{20}} - (\mu - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}. \tag{10}$$

It is noted that P_{III} is the joint probability of both bond spreads breaching their thresholds.

According to Eqs.(6)-(9), the rates of change of probabilities (RCProb) in the four areas are given by:

$$\frac{dP_I}{dt} = P_{A_1} \frac{\partial A_1}{\partial \tau} N\left(\frac{A_2 - \rho A_1}{\sqrt{1 - \rho^2}}\right) + P_{A_2} \frac{\partial A_2}{\partial \tau} N\left(\frac{A_1 - \rho A_2}{\sqrt{1 - \rho^2}}\right) \tag{11}$$

$$\frac{dP_{II}}{dt} = P_{A_1} \frac{\partial A_1}{\partial \tau} \left[1 - N\left(\frac{A_2 - \rho A_1}{\sqrt{1 - \rho^2}}\right)\right] - P_{A_2} \frac{\partial A_2}{\partial \tau} N\left(\frac{A_1 - \rho A_2}{\sqrt{1 - \rho^2}}\right) \tag{12}$$

$$\frac{dP_{III}}{dt} = -P_{A_1} \frac{\partial A_1}{\partial \tau} \left[1 - N\left(\frac{A_2 - \rho A_1}{\sqrt{1 - \rho^2}}\right)\right] - P_{A_2} \frac{\partial A_2}{\partial \tau} \left[1 - N\left(\frac{A_1 - \rho A_2}{\sqrt{1 - \rho^2}}\right)\right] \tag{13}$$

$$\frac{dP_{IV}}{dt} = -P_{A_1} \frac{\partial A_1}{\partial \tau} N\left(\frac{A_2 - \rho A_1}{\sqrt{1 - \rho^2}}\right) + P_{A_2} \frac{\partial A_2}{\partial \tau} \left[1 - N\left(\frac{A_1 - \rho A_2}{\sqrt{1 - \rho^2}}\right)\right] \tag{14}$$

where

$$P_{A_1} = \frac{1}{\sqrt{2\pi}} e^{-\frac{A_1^2}{2}}, \quad P_{A_2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{A_2^2}{2}},$$

$$\frac{\partial A_1}{\partial \tau} = -\frac{\ln \frac{H_1}{R_{10}} + \left(\mu - \frac{1}{2}\sigma_1^2\right)\tau}{2\sigma_1\sqrt{\tau^3}}, \quad \frac{\partial A_2}{\partial \tau} = -\frac{\ln \frac{H_2}{R_{20}} + \left(\mu - \frac{1}{2}\sigma_2^2\right)\tau}{2\sigma_2\sqrt{\tau^3}}.$$

With $H_1 > R_{10}$ and $H_2 > R_{20}$, both $\frac{\partial A_1}{\partial \tau}$ and $\frac{\partial A_2}{\partial \tau}$ are negative.⁵ Therefore, we have

$$\frac{dP_I}{dt} < 0 \quad \text{and} \quad \frac{dP_{III}}{dt} > 0.$$

A contagion-induced funding liquidity shock in the sovereign debt market occurs when country 1 is in danger (i.e., $R_1(t) > H_1$) conditional on country 2 being under stress (i.e., $R_2(t) > H_2$). RCProb(III) in area III (dP_{III}/dt) which is the rate of change of the joint probability of both bond spreads breaching their thresholds is therefore the key measure for analyzing contagion-induced systemic funding liquidity risk. It captures a potential sharp change of liquidity with respect to time.

The second risk measure derived from the probability density function is the distress correlation ρ_D of the probabilities of both of the two countries' bond spreads breaching their thresholds. The distress correlation is expressed as:

$$\begin{aligned} \rho_D &= \frac{P_{III} - P_{R1}P_{R2}}{\sqrt{P_{R1}(1 - P_{R1})}\sqrt{P_{R2}(1 - P_{R2})}} \\ &= \frac{N_2(A_1, A_2, \rho) - N(A_1) \cdot N(A_2)}{\sqrt{N(A_1)(1 - N(A_1))}\sqrt{N(A_2)(1 - N(A_2))}} \end{aligned} \quad (15)$$

where:

$$P_{R1} = 1 - N(A_1) \quad (16)$$

$$P_{R2} = 1 - N(A_2). \quad (17)$$

This risk measure identifies the timing of the distress spillover between countries 1 and 2 when funding liquidity risk in the sovereign debt market intensifies.

⁵ Since H_1 is much larger than R_{10} in general, $\ln H_1/R_{10}$ is greater than $\left(\mu - \frac{1}{2}\sigma_1^2\right)\tau$ for short time, so $\frac{\partial A_1}{\partial \tau}$ is negative. The same argument is applied to R_2 . Without loss of generality, μ is set as 0 throughout this paper.

2.2 Multi-Systemically-Important Country (SIC) Model

When there are two or more (i.e., $N-1$ where $N > 2$) systemically important countries which are correlated, the single-SIC model can be extended to a multi-SIC model. It is assumed that a systemic funding liquidity shock will occur when one or a number of systemically important countries are under severe stress as reflected in their sovereign bond spreads, and that market participants assess the risk arising from those countries as a whole. The sum of the bond spreads of those countries is thus used as an aggregate measure for constructing an analytically tractable and computationally feasible model. Similar to the single-SIC model, country N is not systemically important in the multi-SIC model but more vulnerable than the other $N-1$ systemically important countries, and could have contagion effects on them.

It is well known that no analytical solution of a PDF associated with the sum of asset prices under a lognormal process is available in closed form and one needs to resort to numerical methods, which can be a formidable task. In Appendix A, we derive closed-form approximate PDF formulas for the multi-SIC model by means of the Lie-Trotter operator splitting method, which bears a resemblance to the PDF formulas for the single-SIC model. The solutions of the two risk measures, the rate of change of probabilities (RCProb) and distress correlation ρ_D , of the multi-SIC model in area I, II, III, and IV as shown in Figure 2, are given in Appendix B.

3. Numerical Results during the Euro-Area Sovereign Debt Crisis

3.1 Data Description and Model Parameter Estimations

We collect 10-year sovereign bond spreads of Ireland, Italy, Portugal, and Spain relative to 10-year German government bonds, with active market quotes from Bloomberg covering the period of 2 September 2009 to 4 November 2011 as shown in Figure 1. Italy and Spain are the systemically important countries. Their bond spreads breached the 500-bp threshold in early November 2011 and triggered sharply higher illiquidity in the euro-area debt market. Based on the observations and empirical findings, the threshold for the systemically important countries' bond spread R_1 in the model is set at $H_1 = 500$ bp. Following a worsening sovereign debt situation, the EU and IMF agreed on bailout packages for Portugal in May 2011. The Portuguese bond spread surpassed 1000 bp in July 2011 and stayed at this level for the rest of the year. Consistent with the finding by Kalbaska and Gatkowski (2012) that Portugal was the most vulnerable country in the sample of PIIGS, Portugal is recognized to be vulnerable and have contagion effects on Italy and Spain in the model. The threshold H_2 for the Portuguese bond spread R_2 is set at 1200 bp which was about the highest level during the period.

The three model parameters are the drift μ , volatility σ and correlation ρ of the corresponding bond spreads. We assume the drift to be zero. To estimate the volatility σ and correlation ρ , we adopt the dynamic conditional correlation multivariate GARCH model (to be labelled as DCC_GARCH hereafter)

proposed by Engle and Sheppard (2001) to obtain the daily time series of the realized volatilities of the individual countries' bond spreads as well as their pairwise realized correlations. However, the realized volatilities estimated by using historical data is backward looking. To overcome this deficiency, option-implied volatility can be used given that options have the desirable property of being forward looking in nature and thus are a useful source of information for gauging market sentiment about future prices of financial assets and their dynamics. While there is no liquid sovereign bond spread or CDS option market, Hui and Chung (2011) and Hui and Fong (2015) find evidence of interconnectedness between price dynamics between the euro-area sovereign CDS and the US dollar (USD)-euro currency options. The creditworthiness of euro-area countries distinct from other macro-financial factors affects market expectations of the USD-euro exchange rate. Pu and Zhang (2012) show that the euro-area sovereign CDS spreads contain important information for the euro exchange rate dynamics at various phases of the crisis. In view of the relationship between bond spreads and CDS spreads, and the empirical relationship between CDS spreads and the USD-euro exchange rate dynamics, the foreign exchange (FX)-implied volatility of individual countries' bond spreads can thus be estimated using the linear relationship between the 3-month USD-euro exchange rates and bond spreads.⁶

Specifically, the realized volatilities of the spot USD-euro exchange rate and bond spreads are obtained by estimating an AR(5)-GARCH(1,1) process using their time series.⁷ Secondly, these realized volatility pairs are then linked up by regressing the realized volatility of bond spreads on the spot exchange rate. This estimated linear regression is then used to approximate the linear regression of the FX-implied volatility of bond spreads on the at-the-money option-implied volatility of the USD-euro exchange rate, by assuming that the coefficients of the latter regression are equal to those of the former one.⁸ The estimated FX-implied volatilities and the correlations of the sovereign bond spreads of Ireland, Italy, Portugal and Spain are shown in Figure 3.

3.2 Results of Single-SIC Model

We assume that Italy is country 1 and Portugal is country 2 in the model. The thresholds are set at $H_1 = 500$ bp for the Italian bond spread (R_1) and $H_2 = 1200$ bp for the Portuguese bond spread (R_2)

⁶ The 3-month maturity of the currency options is commonly used as the benchmark because it conveys both short-term and long-term views of market participants.

⁷ The AR(5)-GARCH(1,1) model of log-return r_t at time t is specified as $r_t = \theta_0 + \sum_{i=1}^5 \theta_i r_{t-i} + \varepsilon_t$ where ε_t is normal distributed with a conditional variance $\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$.

⁸ The linear regression of the realised volatility of the yield spread on that of the exchange rate at time t (denoted by YS_t^{RV} and FX_t^{RV} respectively) is specified as:

$$YS_t^{RV} = \alpha + \beta FX_t^{RV} + \varepsilon_t$$

where ε_t is an error term. The estimated α and β (denoted by $\hat{\alpha}$ and $\hat{\beta}$ respectively) are used to calculate the implied volatility of the yield spread (denoted by YS_t^{IV}) based on the following relationship:

$$YS_t^{IV} = \hat{\alpha} + \hat{\beta} FX_t^{IV}$$

where FX_t^{IV} is the implied volatility of the exchange rate.

according to the discussion in section 3.1 above. The time horizon is three months ($\tau = 0.25$).

Figure 4 shows the probabilities P (panel A) and $\text{RCProb } dP/dt$ (panel B) in the four areas – I, II, III and IV as shown in Figure 2 (see Eqs.(6)-(9) and Eqs.(11)-(14)), which are compared with those obtained by assuming zero correlation (i.e., $\rho = 0$) between the two countries' bond spreads. The results in Figure 4 demonstrate that when sovereign credit risk deteriorated in 2011, the probability P_I in area I decreased and the probability P_{II} in area II increased in April 2011. However, the probabilities P_{III} and P_{IV} in areas III and IV increased in July 2011, later than the changes in P_{I-II} . The timing of the changes in P_{III} and P_{IV} reflect the fact that as P_{III-IV} measure the likelihood of the Italian bond spread breaching the threshold $H_1 = 500$ bp, there was a material likelihood only when the bond spread rose to a sufficiently high level (about 220 bp) in July 2011.

RCProb(III) , which is the rate of change of the joint probability of R_1 and R_2 in area III and considered as the key measure for contagion-induced systemic funding liquidity risk, was almost zero before April 2011. Since late April 2011 when Portugal admitted that it could not deal with its finances itself and asked the EU for help, RCProb(III) began to rise to the 0.05 level and gave an earlier signal of risk compared with P_{III} and P_{IV} . This measure rose to the 0.2 level in July 2011 and stayed around that level until the Italian bond spread R_1 breached the 500-bp threshold in November 2011 when the illiquidity shock occurred. The rise in RCProb(III) indicates that the contagion-induced funding liquidity risk began to intensify in April 2011 when the market anticipated a bailout for Portugal, and worsened in the following months. It is consistent with concerns about the spread of sovereign default risk to Italy and Spain. If the correlation is set at $\rho = 0$, RCProb(III) remained close to zero until June 2011, similar to P_{III} . This comparison illustrates that the correlation between the two countries' bond spreads became important for the contagion effect only when the bond spreads began to move towards their thresholds.

Similarly, RCProb(IV) increased from almost zero in July 2011, later than the movement of RCProb(III) , but RCProb(IV) had a spike in November 2010 which was due to a short period of increases in the Italian bond spread. When the Portuguese bond spread R_2 passed the 600-bp level in April 2011, RCProb(II) started to increase, which is consistent with the drop in RCProb(I) . While R_2 increased quite fast in the first quarter of 2011, it did not induce an increase in the Italian bond spread before April 2011. Comparing RCProb(I-IV) with the probabilities P_{I-IV} , RCProb(I-IV) gave an earlier signal of risk than the probabilities P_{I-IV} , especially when the systemic funding liquidity risk began to intensify in April 2011. Among RCProb(I-IV) , RCProb(III) provides the clearest and forward-looking signal of the contagion-induced funding liquidity risk.

To further analyze the contribution of the dynamics of the Italian bond spread R_1 to RCProb(III) , Figure 5 shows the unconditional rate of change of probability (dP_{R_1}/dt) of R_1 breaching the threshold $H_1 = 500$ bp. This measure was non-zero in some periods before April 2011, reflecting material unconditional probabilities of R_1 breaching the threshold, and is different from RCProb(III) conditional on the dynamics of R_2 during the same periods. In late April 2011, these two measures started to have

similar movements, indicating that contagion between the Portuguese and Italian bond spreads began to contribute to the contagion-induced funding liquidity risk.

The two measures RCProb(II) and RCProb(III) were almost zero until material values emerged around April 2011, reflecting the existence of endogenous critical levels for R_1 and R_2 above which the two measures will be well above zero, i.e., signals appear. Such critical levels are estimated in Appendix C and given as:

$$\frac{R_{i0}^c}{H_i} = e^{-\left[1 - \frac{\sqrt{2\sigma_i^2(t-t_0)}}{4\xi_0}\right]\xi_0\sqrt{2\sigma_i^2(t-t_0)}} \approx e^{-\xi_0\sqrt{2\sigma_i^2(t-t_0)}} \quad (18)$$

for $\sqrt{2\sigma_i^2(t-t_0)} \ll 4\xi_0$. Based on this analysis and the coverage of the probability density, $\xi_0 = 2.5$ is used for estimating the critical level defined by Eq.(18).

Figure 5 shows that when the Italian bond spread R_1 is higher than the corresponding critical level R_1^c , the measure RCProb(R1) (dP_{R1}/dt , the rate of change of the probability of R_1 breaching the threshold H_1) is well above zero, otherwise it remains at zero. The result is consistent with the assumption of the existence of the endogenous critical level defined by Eq.(18). While the non-zero RCProb(R1) appeared in late 2010, the contagion liquidity risk measure RCProb(III) was almost zero, indicating that a systemic funding liquidity shock had not built up at that time due to weak contagion from the Portuguese bond spread. Figure 6 presents the Portuguese bond spread R_2 , its corresponding critical level R_2^c and the measure RCProb(R2) (dP_{R2}/dt , the rate of change of the probability of R_2 breaching the threshold H_2). Before early-March 2011, the Portuguese bond spread was lower than the critical level except in a few short periods of time and RCProb(R2) was near zero. Subsequently, RCProb(R2) increased to a material level when the bond spread was higher than the critical level. The signal of accumulating funding liquidity risk appeared in April 2011 when both the Italian and Portuguese bond spreads were above their respective critical levels, indicating a contagion effect between their bond spreads.

Figure 7 shows how the levels of H_1 and H_2 affect the probability density measures P and RCProb in the four areas. As expected, when H_1 increases, both P_I and P_{II} increase, while P_{III} and P_{IV} decrease due to the diminishing probability of R_1 breaching H_1 . Similarly, when H_2 increases, both P_I and P_{IV} increase, while P_{II} and P_{III} decrease owing to a diminishing probability of R_2 breaching H_2 . However, the changes of RCProb with H_1 and H_2 are not monotonic in the four areas, and have their maxima and minima at certain values of H_1 and H_2 . The existence of the maxima and minima implies that when R_1 and R_2 are both higher than the respective critical level R_1^c and R_2^c , RCProb(III) will attain its local maximum values at some combination of values of R_1 and R_2 which are not necessarily close to their thresholds H_1 and H_2 . This explains why a contagion-induced funding liquidity risk emerges some time before the Italian bond spread breaching the threshold H_1 in November 2011.

In summary, the numerical results show that the signals of the contagion-induced systemic funding liquidity risk using the probability density measure of $RCProb(III)$ appeared in April 2011, before the Italian bond spread breached the 500-bp threshold in November 2011 when the funding liquidity shock in the sovereign bond market occurred. The appearance of the signal of $RCProb(III)$ depends on whether the Italian and Portuguese sovereign bond spreads (R_1 and R_2) are higher than their respective endogenous critical levels which are determined by the corresponding thresholds H_i and volatility. The results also demonstrate that their correlation was an important factor in determining the contagion-induced funding liquidity risk only when the bond spreads were above their endogenous critical levels.

The distress correlation shown in Figure 8 identifies that distress spillovers between the bond spreads of Italy and Portugal occurred in May 2010 and the last quarter of 2010. Once it appeared in March 2011, its magnitude increased in the following months when funding liquidity risk in the sovereign debt market intensified. Given that the distress correlation measures the correlation of the probabilities of both of the two countries' bond spreads breaching their thresholds, the magnitude of the correlation could be material but the probabilities are low. This explains why the distress correlation appeared in 2010 when the funding liquidity shock was not an imminent concern and bond spreads were low relative to their thresholds in 2010.

3.3 Results of Multi-SIC Model

Italy and Spain are the two systemically important countries in the multi-SIC model. As in the single-SIC model, Portugal is considered as country 3 which is not systemically important but more vulnerable than the other two systemically important countries. The aggregate thresholds are set at $H_+ = 1000$ bp for the sum of the Italian and Spanish bond spreads (R_+) and $H_3 = 1200$ bp for the Portuguese bond spread (R_3). Figure 9 shows the probabilities P (panel A) and $RCProb$ dP_N/dt (panel B) of the four areas (I, II, III and IV as shown in Figure 2) defined by Eqs.(B.5)-(B.8) and Eqs.(B.10)-(B.13), which are compared with those of zero correlation among the three countries' bond spreads.

The results in Figure 9 are qualitatively similar to those of the single-SIC model in Figure 4 in which Italy is the single systemically important country. Regarding the measures $RCProb(II)$ and $RCProb(III)$, they increased from the zero level from April 2011 onwards. While the timing of the increases in the measures of the multi-SIC model is the same as that based on the single-SIC model in Figure 4, the signals in the multi-SIC model are stronger in magnitude during May-June 2011. The stronger signal of contagion-induced systemic funding liquidity risk in the multi-SIC model is due to a higher level and volatility of Spanish bond spreads and a stronger correlation with Portuguese bond spreads compared with those of Italian bond spreads, as shown in Figures 1 and 3. This is because the aggregate measure captures the combined effect of the dynamics of Italian and Spanish bond spreads.

Following the derivations of the risk measures in section 2, the critical level in the multi-SIC model can

be derived as in Eq.(18) by substituting R_1^c by R_+^c . Figure 10 shows that when the aggregate bond spread R_+ is higher than the corresponding critical level R_+^c , the measure $\text{RCProb}(R_+)$ (dP_{R_+}/dt , the rate of change of the probability of R_+ breaching the threshold H_+) is well above zero, otherwise it is zero. The result is consistent with analysis of the endogenous critical level in the single-SIC model. While the non-zero $\text{RCProb}(R_1)$ appeared in 2010, the contagion liquidity risk measure $\text{RCProb}(\text{III})$ remained almost zero (except a spike in May 2010) during the same period, indicating that a systemic funding liquidity shock had not built up at that time. Similar to the results of the single-SIC model, the results with finite and zero correlation of the multi-SIC model in Figure 9 illustrate that the correlation among the three countries' bond spreads played an important role in the contagion-induced funding liquidity risk measure when R_+ and R_3 were both above their respective critical levels (see Figure 6 for the Portuguese bond spread).

Figure 11 shows the distress correlation between the aggregate bond spreads of Italy and Spain (R_+) and Portuguese bond spreads (R_3). The measure was material in some periods of 2010, reflecting the fact that the correlation of the probabilities of the countries' bond spreads breaching their thresholds was substantial even with relatively low probabilities. After a drop in December 2010, the distress correlation surged from zero in January 2011 to about 0.3 in July 2011 and remained at this level until November 2011 when the illiquidity shock occurred. The timing of the occurrence of distress spillovers was earlier than the appearance of a signal from $\text{RCProb}(\text{III})$ in March 2011. Similar to the result in Figure 8, Figure 11 demonstrates that the distress correlation among the three countries provides forward-looking information for monitoring distress spillovers of the systemic funding liquidity risk in the sovereign debt market.

To illustrate the multi-SIC model with more systemically important countries, we add the bond spread of Ireland as a proxy country. The aggregate threshold is set at $H_+ = 2000$ bp for the sum (R_+) of the Italian, Spanish and Irish bond spreads given that the Irish bond spread breached the 1000-bp level shown in Figure 1. Figure 12 reports the corresponding RCProb in the four areas (I, II, III and IV). While the signal of $\text{RCProb}(\text{III})$ appeared in March 2011 similar to that in Figure 9, the intensity of $\text{RCProb}(\text{III})$ and the differences between the intensities with finite and null correlation in Figure 12 were larger than those in Figure 9, indicating the additional effects from the dynamics of the Irish bond spread.

4. Determinants of Contagion-Induced Systemic Funding Liquidity Risk Measure $\text{RCProb}(\text{III})$

4.1 Data and Method

To understand better the contemporaneous interaction between the contagion-induced systemic funding liquidity risk measure $\text{RCProb}(\text{III})$ and information in other markets, we use regression analysis to identify the major determinants of the measures. As the risk measure is based on the dynamics of sovereign bond spreads, previous studies on the determinants of sovereign CDS

spreads and market/funding liquidity indicators can provide guidance in choosing the related macro-financial variables. For example, Pan and Singleton (2008) find that the sovereign CDS spreads are related to global risk appetite, market volatility, and macroeconomic policy. Longstaff et al. (2011) show that sovereign CDS spreads are primarily driven by the equity market. In view of these findings, we examine the following five macro-financial variables:⁹

- (i) Returns of the Euro Stoxx50 index (STOXX50) measure the stock market systematic risk. The performance of the equity market reflects directly the economic outlook within the euro area;
- (ii) Volatility index (VIX) which is the market volatility of the US S&P 500 index gauges the global risk appetite in the financial market. An increase in the VIX index is usually associated with heightened volatility across different asset classes in particular equities. The VIX index is a measure of investors' aversion to volatility exposure and hence their willingness to put capital at risk;
- (iii) Euribor-Eonia spread (SPREADEURIBOR), which is measured as the difference between the 3-month Euro Area Inter-Bank Offered Rate (Euribor) for the euro and the 3-month Euro OverNight Index Average (Eonia), proxies the general increase in the cost of funding by banks in the euro area;
- (iv) Eonia-German T-Bill spread (SPREADGTB3M), which is the difference between the 3-month Eonia and the yield of the 3-month German Treasury bill, measures the funding cost in the financial market; and
- (v) 1-year euro-US dollar cross currency swap (CCBSS1Y) measures the macro-funding constraints in the euro versus the US dollar markets. As studied by Duffie and Singleton (1997), the market prices of the swaps contain information about the funding liquidity risk which is determined by both default and market liquidity risks.

RCProb(III) is expected to have a negative relationship with STOXX50 and to be positively related to VIX, SPREADEURIBOR, SPREADGTB3M and CCBSS1Y.

The analysis of variance (ANOVA) is employed to test the effect of each macro-financial variable on RCProb(III). Specifically, ANOVA compares the average RCProb(III) of several intervals of individual macro-financial variables, other things equal. For instance as shown in Table 1, the average RCProb(III) is 0.1414 given the STOXX50 in the range of 2,000 and 2,500, while the average RCProb(III) falls to 0.0014 when the STOXX50 hovers between 3,000 and 3,500. If all the averages are found to be statistically equal, RCProb(III) and the macro-financial variable are regarded independent. Conversely, if the average RCProb(III) for a certain interval of the macro-financial

⁹ The data on the macro-financial variables are collected from JP Morgan DataQuery and Bloomberg.

variable is statistically larger or smaller, RCProb(III) and the macro-financial variable are regarded as dependent. This dependence is used to investigate how RCProb(III) is related to the macro-financial variables based on the magnitude of the average RCProb(III). Moreover, we can identify the level at which the macro-financial variables will trigger sharp increases in RCProb(III) by properly grouping all the average RCProb(III) based on a pairwise t-test. ANOVA is specified as the following regression form:

$$y_i = \alpha + \sum_{k=1}^K \beta_k D_k + \epsilon_i \quad (19)$$

where y_i is the i -th observed probability, ϵ_i is a random error, and D_k is a dummy variable which is equal to 1 if y_i falls into the k -th interval of the macro-financial variable and zero otherwise. In the example of STOXX50 with three intervals (i.e., $K=3$), given an observed probability y_i , D_1 , D_2 , and D_3 are defined as:

$D_1 = 1$ when STOXX50 $\in [1,500, 2,500)$ and zero otherwise;

$D_2 = 1$ when STOXX50 $\in [2,500, 3,000)$ and zero otherwise; and

$D_3 = 1$ when STOXX50 $\in [3,000, 3,500)$ and zero otherwise.

Under this specification, the magnitude of β_k is used to test the differences among the least squares means at different intervals.

Before conducting the ANOVA test, each macro-financial variable is first grouped into no more than five intervals and each interval has an equal range. Covering the sample period from 2 September 2009 to 4 November 2011, all groupings of macro-financial variables are reported in Table 1. Since STOXX50 below 2,000 has only two observations, it is combined with the interval of $[2,000, 2,500)$ to avoid any small sample bias.

4.2 Empirical Results

Table 2 reports the least squares means and F-statistics. A least squares mean refers to an average RCProb(III) after controlling for the effect of individual macro-financial variables. An F-statistic is a test of the null hypothesis that all least squares means are equal. The results show that all the F-statistics have a very small p-value (i.e., <0.0001), indicating that all the macro-financial variables have significant effects on RCProb(III). Comparing the magnitude of the least squares means, the results show that a lower level of STOXX50 is associated with a higher least squares means, suggesting that RCProb(III) is negatively correlated with STOXX50 as expected. For the other variables (i.e., VIX, SPREADEURIBOR, SPREADGTB3M, and CCBSS1Y), their relations with RCProb(III) are positive as expected.

We further check the level at which each variable triggers a higher average RCProb(III). Using pairwise t-tests, we group all least squares means of individual variables which are statistically indifferent. For example, the least squares means under the STOXX50's intervals of [2500, 3000) and [3000, 3500) which are 0.0091 and 0.0014 respectively are statistically equal but different from that of 0.1379 under the interval of [1500, 2500). By comparing the intervals of the other variables, the least squares means are found to be particularly high when (i) the STOXX50 falls below 2,500; (ii) VIX rises above 30%; (iii) SPREADEURIBOR rises above 0.6%; (iv) SPREADGTB3M rises above 0.6%; and (v) CCBSS1Y rises above 60 bp.

In addition to these contemporaneous relationships, we also find evidence of a lead-lag relationship between the RCProb(III) and STOXX50, but no evidence for the other macro-financial variables based on Granger-causality tests. Testing the variables' changes at a 10% level of significance, the test result shows that STOXX50 granger causes RCProb(III) significantly (with the F-statistic 2.1779 and p-value 0.0553) but not the other way round (with the F-statistic 1.5141 and p-value 0.1836). This suggests that the equity market can predict stress in the sovereign bond market to some extent, probably reflecting investors' flight-to-quality behaviour by selling equities and holding less risky sovereign bonds in times of market stress.

These empirical results demonstrate that the contagion-induced systemic funding liquidity risk in the euro-area sovereign debt market is driven by market liquidity in the cross-currency swap market, funding cost in the euro-area banks, risk aversion level and equity market performance. The results also identify that when the macro-financial variables pass through certain levels, there will be a sharp rise in the risk measure RCProb(III). Investors' flight-to-quality behaviour affecting equity and sovereign bond markets was also observed.

5. Conclusion

The euro-area sovereign debt crisis demonstrated how systemic funding liquidity risk built up in the sovereign debt market when the sovereign credit risk of Portugal caused contagion to Italy and Spain which are systemically important sovereigns. The crisis also suggested the existence of a 500-bp threshold of Italian bond spreads, above which a systemic funding liquidity shock occurred. In view of these observations, this paper proposes a model based on the probability density associated with the dynamics of sovereign bond spreads to measure contagion-induced funding liquidity risk in the euro-area sovereign debt market.

The numerical results suggest that there are two useful risk measures associated with closed-form formulas derived from the model. These are: (1) the rate of change of the joint probabilities (RCProb) above the threshold of sovereign bond spreads of systemically important countries (Italy and Spain) and the small country (Portugal); and (2) the distress correlation of the probabilities of the thresholds being breached, which can provide a forward-looking signal of contagion-induced systemic funding liquidity risk. The RCProb was almost zero level before its signal materialized in April 2011, when the

sovereign bond spreads rose above the endogenous critical levels for the signal but the thresholds had not yet been breached. Subsequently, a liquidity shock occurred in the sovereign debt market during November 2011 as the Italian bond spread breached the 500-bp threshold. The results suggest that a liquidity shock began to build up when the sovereign bond spreads passed an endogenous critical level and rose towards their thresholds. Their correlation was an important factor in determining the contagion-induced funding liquidity risk only when bond spreads were above their critical levels. While the distress correlation provided a forward-looking signal of the liquidity shock in March 2011, the signal also appeared in 2010 when the illiquidity shock was not an imminent concern given the low level of bond spreads relative to their thresholds.

The empirical results show that contagion-induced systemic funding liquidity risk (measured by the RCProb above the thresholds of the sovereign bond spreads of Italy and Portugal) in the euro-area sovereign debt market is driven by market liquidity in the cross-currency swap market, funding costs in the euro-area banks, risk aversion levels and equity market performance. When the macro-financial variables associated with these determinants pass through certain levels, the funding liquidity risk rises sharply. The numerical and empirical results demonstrate that the proposed risk measures derived from the model are useful for gauging contagion-induced funding liquidity risk. Further research can be conducted to test the applications of the proposed model for measuring risks in other financial markets.

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Table 1. Grouping of Macro-Financial Variables

<u>STOXX50</u>	<u>Count</u>	<u>Average RCProb(III)</u>	<u>VIX</u>	<u>Count</u>	<u>Average RCProb(III)</u>
[1500, 2000)	2	0.0260	[10, 20)	216	0.0102
[2000, 2500)	65	0.1414	[20, 30)	245	0.0121
[2500, 3000)	450	0.0091	[30, 40)	69	0.0965
[3000, 3500)	27	0.0014	[40, 50)	14	0.1083
Total	544	0.0245	Total	544	0.0245
<u>SPREADG3M</u>	<u>Count</u>	<u>Average RCProb(III)</u>	<u>CCBSS1Y</u>	<u>Count</u>	<u>Average RCProb(III)</u>
[0, 0.2)	159	0.0011	[0, 20)	33	0.0078
[0.2, 0.4)	295	0.0093	[20, 40)	349	0.0119
[0.4, 0.6)	68	0.0988	[40, 60)	120	0.0231
[0.6, 0.8)	22	0.1684	[60, 80)	42	0.1468
Total	544	0.0245	Total	544	0.0245
<u>SPREADEURIBOR</u>	<u>Count</u>	<u>Average RCProb(III)</u>			
[0, 0.2)	20	0.0136			
[0.2, 0.4)	439	0.0083			
[0.4, 0.6)	23	0.0371			
[0.6, 0.8)	45	0.1424			
[0.8, 1)	17	0.1293			
Total	544	0.0245			

Table 2. Variables' Significance and Least Squares Means of RCProb(III) by Interval of the Variables

Variable	F-stat	p-value	Least squares means of RCProb(III) under intervals of the variable					
			<u>Group 1</u>	<u>Group 2</u>		<u>Group 3</u>		<u>Group 4</u>
			<u>[1500, 2500)</u>	<u>[2500, 3000)</u>	<u>[3000, 3500)</u>			
STOXX50	398.4	<0.0001	0.1379	0.0091	0.0014			
			<u>[10, 20)</u>	<u>[20, 30)</u>	<u>[30, 40)</u>	<u>[40, 50)</u>		
VIX	86.9	<0.0001	0.0102	0.0121	0.0965	0.1083		
			<u>[0, 0.2)</u>	<u>[0.2, 0.4)</u>	<u>[0.4, 0.6)</u>	<u>[0.6, 0.8)</u>	<u>[0.8, 1)</u>	
SPREADEURIBOR	175.3	<0.0001	0.0136	0.0083	0.0371	0.1424	0.1293	
			<u>[0, 0.2)</u>		<u>[0.2, 0.4)</u>	<u>[0.4, 0.6)</u>		<u>[0.6, 0.8)</u>
SPREADGTB3M	266.5	<0.0001	0.0011		0.0093	0.0988		0.1684
			<u>[0, 20)</u>	<u>[20, 40)</u>	<u>[40, 60)</u>	<u>[60, 80)</u>		
CCBSS1Y	130.4	<0.0001	0.0078	0.0119	0.0231	0.1468		

Note: For each variable, all the least squares means are statistically the same within group but different between groups at a significance level of 0.05.

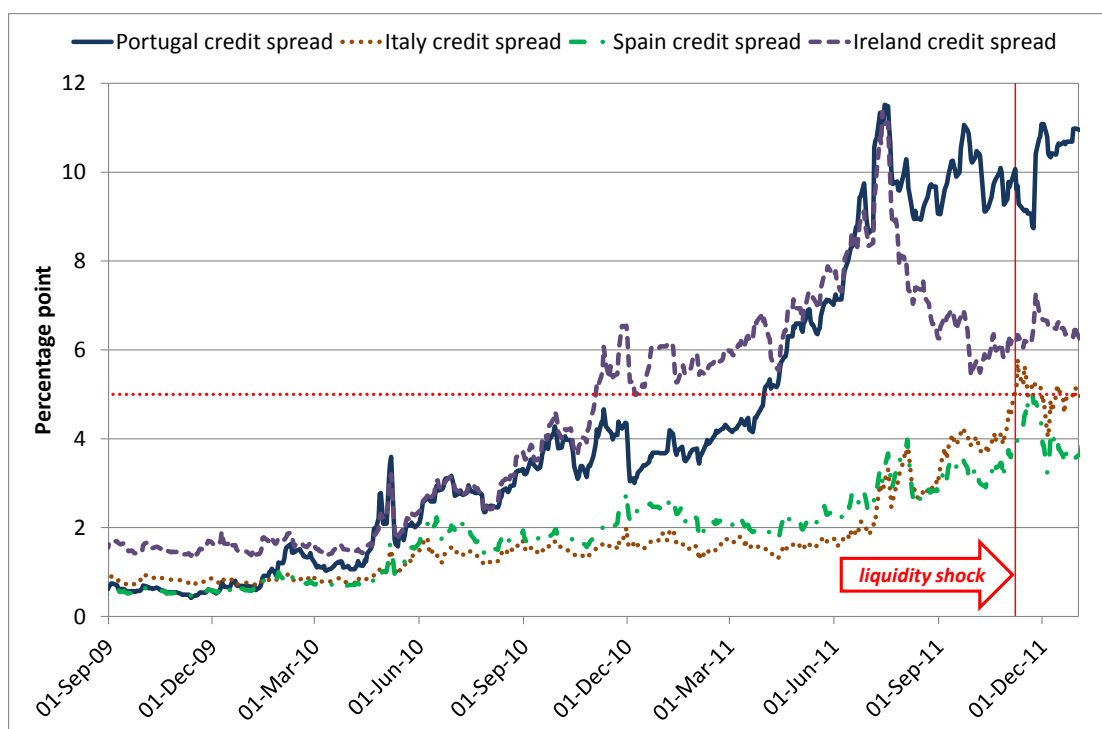
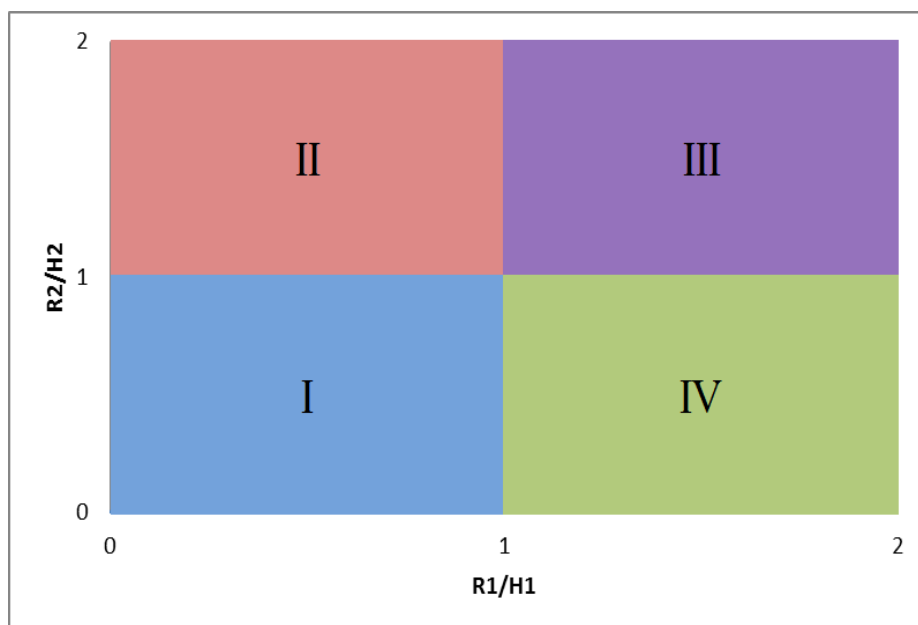
Figure 1. 10-Year Sovereign Bond Spreads of Ireland, Italy, Portugal and Spain**Figure 2. Four Areas According to the Levels of Two Countries' Bond Spreads (R_1 or R_+ and R_2 or R_N) Divided by their Respective t Thresholds (H_1 or H_+ and H_2 or H_N)**

Figure 3. FX-Implied Volatility (Upper Panel) and Correlation Computed by DCC_GARCH (Lower Panel) Of Sovereign Bond Spreads (CS) of Ireland (IR), Italy (IT), Portugal (P) and Spain (S)

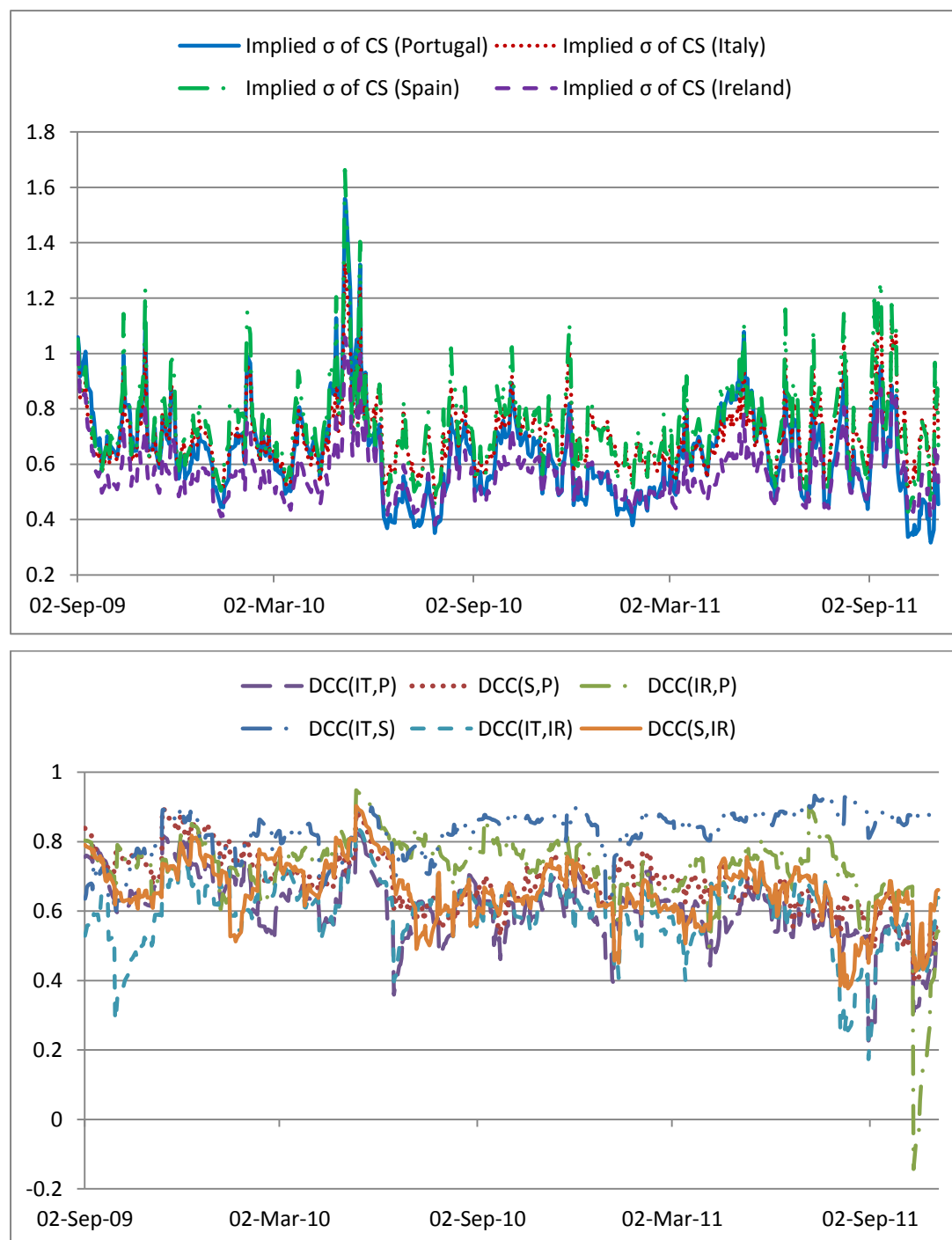
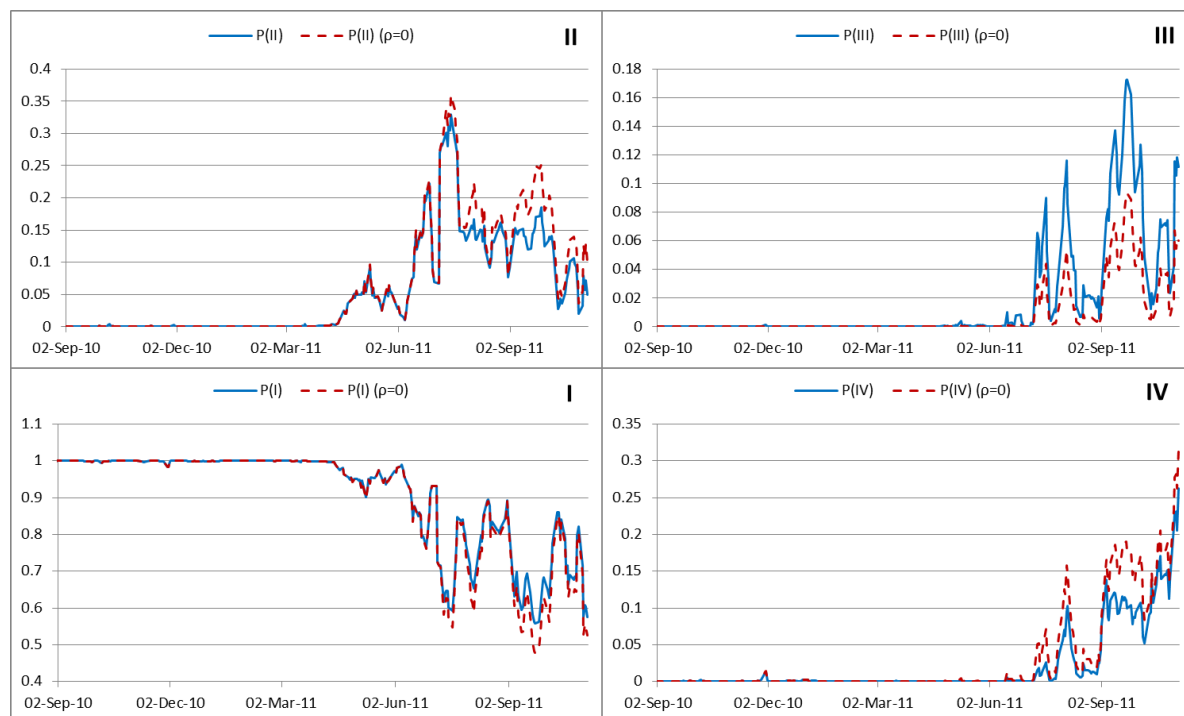
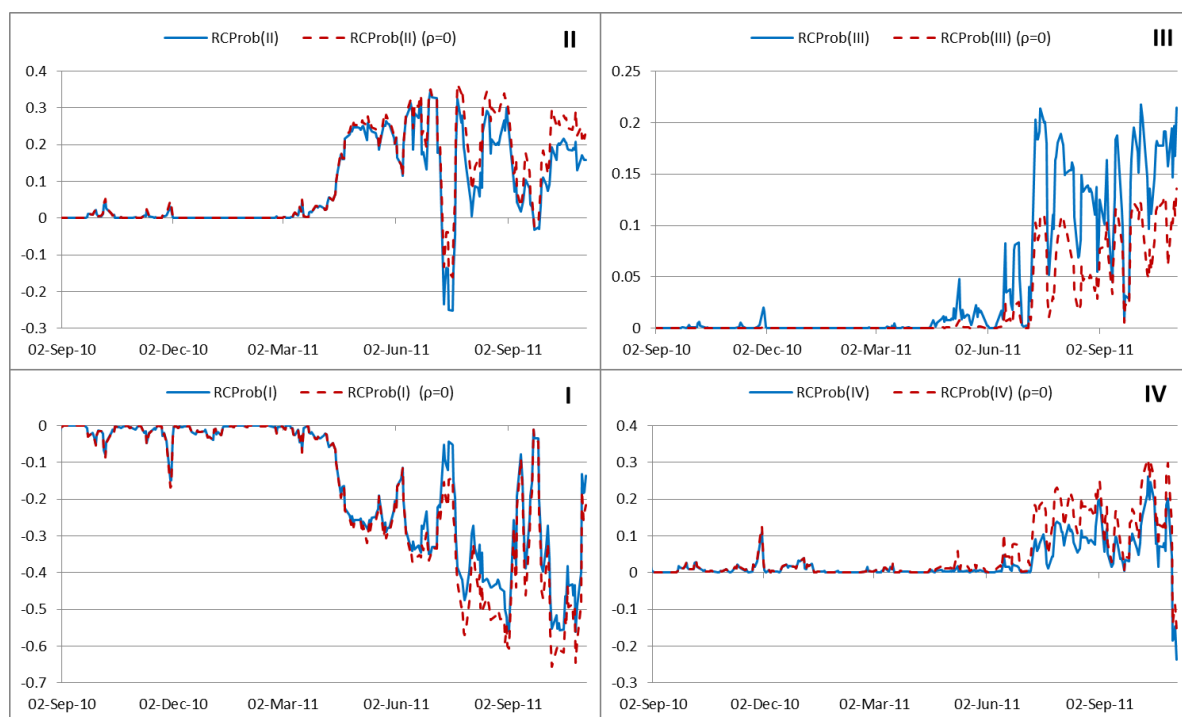


Figure 4. Probability (P) (Panel A) and RCProb (Panel B) with Correlation and Without Correlation ($\rho=0$) for Sovereign Bond Spreads of Italy (R_1) and Portugal (R_2) with $H_1=5\%$ and $H_2=12\%$, and $\tau=0.25$ year.



(A) Probability P



(B) Rate of change of probability $RCProb$ dP/dt

Figure 5. RCPProb(III), RCPProb(R1), R_{10} and R_1^c Normalized by H_1 for Sovereign Bond Spreads of Italy (R_1) with $H_1 = 5\%$ and $H_2 = 12\%$, and $\tau = 0.25$ year.

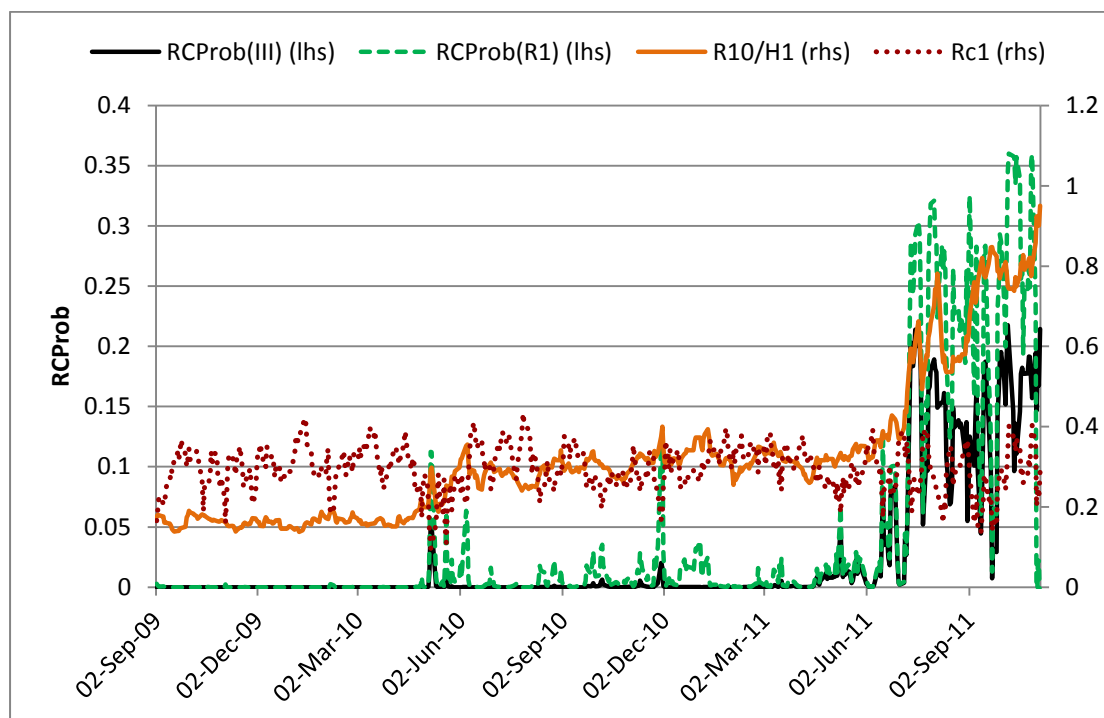


Figure 6. RCPProb(III), RCPProb(R2), R_{20} and R_2^c Normalized by H_2 for Sovereign Bond Spreads of Portugal (R_2) with $H_1 = 5\%$ and $H_2 = 12\%$, and $\tau = 0.25$ year.

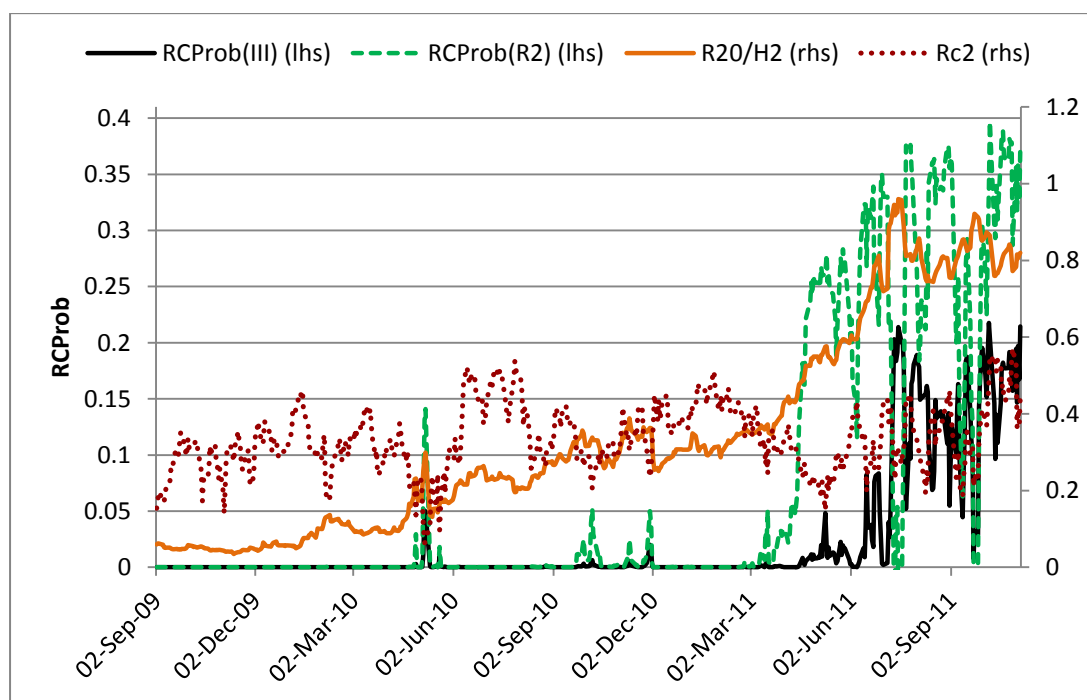


Figure 7. Effects of H_1 and H_2 on Probability (P) and RCProb with $R_1 = 3.17\%$ for Italy and $R_2 = 11.54\%$ for Portugal as at 11 July 2011 and $\tau = 0.25$ year.

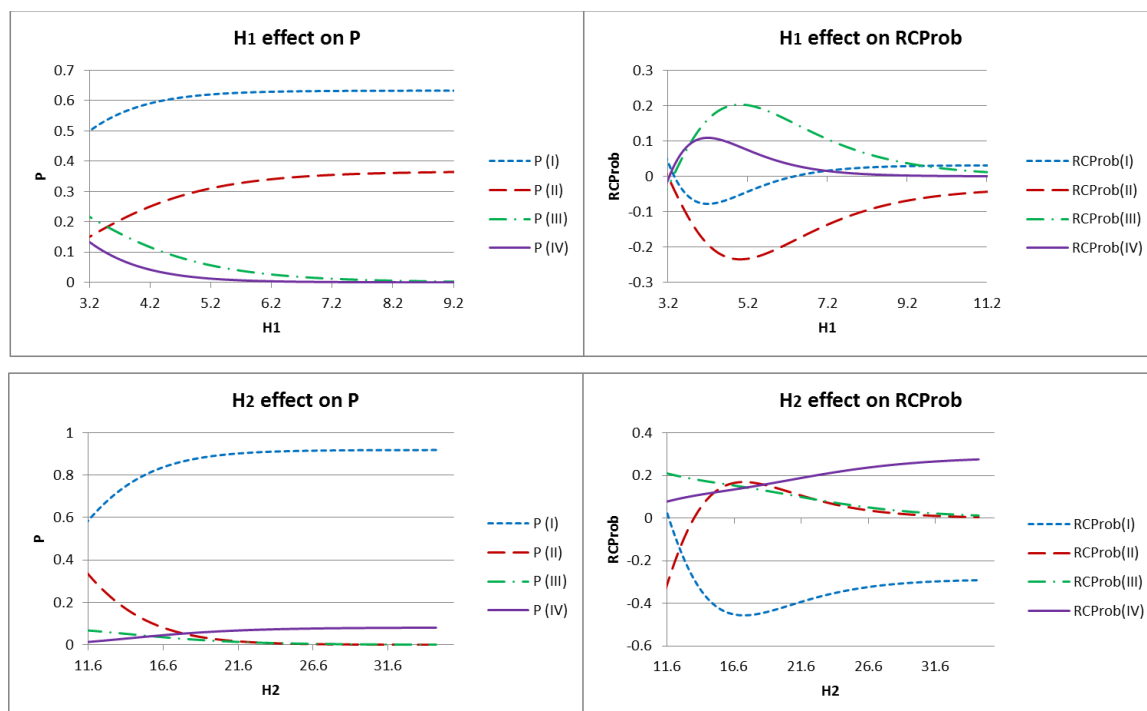


Figure 8. Distress Correlation for Sovereign Bond Spreads of Italy (R_1) and Portugal (R_2) with $H_1 = 5\%$ and $H_2 = 12\%$, and $\tau = 0.25$ year.

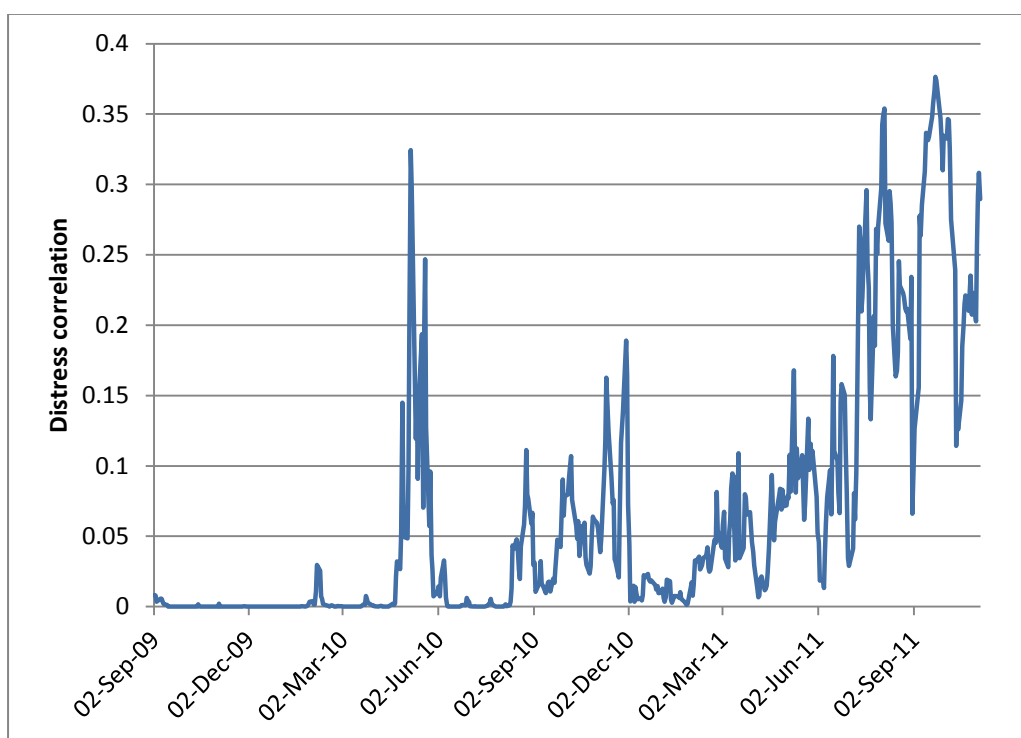
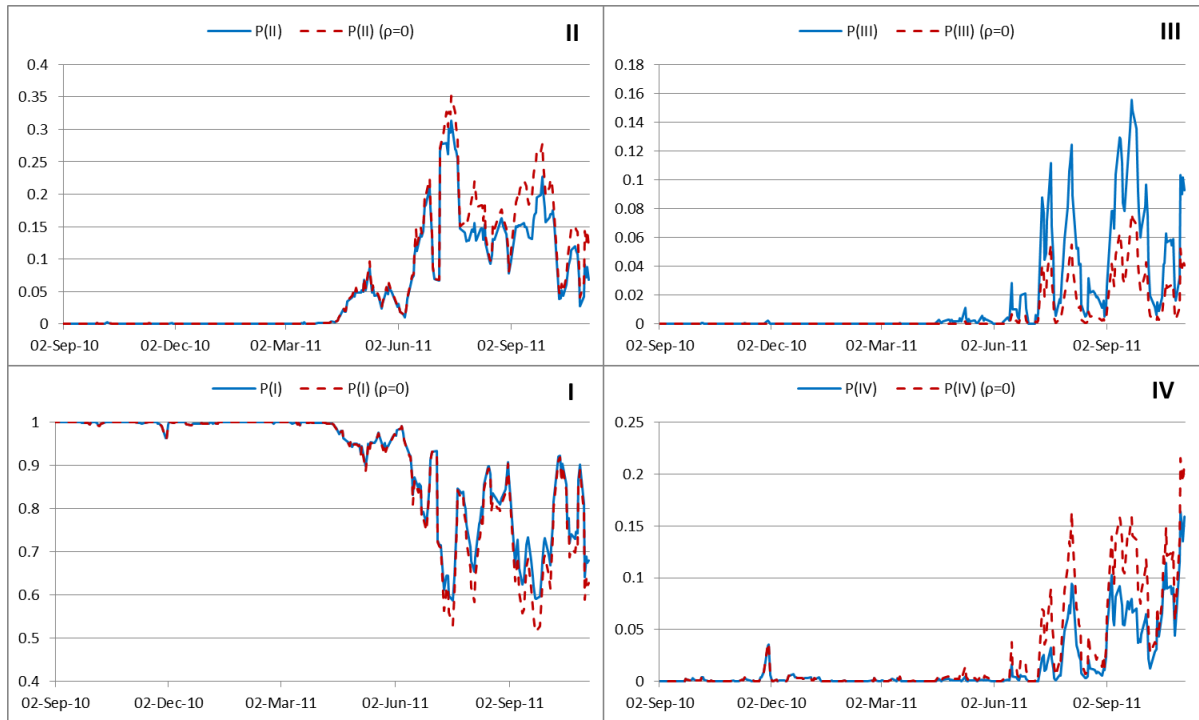
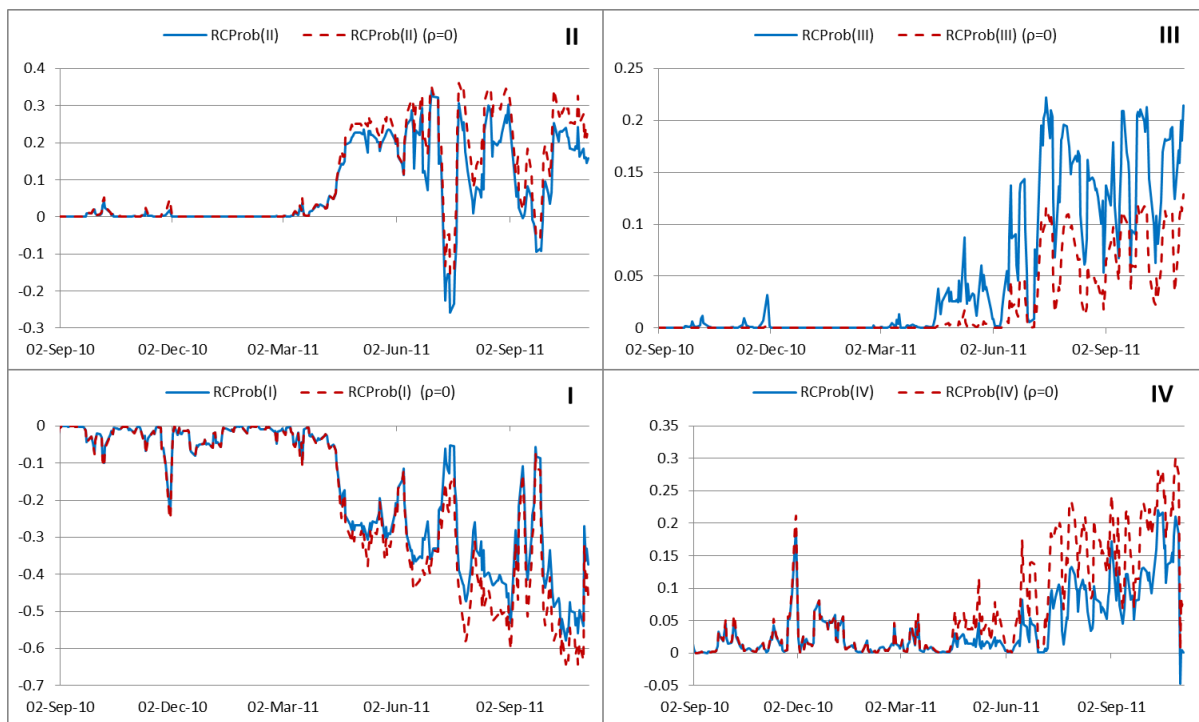


Figure 9. Probability (P) (panel A) and RCProb (panel B) with Correlation and Without Correlation ($\rho=0$) for Sovereign Bond Spreads of Italy and Spain (R_+) and Portugal (R_2) with $H_+ = 10\%$ and $H_3 = 12\%$, and $\tau = 0.25$ year.



(A) Probability P



(B) Rate of change of probability RCProb dP/dt

Figure 10. RCProb(III), RCProb(R+), R_{+0} and R_{+}^c Normalized by H_{+} for Sovereign Bond Spreads of Italy and Spain (R_{+}) and Portugal (R_3) with $H_{+} = 10\%$ and $H_3 = 12\%$, and $\tau = 0.25$ year.

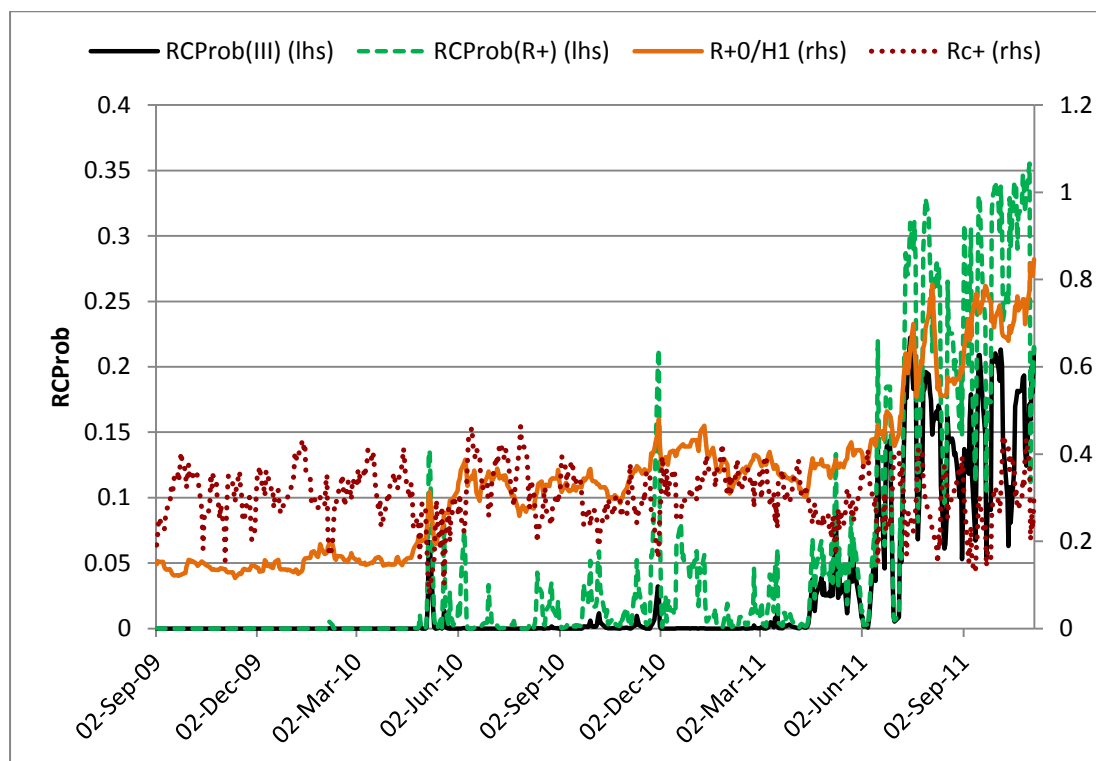


Figure 11. Distress Correlation for Sovereign Bond Spreads of Italy and Spain (R_{+}) and Portugal (R_2) with $H_{+} = 10\%$ and $H_3 = 12\%$, and $\tau = 0.25$ year.

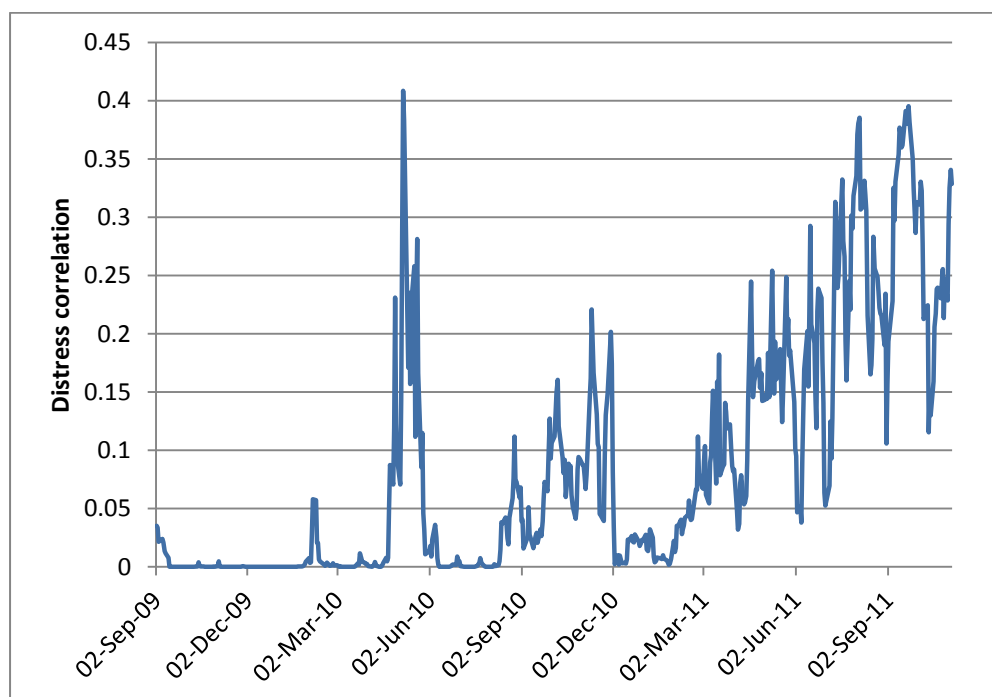
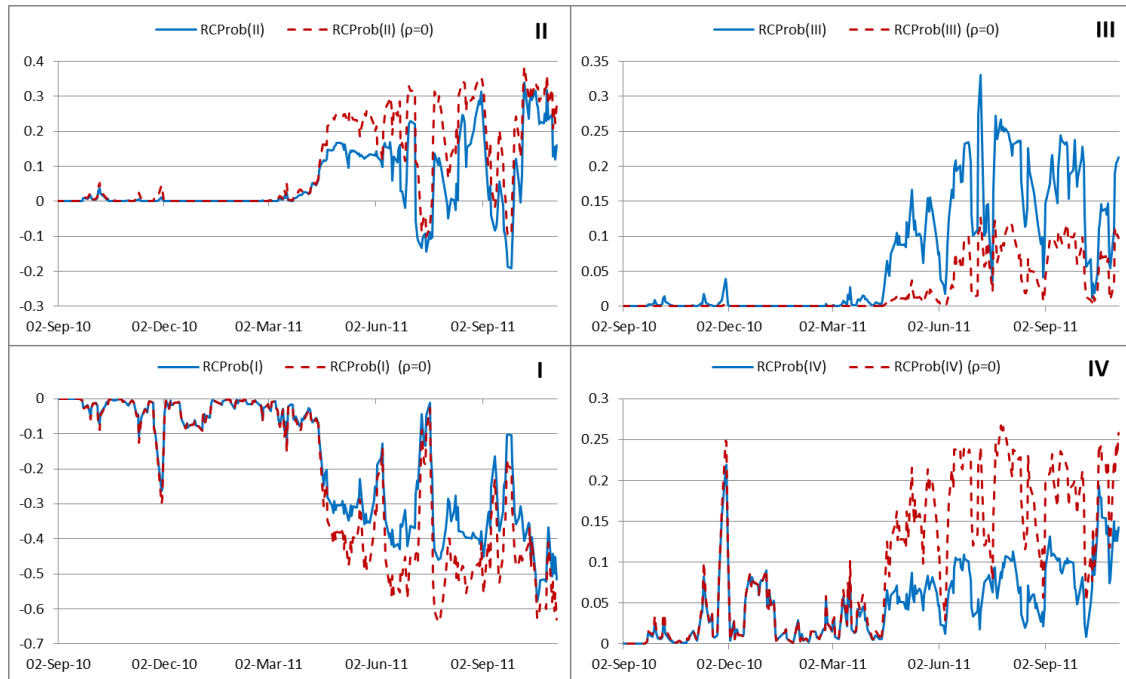


Figure 12. RCProb with Correlation and Without Correlation ($\rho=0$) for Sovereign Bond Spreads of Ireland, Italy and Spain (R_+) and Portugal ($R_{N=4}$) with $H_+=20\%$ and $H_{N=4}=12\%$, and $\tau=0.25$ year.



Appendix A

The approximate probability distribution function (PDF) for the multi-SIC model in Eq.(B.1) can be derived as follows. Given N correlated lognormal variables R_i 's obeying the stochastic differential equations:

$$\frac{dR_i}{R_i} = \mu dt + \sigma_i dZ_i, \quad i = 1, 2, 3, \dots, N, \quad (\text{A.1})$$

where μ is the drift, σ_i is the volatility of R_i , dZ_i denotes a standard Weiner process associated with R_i , and $dZ_i dZ_j = \rho_{ij} dt$, the time evolution of the joint PDF $P(\{R_i\}, t; \{R_{i0}\}, t_0)$ of the N lognormal variables is governed by the backward Kolmogorov equation:

$$\left\{ \frac{\partial}{\partial t_0} + \hat{L} \right\} P(\{R_i\}, t; \{R_{i0}\}, t_0) = 0 \quad \text{for } t > t_0, \quad (\text{A.2})$$

where:

$$\hat{L} = \sum_{i,j=1}^N \frac{1}{2} \rho_{ij} \sigma_i \sigma_j R_{i0} R_{j0} \frac{\partial^2}{\partial R_{i0} \partial R_{j0}} + \mu \sum_{i=1}^N R_{i0} \frac{\partial}{\partial R_{i0}}, \quad (\text{A.3})$$

subject to the boundary condition:

$$P(\{R_i\}, t; \{R_{i0}\}, t_0 \rightarrow t) = \prod_{i=1}^N \delta(R_i - R_{i0}), \quad (\text{A.4})$$

where $\delta(\cdot)$ is the Dirac delta function. This joint PDF tells us how probable the N lognormal variables assume the values $\{R_i\}$ at time $t > t_0$, provided that their values at t_0 are given by $\{R_{i0}\}$. Since $P(\{R_i\}, t; \{R_{i0}\}, t_0)$ is known in closed form, the joint PDF $\bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0)$ of R_N and the sum $R_+ \equiv \sum_{i=1}^{N-1} R_i$ can be obtained by evaluating the integral:

$$\bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0) = \int_0^\infty dR_1 \int_0^\infty dR_2 \cdots \int_0^\infty dR_{N-1} P(\{R_i\}, t; \{R_{i0}\}, t_0) \delta\left(\sum_{i=1}^{N-1} R_i - R_+\right). \quad (\text{A.5})$$

The problem can be addressed by many numerical methods, however, a closed-form representation for the joint PDF is still missing. Moreover, for large N the numerical integration becomes a formidable task. Hence, in the following we apply the Lie-Trotter operator splitting method (Trotter, 1958 and 1959) to derive an approximation for the joint PDF. This approach has been recently used to study the dynamics of the sum and difference of two correlated stochastic variables, namely (i) two lognormal

variables (Lo, 2012 and 2013a) and (ii) two constant elasticity of variance variables (Lo, 2013b), and the valuation of multi-asset spread options (Lo, 2014).

It is observed that the joint PDF $\bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0)$ also satisfies the same backward Kolmogorov equation given in Eq.(A.2), but with a different boundary condition:

$$\bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0 \rightarrow t) = \delta(R_N - R_{N0}) \delta\left(R_+ - \sum_{i=1}^{N-1} R_{i0}\right). \quad (\text{A.6})$$

To solve for $\bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0)$, we first note that the solution takes the form

$$\begin{aligned} & \bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0) \\ &= \exp\left\{(t - t_0)\mu \sum_{i=1}^N R_{i0} \frac{\partial}{\partial R_{i0}}\right\} \tilde{P}(R_+, R_N, t; \{R_{i0}\}, t_0) \\ &= \tilde{P}(R_+, R_N, t; \{R_{i0} e^{(t-t_0)\mu}\}, t_0) \quad , \end{aligned} \quad (\text{A.7})$$

and $\tilde{P}(R_+, R_N, t; \{R_{i0}\}, t_0)$ satisfies the partial differential equation

$$\left\{\frac{\partial}{\partial t_0} + \hat{L}_0\right\} \tilde{P}(R_+, R_N, t; \{R_{i0}\}, t_0) = 0 \quad \text{for } t > t_0 \quad , \quad (\text{A.8})$$

where

$$\hat{L}_0 = \sum_{i,j=1}^N \frac{1}{2} \rho_{ij} \sigma_i \sigma_j R_{i0} R_{j0} \frac{\partial^2}{\partial R_{i0} \partial R_{j0}} \quad . \quad (\text{A.9})$$

Then, in terms of the new variables:

$$R_{+0} \equiv \sum_{i=1}^{N-1} R_{i0} \quad \text{and} \quad \bar{R}_{i0} \equiv R_{i0} - R_{+0} \quad (\text{A.10})$$

for $i = 1, 2, 3, \dots, N-2$, Eq.(A.8) can be rewritten as

$$\left\{\frac{\partial}{\partial t_0} + \hat{L}_{LT} + \hat{L}_R\right\} \tilde{P}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0) = 0 \quad (\text{A.11})$$

Where

$$\hat{L}_{LT} = \frac{1}{2} \sigma_N^2 R_{N0}^2 \frac{\partial^2}{\partial R_{N0}^2} + \rho_{N+} \sigma_N \sigma_+ R_{N0} R_{+0} \frac{\partial^2}{\partial R_{N0} \partial R_{+0}} + \frac{1}{2} \sigma_+^2 R_{+0}^2 \frac{\partial^2}{\partial R_{+0}^2} \quad (\text{A.12})$$

$$\sigma_+ = \sqrt{\sum_{i,j=1}^{N-1} \rho_{ij} \sigma_i \sigma_j \left(\frac{R_{i0}}{R_{+0}} \right) \left(\frac{R_{j0}}{R_{+0}} \right)} \quad (\text{A.13})$$

$$\rho_{N+} = \frac{1}{\sigma_+} \sum_{i=1}^{N-1} \rho_{iN} \sigma_i \left(\frac{R_{i0}}{R_{+0}} \right) . \quad (\text{A.14})$$

and $\hat{L}_R$ contains terms involving partial derivatives with respect to $\bar{R}_{i0}$'s. The corresponding boundary condition is given by

$$\tilde{P}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0 \rightarrow t) = \delta(R_N - R_{N0}) \delta(R_+ - R_{+0}) . \quad (\text{A.15})$$

Accordingly, the formal solution of Eq.(A.11) is given by

$$\begin{aligned} & \tilde{P}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0) \\ &= \exp\{(t - t_0)(\hat{L}_{LT} + \hat{L}_R)\} \delta(R_N - R_{N0}) \delta(R_+ - R_{+0}) . \end{aligned} \quad (\text{A.16})$$

Since the exponential operator $\exp\{(t - t_0)(\hat{L}_{LT} + \hat{L}_R)\}$ is difficult to evaluate, we apply the Lie-Trotter operator splitting method (Trotter, 1958 and 1959) to approximate the operator by¹⁰

$$\hat{O}^{LT} = \exp\{(t - t_0)\hat{L}_{LT}\} \exp\{(t - t_0)\hat{L}_R\} , \quad (\text{A.17})$$

and obtain an approximation to the formal solution $\tilde{P}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0)$, namely

$$\begin{aligned} & \tilde{P}^{LT}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0) \\ &= \hat{O}^{LT} \delta(R_N - R_{N0}) \delta(R_+ - R_{+0}) \end{aligned}$$

¹⁰ Suppose that one needs to exponentiate an operator $\hat{C}$ which can be split into two different parts, namely $\hat{A}$ and $\hat{B}$. For simplicity, let us assume that $\hat{C} = \hat{A} + \hat{B}$, where the exponential operator $\exp(\hat{C})$ is difficult to evaluate but $\exp(\hat{A})$ and $\exp(\hat{B})$ are either solvable or easy to deal with. Under such circumstances the exponential operator $\exp(\varepsilon \hat{C})$, with ε being a small parameter, can be approximated by the Lie-Trotter splitting formula (Trotter, 1958 and 1959):

$$\exp(\varepsilon \hat{C}) = \exp(\varepsilon \hat{A}) \exp(\varepsilon \hat{B}) + O(\varepsilon^2) .$$

This can be seen as the approximation to the solution at $t = \varepsilon$ of the equation $d\hat{Y}/dt = (\hat{A} + \hat{B})\hat{Y}$ by a composition of the exact solutions of the equations $d\hat{Y}/dt = \hat{A}\hat{Y}$ and $d\hat{Y}/dt = \hat{B}\hat{Y}$ at time $t = \varepsilon$.

$$= \exp\{(t - t_0)\hat{L}_{LT}\}\delta(R_N - R_{N0})\delta(R_+ - R_{+0}) \quad (\text{A.18})$$

where the relation

$$\begin{aligned} \hat{L}_R \delta(R_N - R_{N0})\delta(R_+ - R_{+0}) &= 0 \\ \Rightarrow \exp\{(t - t_0)\hat{L}_R\}\delta(R_+ - R_{+0}) &= \delta(R_+ - R_{+0}) \end{aligned} \quad (\text{A.19})$$

is utilized. As both σ_+ and ρ_{N+} are functions of R_{+0} , the approximate solution cannot be the joint PDF of two correlated lognormal variables. Nevertheless, if both σ_+ and ρ_{N+} are slowly-varying functions of R_{+0} , i.e.

$$\frac{R_{+0}}{\sigma_+} \left| \frac{\partial \sigma_+}{\partial R_{+0}} \right| \ll 1 \quad \text{and} \quad \frac{R_{+0}}{\rho_{N+}} \left| \frac{\partial \rho_{N+}}{\partial R_{+0}} \right| \ll 1, \quad (\text{A.20})$$

then we may apply the idea of the WKB method, which is a powerful tool for obtaining a global approximation to the solution of a linear ordinary differential equation¹¹, to approximate $\tilde{P}^{LT}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0)$ by

$$\begin{aligned} &\tilde{P}^{LT}(R_+, R_N, t; R_{+0}, \{\bar{R}_{i0}\}, R_{N0}, t_0) \\ &\approx \frac{1}{2\pi\sigma_+\sigma_N\tau\sqrt{1-\rho_{N+}^2}R_+R_N} \exp\left\{-\frac{[\ln(R_{+0}/R_+) - \sigma_+^2\tau/2]^2}{2(1-\rho_{N+}^2)\sigma_+^2\tau} + \right. \\ &\quad \left. \frac{\rho_{N+}[\ln(R_{N0}/R_N) - \sigma_N^2\tau/2][\ln(R_{+0}/R_+) - \sigma_+^2\tau/2]}{(1-\rho_{N+}^2)\sigma_N\sigma_+\tau} - \right. \end{aligned}$$

¹¹ The WKB (Wentzel–Kramers–Brillouin) method provides approximate solutions of differential equations of the form

$$\frac{d^2 y(x)}{dx^2} + k(x)^2 y(x) = 0,$$

provided that $k(x)$ is slowly varying, i.e.

$$\left| \frac{1}{k(x)} \frac{dk(x)}{dx} \right| \ll 1.$$

The completed approximate solution is given by

$$y(x) \approx \frac{1}{\sqrt{k(x)}} \exp\left\{i \int k(x) dx\right\}.$$

It is obvious that the approximate solution will be reduced to the usual plane-wave solution if $k(x)$ is replaced by a constant. Details of the method can be found in Morse and Feshbach (1953), Mathews and Walker (1973), and Bender and Orszag (1978).

$$\frac{[\ln(R_{N0}/R_N) - \sigma_N^2\tau/2]^2}{2(1 - \rho_{N+}^2)\sigma_N^2\tau} \Bigg\} \quad (\text{A.21})$$

which resembles the joint PDF of two correlated lognormal variables very closely. A more detailed discussion about the validity of this approach can be found in Lo (2013a and 2014). As a result, it is obvious that the joint PDF $\bar{P}(R_+, R_N, t; \{R_{i0}\}, t_0)$ in Eq.(A.7) is identical to the approximate PDF for the multi-SIC model in Eq.(B.1).

Appendix B

We have the following approximate PDF:

$$PDF(R_{+0}, R_{N0}; R_+, R_N; \tau) = \frac{1}{R_+ R_N} \frac{1}{\sqrt{2\pi\sigma_+^2\tau}} \frac{1}{\sqrt{2\pi(1-\rho_+^2)\sigma_N^2\tau}} \times e^{-\left[\frac{\left(\ln \frac{R_{+0}}{R_+} + (\mu - \frac{1}{2}\sigma_+^2)\tau\right)^2}{2(1-\rho_+^2)\sigma_+^2\tau} + \frac{\left(\ln \frac{R_{N0}}{R_N} + (\mu - \frac{1}{2}\sigma_N^2)\tau\right)^2}{2(1-\rho_+^2)\sigma_N^2\tau} - 2\rho_+ \frac{\left(\ln \frac{R_{+0}}{R_+} + (\mu - \frac{1}{2}\sigma_+^2)\tau\right)\left(\ln \frac{R_{N0}}{R_N} + (\mu - \frac{1}{2}\sigma_N^2)\tau\right)}{2(1-\rho_+^2)\sigma_+\sigma_N\tau} \right]} \quad (B.1)$$

where

$$R_+ = \sum_{j=1}^{N-1} R_j \quad (B.2)$$

$$\sigma_+ = \frac{\sqrt{\sum_{i,j=1}^{N-1} \rho_{ij}\sigma_i\sigma_j R_{i0}R_{j0}}}{\sum_{i=1}^{N-1} R_{i0}} \quad (B.3)$$

$$\rho_+ = \frac{\sum_{i,j=1}^{N-1} \rho_{iN}\sigma_i R_{i0}}{\sigma_+ \sum_{i=1}^{N-1} R_{i0}}. \quad (B.4)$$

As in the single-SIC model, a liquidity shock is triggered when the aggregate measure of the $(N-1)$ countries breach the threshold (i.e., $R_+(t) > H_+$) conditional on country N under stress (i.e., $R_N(t) > H_N$). The corresponding probabilities of the multi-SIC model in area I, II, III, and IV as shown in Figure 2 are given by:

$$\begin{aligned} P_{NI} &= \int_0^{H_+} \int_0^{H_N} PDF(R_{+0}, R_{N0}; R_+, R_N; \tau) dR_+ dR_N \\ &= N_2(A_+, A_N, \rho_+) \end{aligned} \quad (B.5)$$

$$\begin{aligned} P_{NII} &= \int_0^{H_+} \int_{H_N}^{\infty} PDF(R_{+0}, R_{N0}; R_+, R_N; \tau) dR_+ dR_N \\ &= N(A_+) - N_2(A_+, A_N, \rho_+) \end{aligned} \quad (B.6)$$

$$\begin{aligned}
P_{NIII} &= \int_{H_+}^{\infty} \int_{H_N}^{\infty} PDF(R_{+0}, R_{N0}; R_+, R_N; \tau) dR_+ dR_N \\
&= 1 - N(A_+) - N(A_N) + N_2(A_+, A_N, \rho_+)
\end{aligned} \tag{B.7}$$

$$\begin{aligned}
P_{NIV} &= \int_{H_+}^{\infty} \int_0^{H_N} PDF(R_{+0}, R_{N0}; R_+, R_N; \tau) dR_+ dR_N \\
&= N(A_N) - N_2(A_+, A_N, \rho_+)
\end{aligned} \tag{B.8}$$

where

$$A_+ = \frac{\ln \frac{H_+}{R_{+0}} - (\mu - \frac{1}{2} \sigma_+^2) \tau}{\sigma_+ \sqrt{\tau}}; \quad A_N = \frac{\ln \frac{H_N}{R_{N0}} - (\mu - \frac{1}{2} \sigma_N^2) \tau}{\sigma_N \sqrt{\tau}} \tag{B.9}$$

Accordingly, RCProb are expressed as:

$$\frac{dP_{NI}}{dt} = P_{A_+} \frac{\partial A_+}{\partial \tau} N\left(\frac{A_N - \rho_+ A_+}{\sqrt{1 - \rho_+^2}}\right) + P_{A_N} \frac{\partial A_N}{\partial \tau} N\left(\frac{A_+ - \rho_+ A_N}{\sqrt{1 - \rho_+^2}}\right) \tag{B.10}$$

$$\frac{dP_{NII}}{dt} = P_{A_+} \frac{\partial A_+}{\partial \tau} \left[1 - N\left(\frac{A_N - \rho_+ A_+}{\sqrt{1 - \rho_+^2}}\right)\right] - P_{A_N} \frac{\partial A_N}{\partial \tau} N\left(\frac{A_+ - \rho_+ A_N}{\sqrt{1 - \rho_+^2}}\right) \tag{B.11}$$

$$\frac{dP_{NIII}}{dt} = -P_{A_+} \frac{\partial A_+}{\partial \tau} \left[1 - N\left(\frac{A_N - \rho_+ A_+}{\sqrt{1 - \rho_+^2}}\right)\right] - P_{A_N} \frac{\partial A_N}{\partial \tau} \left[1 - N\left(\frac{A_+ - \rho_+ A_N}{\sqrt{1 - \rho_+^2}}\right)\right] \tag{B.12}$$

$$\frac{dP_{NIV}}{dt} = -P_{A_+} \frac{\partial A_+}{\partial \tau} N\left(\frac{A_N - \rho_+ A_+}{\sqrt{1 - \rho_+^2}}\right) + P_{A_N} \frac{\partial A_N}{\partial \tau} \left[1 - N\left(\frac{A_+ - \rho_+ A_N}{\sqrt{1 - \rho_+^2}}\right)\right] \tag{B.13}$$

where

$$\frac{\partial A_+}{\partial \tau} = -\frac{\ln \frac{H_+}{R_{+0}} + (\mu - \frac{1}{2} \sigma_+^2) \tau}{2\sigma_+ \sqrt{\tau^3}}, \quad \frac{\partial A_N}{\partial \tau} = -\frac{\ln \frac{H_N}{R_{N0}} + (\mu - \frac{1}{2} \sigma_N^2) \tau}{2\sigma_N \sqrt{\tau^3}}$$

$$P_{A_+} = \frac{1}{\sqrt{2\pi}} e^{-\frac{A_+^2}{2}}, \quad P_{A_N} = \frac{1}{\sqrt{2\pi}} e^{-\frac{A_N^2}{2}}.$$

The distress correlation for R_+ and R_N is expressed as:

$$\begin{aligned}
\rho_{ND} &= \frac{P_{NIII} - P_{R+}P_{RN}}{\sqrt{P_{R+}(1 - P_{R+})}\sqrt{P_{RN}(1 - P_{RN})}} \\
&= \frac{N_2(A_+, A_N, \rho_+) - N(A_+) \cdot N(A_N)}{\sqrt{N(A_+)(1 - N(A_+))}\sqrt{N(A_N)(1 - N(A_N))}} \quad (B.14)
\end{aligned}$$

where

$$P_{R+} = 1 - N(A_+) \quad (B.15)$$

$$P_{RN} = 1 - N(A_N). \quad (B.16)$$

Appendix C

We let $x_i = \ln(R_i/H_i)$ such that Eq.(1) becomes:

$$dx_i = \left(\mu - \frac{1}{2} \sigma_i^2 \right) dt + \sigma_i dz_i . \quad (\text{C.1})$$

Then we have the following statistical measures for x_i at time t :

1. Means or first moments:

$$\langle x_i \rangle = x_{i0} + \left(\mu - \frac{1}{2} \sigma_i^2 \right) (t - t_0) \quad (\text{C.2})$$

2. Second moments:

$$\langle x_i^2 \rangle = \langle x_i \rangle^2 + 2\sigma_i^2(t - t_0) \quad (\text{C.3})$$

3. Variances:

$$\langle x_i^2 \rangle - \langle x_i \rangle^2 = 2\sigma_i^2(t - t_0) \quad (\text{C.4})$$

4. Standard Deviations:

$$\Delta x_i = \sqrt{\langle x_i^2 \rangle - \langle x_i \rangle^2} = \sqrt{2\sigma_i^2(t - t_0)}. \quad (\text{C.5})$$

Given the initial value x_{i0} , the expected value of x_i at $t > t_0$ most likely lies between $\langle x_i \rangle - \xi \Delta x_i$ and $\langle x_i \rangle + \xi \Delta x_i$, where $1 < \xi \leq 2.5 \equiv \xi_0$. According to Eqs.(10) and (B.9), given $\sigma_i = 0.5$ which is about the lower limit of the volatility associated with the bond spreads in Figure 3, the range of expected values cover 99.987% of the distribution with $\xi_0 = 2.5$ and $(t - t_0) = 0.25$. Thus, for x_i breaching the boundary at the origin, we require that

$$\langle x_i \rangle + \xi_0 \Delta x_i > 0 \quad (\text{C.6})$$

$$\Rightarrow x_{i0} > -\left(\mu - \frac{1}{2} \sigma_i^2 \right) (t - t_0) - \xi_0 \sqrt{2\sigma_i^2(t - t_0)} \quad (\text{C.7})$$

$$\Rightarrow \frac{R_{i0}}{H_i} > e^{-\left(\mu - \frac{1}{2} \sigma_i^2 \right) (t - t_0) - \xi_0 \sqrt{2\sigma_i^2(t - t_0)}} \quad (\text{C.8})$$

As a result, for $\mu = 0$ the critical value of R_{i0} is defined by Eq.(18).