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MEASUREMENT ERROR AND POLICY EVALUATION IN THE FREQUENCY DOMAIN

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Abstract

This paper explores frequency-specific implications of measurement error for the design of stabilization policy rules. Policy evaluation in the frequency domain is interesting because the characterization of policy effects frequency by frequency gives the policymaker additional information about the effects of a given policy. Further, some important aspects of policy analysis can be better understood in the frequency domain than in the time domain. In this paper, I develop a rich set of design limits that describe fundamental restrictions on how a policymaker can alter variance at different frequencies. I also examine the interaction of measurement error and model uncertainty to understand the effects of different sources of informational limit on optimal policymaking. In a linear feedback model with noisy state observations, measurement error seriously distorts the performance of the policy rule that is optimal for the noise-free system. Adjusting the policy to appropriately account for measurement error means that the policymaker becomes less responsive to the raw data. For a parameterized example which corresponds to the choice of monetary policy rules in a simple AR (1) environment, I show that an additive white noise process of measurement error has little impact at low frequencies but induces less active control at high frequencies, and even may lead to more aggressive policy actions at medium frequencies. Local robustness analysis indicates that measurement error reduces the policymaker's reaction to model uncertainty, especially at medium and high frequencies.

Keywords: Policy Evaluation, Measurement Error, Spectral Analysis, Design Limits, Model Uncertainty, Monetary Policy Rules

JEL Classification: C52, E52, E58

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1. Introduction

Measurement error is well understood to exist in most macroeconomic data. The fact that data are *ex post* revised from time to time indicates how common measurement error can be. For example, the U.S. Bureau of Economic Analysis monthly releases its updated measure of GDP and price indices of recent quarters. In 1983-2009, the average revision without regard to sign is about 1.1% for current-dollar quarterly GDP and 1.3% for real quarterly GDP [Fixler *et al* (2011, Page 12)]. Historical data are also *ex post* revised based on more complete information, as well as changes to methodology intended to more accurately reflect economic activities. About every five years, the U.S. government issues comprehensive revisions to past estimates of GDP. The latest July 2009 revisions reach back to 1929.[1] This is of course hardly unique to the United States. One striking example is China's 2005 GDP revision. In light of the country's first nationwide economic census, China's statistics bureau revised its measure of 2004 national GDP upward by 16.8%. This substantial revision moved China above Italy as the sixth-largest economy in the world in 2004. For variables that are defined as the differences between actual and baseline values, the measurement problems become even acuter under structural change when baseline values may vary unpredictably. As an example, Orphanides *et al* (2000) and Orphanides and van Norden (2002) empirically documented errors in the measurement of the output gap for the U.S. economy, a part of which arise from the unobservable baseline of potential GDP. This type of data noise is serious when it is difficult to distinguish temporary shocks from permanent changes.

Monetary policy must be made in real time and so necessarily uses noisy data. Standard policy rules represent mappings from current and past economic conditions to monetary policy instruments such as the money supply or interest rates. For example, the famous Taylor (1993) rule is a linear mapping of observations of inflation and the output gap to the federal funds rate. At the time of the interest rate choice, the data available are therefore preliminary and with considerable measurement noise. In short, policymakers must live with and account for measurement error. But how should measurement error affect policy choices? How does measurement error affect the robustness properties of policy rules when the knowledge about fundamental economic structures is imperfect? Does this information constraint justify policy cautiousness, and if it does, how? Considering the possibly large welfare costs and long-lasting economic consequences associated with inflation and economic fluctuations, these questions are important in assessing alternative monetary policies.

To address these issues, this paper contributes to the policy evaluation literature by investigating the implications of measurement error for the design of stabilization policy rules in the frequency domain. As pointed out by Orphanides (2003), this informational limitation on the true macroeconomic

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[1] The Federal Reserve Bank of Philadelphia maintains a real-time dataset of the U.S. economy which consists of 23 quarterly macroeconomic variables from 1965 to the present and includes historical revisions to these variables in great details [Croushore and Stark (2001)].

variables facing policymakers has been noticed in policy analysis since at least Friedman (1947). Orphanides' (2001, 2003) own work largely reignites research interest in monetary policy evaluation with noisy information; recent contributions include Aoki (2003), Coenen *et al* (2005), Croushore and Evans (2006), Molodtsova *et al* (2008), and Orphanides and Williams (2002) among others. Many of these studies focus on the use of real-time data and evaluate the performance of real-time policies against *ex post* revised data.[2] However, there has yet to be any systematic examination of the role of measurement error on policy choice in the frequency domain. Frequency domain approaches have been part of macroeconomic analysis for several decades – Hansen and Sargent (1980), Whiteman (1985, 1986), and Sargent (1987) are standard examples – and have recently experienced a resurgence in the context of policy evaluation [*e.g.*, Brock and Durlauf (2005), Brock *et al* (2008a) and Hansen and Sargent (2008, Chapter 8)]. The current paper develops strategies to characterize frequency-specific performance of alternative policy rules in exposure to data noise.

There are significant reasons why frequency-specific analysis is important for policy evaluation. First, a full characterization of policy effects frequency by frequency is informative to policymakers. In the frequency domain, stabilization policy may be understood as determining the spectral density matrix of the state variables concerned. A full understanding of the effects of a policy rule requires evaluating how cycles at all frequencies are reshaped by the policy. When variances at some frequencies have greater social welfare costs than variances at other frequencies, it is necessary to know frequency-specific performance of alternative policies in order to make sound policy recommendations. This differential weighting of variance by frequency will occur, for example, when the social loss function involves non-time-separable preferences [Otrok (2001)]. In the case of committee policymaking, it is possible that some committee members care more about performance at low frequencies while others care more about performance at high frequencies, hence this information is needed to allow for successful group decisionmaking.

Second, a number of properties of stabilization policies can really only be understood in the frequency domain. Policies that perform well at all frequencies are naturally appealing to policymakers regardless of their preferences. However, it turns out that such policies do not exist. Even if a policy reduces aggregate variance relative to some baseline, for the framework I study this will necessitate increasing variance at some frequencies in exchange for reducing variance at others. These tradeoffs are known as design limits. They were first identified by Bode (1945) in the engineering literature of linear system control and were introduced into the study of feedback policy rules in macroeconomics by Brock and Durlauf (2004, 2005) with extension to the vector case with forward-looking elements developed by Brock *et al* (2008b). These design limits are sufficiently complicated in the time domain as to render use impractical outside of the frequency domain.

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[2] For example, Orphanides *et al* (2000) and Orphanides and van Norden (2002) showed that measurement errors are significantly large in real-time estimates of the output gap so as to render the estimates highly unreliable as guides to policymaking if data noise is not appropriately accounted for.

In this direction, the current paper contributes to the existing literature by studying design limits in the presence of measurement error. As such, the paper extends the study of design limits to the empirically salient case in which a policymaker is ignorant the true state of the economy due to measurement imperfections. In the linear feedback control system, the presence of measurement error creates new design limits than those that have been identified. Intuitively, a feedback policy rule introduces undesirable side noise into the system responding to noisy data, when it exerts influences on the state variable to stabilize the economy. And when the responses are aggressive, the side noise effects are also strong. Therefore, good variance-reducing control has to be traded off against suppression of side noise. Put differently, facing noisy data the policymaker has to make tradeoffs across frequencies as well as between the channels of stabilizing control effects and side noise effects. These constraints are summarized by two concepts – Bode's (1945) integral formula and the complementary principle [Skogestad and Postlethwaite (1996, Chapter 5)], and are thus amenable to analytical treatment. This paper is the first to put them to work in the practice of policy evaluation.

I further examine the effects of measurement error on policy design in the presence of model uncertainty. When policymakers are uncertain about the true model of the economy, the conventional wisdom is that policy reactions to the observed state variables should be less aggressive; see Brainard (1967) for a classic example with parameter uncertainty and a known probability density on the parameters, and Giannoni (2002) for a recent case of parameter uncertainty but unknown distribution. However, little has been known about policy behavior when the observations are also noisy. Allowing for both sources of uncertainty – measurement error and model uncertainty, the analysis presented in this paper sheds light on robustness of efficient policy rules that recognize the presence of data noise. Following the literature on robustness pioneered by Hansen and Sargent (2001, 2003, 2008), I assume that the true model is local to a baseline model and employ the minimax decision criterion to evaluate policies against potential deviations from the baseline. The minimax criterion is appealing in this context because deviations from the baseline are, given the assumption of local model uncertainty, empirically indistinguishable so that, unlike in Brainard (1967), one has no basis outside of prior beliefs for assigning probabilities to alternative models. This paper models potential model misspecifications as variations of the spectral density function of the state variable, which conceptually distinguishes from Giannoni's (2002) parameter uncertainty. This nonparametric approach to model uncertainty allows one to apply design limit results in a straightforward fashion to construct robust feedback policies, *i.e.* policies that work well regardless of which element of the model space is the true model.

Concretely, this paper focuses on linear feedback control rules in a one-equation backwards-looking model with single control input.[3] The control is chosen by a policymaker to stabilize the economy in

[3] Central banks may also need to account for the effects of policies on expectations. Thus, it is important to address measurement error issues in a forward-looking framework as well. I leave this problem for future research. However, as Fuhrer (1997) has shown, expectations of future prices are empirically unimportant in explaining price and inflation behavior. In light of such evidence, the attention of this paper to backward looking models is natural and also relevant.

the sense of minimizing the variance of the state variable of interest. I show how measurement error, which produces a stochastically perturbed optimization problem with noisy control in the time domain, can be represented as a deterministic perturbation in the frequency domain. Further, each feedback rule can be associated with a sensitivity function in the frequency domain which describes how the policy shapes the spectral density of the state variable of interest. I characterize how the sensitivity function behaves differently in the presence and absence of data noise to show the frequency-specific effects of measurement error on optimal and robust policies.

Different policy scenarios are considered and compared. First, the policymaker simply ignores the measurement issue and naively adopts the optimal policy for the standard noise-free problem. I show that this leads to a policy rule that is excessively aggressive. Failing to acknowledge data noise causes undesirable side effects with an activist policy. Second, I consider the case in which the policymaker is aware of measurement error and uses this information to adjust to an efficient policy rule design. The policymaker is more cautious in this case, but the adjustments in optimal policy relative to the noise-free case differ across frequencies. Third, I examine the situation in which the policymaker filters the noisy data to reduce measurement inaccuracy and applies the noise-free optimal rule to the filtered data. The Wiener filter used in this scenario effectively accounts for measurement error, as shown in the numerical exercises. Fourth, following Brock and Durlauf (2005), I perform local robustness analysis by approximating the robust policy solution with a small level of model uncertainty around the equilibrium solution to the baseline model using the minimax criterion. This worst-case study is modeled as a zero-sum game by introducing an adversarial agent who selects a model from a small neighborhood of the baseline model to maximize the loss function against the policymaker. I show the differences between the robust and standard solutions to illustrate the interaction between concerns over measurement noise and model uncertainty in policymaking.

Finally, for numerical implementation and more specific conclusions on monetary policy, I apply the theoretical analysis to an AR(1) monetary model, which is a variant of the two-equation Keynesian model. I start with a simple rule in which the control only depends on the current state; lagged terms are excluded. This is interesting because the effects of measurement error can be characterized by one single coefficient parameter in this simple case. Also, simple instrument rules have received much attention in related literature. The optimal policy adjusted for the measurement turns out to be less responsive than a naive policy that assumes measurement is exact. In frequency domain, adjusting the policy design or filtering the data to account for measurement error, which is assumed to be a white noise process, the policymaker generally becomes less responsive to raw data observations; yet, the data filtering method outperforms the policy adjusting approach. Measurement error has little impact at low frequencies but results in more cautious policy reaction at high frequencies, and even may lead to more active control at medium frequencies. Without measurement error, model uncertainty has similar policy implications; it has little effect at low frequencies, reduces the strength of control at high frequencies, and increases control at medium frequencies. Therefore, Brainard's (1967) intuition that model uncertainty leads to less effective policies follows in the sense of the total effect over the whole frequency domain, but fails at medium frequencies where the robust control is actually becoming more active. Introducing measurement error, however, the policymaker will reduce his reaction to model uncertainty, especially at medium and high frequencies. In other words, facing various types of uncertainty the policymaker's reaction to one type of uncertainty is weakened by his attention to another type, although such effects differ across frequencies.

2. General Framework

To develop a general framework for policy analysis in the presence of measurement error, I start by formulating the policymaker's problem in the frequency domain. Then, I turn to consider scenarios in which the policymaker reacts to mismeasurements differently, and evaluate policy performance in each case.

2.1 Policy Evaluation in the Frequency Domain

Suppose that the economy is governed by a simple scalar version of a backwards-looking dynamic system and the policymaker wishes to stabilize the economy in the sense of minimizing the variance of the economic variable of concern. The system is described as

$$x_t = A(L)x_{t-1} + B(L)u_{t-1} + W(L)\varepsilon_t, \tag{1}$$

where x_t is the unobserved random variable of interest with measure x_t^*, both of which have zero means, and $\{\varepsilon_t\}$ is a process of fundamental innovations with variance σ_ε^2. The control u_{t-1} is restricted to be a linear feedback rule and only feeds back to current and past observations, rather than underlying true realizations, of variable x_t:

$$u_{t-1} = -F(L)x_{t-1}^*. \tag{2}$$

When there is no measurement error in the data, $x_t^* = x_t$ for any t. Otherwise, I model measurement error as an additive process $\{n_t\}$ to $\{x_t\}$, that is,

$$x_t^* = x_t + n_t, \qquad \text{with } n_t = D(L)\eta_t. \tag{3}$$

Here, $\{\eta_t\}$ is another process of fundamental innovations orthogonal to $\{\varepsilon_t\}$, $\mathbb{E}[\eta_t\varepsilon_{t-k}] = 0, \forall k$, and hence uncorrelated with $\{x_t\}$. The variance of η_t is σ_η^2. The linear representation of n_t does not require that that measurement error is white noise, as is usually assumed without justification. Thus, $\{n_t\}$ may be autocorrelated. The objective of stabilization policy design is to minimize the unconditional variance of x_t,

$$V(x) = \mathbb{E}[x_t^2], \tag{4}$$

by choosing an optimal rule u_{t-1}. I consider this very simple loss function to focus on the study of measurement error effects, but the analysis can be extended to the case of more complicated preferences without difficulty, especially when I move to solve the problem in the frequency domain. In the presence of measurement error, the challenge for the policymaker is that he only observes x_t^* but needs to stabilize x_t. If there is no such data limitation, he simply solves a standard feedback control problem with a single input and a single output.

The lag polynomials $A(L)$, $B(L)$, $W(L)$, $F(L)$, and $D(L)$ are assumed to have only nonnegative-power terms. This one-sided specification rules out any forward-looking element in the model, and allows us to avoid complexities that arise from the formation of expectations. Some studies, for example, Fuhrer (1997), have shown that expectations elements are not empirically important in explaining the dynamics of inflation and output. Previous work on measurement error such as Orphanides (2001, 2003) also uses backwards-looking models.

In the time domain, when the control is set to zero, $F(L) = 0$, the *uncontrolled state variable* x_t^{nc} has a moving-average representation as

$$x_t^{nc} = (1 - A(L)L)^{-1} W(L)\varepsilon_t. \tag{5}$$

I assume that the uncontrolled model (5) itself is stationary. This is without loss of generality because the unit root can be removed from a nonstationary model before we take it to consider the policy question. On the other hand, when a noisy feedback rule $F(L)$ as specified by (2) is applied, the *controlled state variable* x_t^c takes the form of

$$x_t^c = -T(L)n_t + S(L)x_t^{nc}, \tag{6}$$

where $S(L) \equiv \frac{1}{1+G(L)}$ and $T(L) \equiv \frac{G(L)}{1+G(L)}$, with $G(L) \equiv [1 - A(L)L]^{-1}[B(L)F(L)L]$. Stabilization imposes the invertibility of $G(L)$ and $1 + G(L)$. It is clear from equation (6) that there are two sources of volatility for the controlled state variable x_t^c: original system and measurement error. Specifically, there are irreducible stochastic components in the state variable, and the feedback control causes undesirable side noise effects when responding to noisy data.

I use the following notations to facilitate work in the frequency domain. The Fourier transform of the coefficients of an arbitrary lag polynomial $C(L)$, $C(e^{-i\omega}) = \sum_{j=-\infty}^{\infty} C_j e^{-ij\omega}$, is denoted as $C(\omega)$, $\omega \in [-\pi, \pi]$. I define the *sensitivity function* $S(\omega)$ and the *complementary sensitivity function* $T(\omega)$ in the frequency domain associated with a given feedback rule $F(L)$ as the Fourier transforms of $S(L)$ and $T(L)$, respectively. Thus,

$$S(\omega) = \frac{1}{1 + G(\omega)} \text{ and } T(\omega) = \frac{G(\omega)}{1 + G(\omega)}, \tag{7}$$

where $G(\omega) = [1 - A(e^{-i\omega})e^{-i\omega}]^{-1}[B(e^{-i\omega})F(e^{-i\omega})e^{-i\omega}]$. In general, both $S(\omega)$ and $T(\omega)$ are complex functions. They sum up to one at each frequency in $[-\pi, \pi]$,

$$T(\omega) + S(\omega) = 1. \tag{8}$$

This is known as the complementary principle [Skogestad and Postlethwaite (1996, Chapter 5)].

To specify a feedback rule $F(L)$ in the time domain is equivalent to characterizing the associated $S(\omega)$ and $T(\omega)$ in the frequency domain. Notice that all elements except $F(L)$ in the expressions of $S(L)$ and $T(L)$ are exogenously determined. Thus, a chosen feedback rule $F(L)$ will determine $S(L)$ and $T(L)$, and in turn determine their Fourier transforms $S(\omega)$ and $T(\omega)$. On the other hand, the coefficients of the lag polynomials $S(L)$ and $T(L)$, and therefore those of $F(L)$, can be discovered from $S(\omega)$ and $T(\omega)$ by the Fourier recovery formula [Priestley (1981, Chapter 4)]. Therefore, one can always infer the feature of a linear feedback rule by reverse engineering, once clear about the behavior of its sensitivity function. For the rest of the paper, instead of deriving feedback policy rules directly, I will focus on the characterization of their sensitivity functions to study the frequency-specific implications of measurement error.

Every second-order stationary stochastic process admits a *spectral density function* that describes how the variance of a time series is distributed with frequency. Let $f_{x^c}(\omega)$, $f_{x^{nc}}(\omega)$, and $f_n(\omega)$ be the spectral density functions of controlled state variable x_t^c, uncontrolled state variable x_t^{nc}, and measurement error n_t. Given the expression (6) of x^c, I can represent its spectral density function $f_{x^c}(\omega)$ as

$$f_{x^c}(\omega) = |T(\omega)|^2 f_n(\omega) + |S(\omega)|^2 f_{x^{nc}}(\omega), \tag{9}$$

where $|\cdot|$ stands for complex modulus, $|S(\omega)|^2 = S(\omega)S^*(\omega)$ and $|T(\omega)|^2 = T(\omega)T^*(\omega)$, with superscript $*$ referring to the complex conjugate. By applying the spectral representation theorem to (6), this result follows immediately from the fact that η_t and ε_t are independent from each other [Priestley (1981, Chapter 4)].[4]

[4] A quick derivation: Let $dZ(\omega)$ be the increment in power, under a Fourier integral, over an infinitesimal interval $d\omega$, then $dZ_{x^c}(\omega) = S(\omega)dZ_{x^{nc}}(\omega) - T(\omega)dZ_n(\omega)$. Since the independence, $\mathbb{E}[dZ_{x^{nc}}(\omega)dZ_n(\omega)] = 0$. It follows that $f_{x^c}(\omega) = \frac{\mathbb{E}|dZ_{x^c}(\omega)|^2}{d\omega} = \frac{\mathbb{E}|S(\omega)dZ_{x^{nc}}(\omega) - T(\omega)dZ_n(\omega)|^2}{d\omega} = |T(\omega)|^2 f_n(\omega) + |S(\omega)|^2 f_{x^{nc}}(\omega)$.
This expression also holds as a special case of the cross spectrum formula [see Sargent (1987, Page 248)].

Equation (9) conveys the nature of the policy design problem with noisy data. The variance is just the integral of a variable's spectral density function over $[-\pi, \pi]$. Functions $f_{x^{nc}}(\omega)$ and $f_n(\omega)$ represent the allocation of variance across frequencies for the uncontrolled state variable x_t^{nc} and measurement error n_t, respectively. The sensitivity function $S(\omega)$ specifies how the control redistributes the variance of the state variable frequency-by-frequency by reshaping $f_{x^{nc}}(\omega)$. Since the control responds to noisy observations, it introduces additional noise into the system. The complimentary sensitivity function $T(\omega)$ captures the size of this side effect at each frequency, which acts on the variance distribution $f_n(\omega)$ of measurement error. At each frequency, the total effect from the two channels is then summed up as given in (9). The variance of the state variable under control is distributed with frequency according to $f_{x^n}(\omega)$.

An ideal control rule would be one that lowers variance at every frequency to zero. However, limitations in control design make it impossible to reduce power uniformly over the whole frequency domain, even without data noise effects. This is known as Bode's (1945) constraint; for a backwards-looking system, the sensitivity function associated with a feedback rule is subject to the integral restriction

$$\int_{-\pi}^{\pi} ln(|S(\omega)|^2)d\omega = K_s, \ with \ K_s \geq 0. \tag{10}$$

If the uncontrolled system is stationary as assumed, then $K_s = 0$; otherwise, $K_s > 0$. Notice that if $|S(\omega)| < 1$ for all $\omega \in [-\pi, \pi]$, then $K_s \geq 0$ cannot hold in equation (10). Therefore, reductions of variance at some frequencies for the state variable induce increases in its variance at other frequencies. Policy design has to make tradeoffs among the magnitudes of different frequency-specific variance contributions. A simple proof of this constraint (10) on sensitivity function can be found in Section A.1 of the Appendix.[5]

Now let me state the policy design problem in the frequency domain. Recall that $\mathbb{E}[x_t^2] = \int_{-\pi}^{\pi} f_{x^c}(\omega)d\omega$ in the loss function (4), which can be conveniently generalized to more complicated – for example, non-time-separable – preferences using a general weighting function of variance by frequency. Notice that $f_{x^c}(\omega)$ is given by (9). Then, I first consider the standard noise-free control problem as a benchmark. Since $f_n(\omega) = 0$ for all $\omega \in [-\pi, \pi]$, $T(\omega)$ becomes irrelevant in the objective function. The policymaker's problem is just to choose a sensitivity function $S(\omega)$, which is equivalent to his choice of a feedback rule $F(L)$ in the time domain, to solve

$$\min_{S(\omega)} \int_{-\pi}^{\pi} [|S(\omega)|^2 f_{x^{nc}}(\omega)]d\omega, \tag{11}$$

[5] Since $S(\omega)$ and $T(\omega)$ sums up to 1 at each frequency, there exists a similar Bode's integral constraint and interpretation for $T(\omega)$. A recent version of proof for Bode's integral constraint on $T(\omega)$ can be found at Okanoa *et al* (2009). I will only work with the constraint on sensitivity function throughout this paper.

subject to Bode's constraint (10) with $K_s = 0$. Following Brock and Durlauf's (2005) argument, the optimal feedback rule is to reduce the system to a white noise process. In this benchmark case, the controlled state variable is

$$x_t^B = [1 - (A(L) - B(L)F(L))L]^{-1}W(L)\varepsilon_t. \tag{12}$$

Then, the optimal feedback rule $F^B(L)$ is

$$F^B(L) = B(L)^{-1}([W(L)L^{-1}]_+ + A(L)), \tag{13}$$

where $[\ ]_+$ is the annihilation operator. The sensitivity function $S^B(\omega)$ associated with this benchmark control satisfies

$$|S^B(\omega)|^2 = \frac{\sigma_\varepsilon^2}{2\pi f_{x^{nc}}(\omega)}. \tag{14}$$

In the frequency domain, the benchmark control targets the constant spectral density function of fundamental innovations for the controlled state variable and achieves that by reallocation of variance across frequencies.

Back to the interesting case with noisy data, $T(\omega)$ enters into equation (9) and hence the objective function (4). The policymaker chooses both $S(\omega)$ and $T(\omega)$ to minimize the loss function,

$$\min_{S(\omega),T(\omega)} \int_{-\pi}^{\pi} [|T(\omega)|^2 f_n(\omega) + |S(\omega)|^2 f_{x^{nc}}(\omega)] d\omega, \tag{15}$$

subject to the complementarity condition (8) and Bode's constraint (10). As noted above, in the presence of measurement error, any control introduces side noise into the system, which is captured by the complementary sensitivity function. The policymaker then has to balance the conflicting effects between stabilizing control and side noise. Put differently, a compromise has to be made in control design when information is noisy; good control and disturbance rejection must be traded off against suppression of side noise process. This is why the complementary principle becomes important in this context. I will consider scenarios in which the policymaker deals with measurement error differently in the next subsections.

Let me close this subsection by a discussion on the notion of *control aggressiveness*. It is natural to expect that measurement error may change the aggressiveness of the control used by the policymaker in response to current and past observations. In the time domain, the coefficients of the feedback rule $F(L)$ reflect responses to the data. Focusing on nonparametric sensitivity functions in the frequency domain, however, there is not such an obvious measure. I therefore propose the following notion of aggressiveness AG:

$$AG \equiv \left[\int_{-\pi}^{\pi} |1 - |S(\omega)||^2 d\omega\right]^{\frac{1}{2}}. \quad (16)$$

In L^2 norm, it measures how close the modulus function $|S(\omega)|$ to the constant function 1. Notice that $|S(\omega)| = 1$ means that the control is inactive at ω, while the more $|S(\omega)|$ deviates from 1, the more powerful the control is at this frequency either shifting up or down the uncontrolled spectral density function. AG measures control aggressiveness by the total deviation of $|S(\omega)|$ from 1 over the whole interval $[-\pi, \pi]$. Furthermore, similar to the benchmark case, I show that the policymaker's behavior when he faces noisy data can still be interpreted as targeting some constant level λ for the spectral density function of the state variable, although this may not be actually achieved due to side noise effects. In the same environment, the higher the level λ the policymaker targets, the less aggressive the control would be.[6] Hence, I consider the target level for the controlled spectral density function as another way for understanding control aggressiveness. In the numerical exercises, I will use both notions.

2.2 Naive Policy Rule

A policymaker's decision depends on his knowledge of measurement error. Let me first examine the case in which he is not aware of measurement issues and simply considers the observations as the true realizations of the state variable.

Since the policymaker knows the model, then he just naively adopts the optimal feedback rule for the noise-free system $F^B(L)$ from (13) even in the presence of measurement error. That is,

$$u_{t-1}^N = -F^B(L)x_{t-1}^*. \quad (17)$$

The associated sensitivity function $S^N(\omega)$ is still the same as (14),

$$|S^N(\omega)|^2 = \frac{\sigma_\varepsilon^2}{2\pi f_{x^{nc}}(\omega)}. \quad (18)$$

I refer to u_{t-1}^N or $S^N(\omega)$ as the *naive policy rule* in Orphanides(2003) terminology. By using this naive rule, the policymaker still targets the constant spectral density function of fundamental innovations for the state variable:

[6] If the original systems are different, then the same target level λ for the spectral density may reflect different degrees of control aggressiveness in the AG measure.

$$\lambda^N = \frac{\sigma_\varepsilon^2}{2\pi}. \tag{19}$$

However, the naive policy rule is inefficient with noisy data. The state variable under the control of the naive rule obeys

$$x_t^N = \varepsilon_t + W(L)^{-1}[1 - W(L) - A(L)L]D(L)\eta_t, \tag{20}$$

which admits spectral density function

$$f_{x^N}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} + |D(\omega)|^2|1 - S^N(\omega)|^2\frac{\sigma_\eta^2}{2\pi}. \tag{21}$$

Therefore, the naive policy rule is unable to achieve its target spectral density as in the noise-free benchmark; rather, it induces extra volatility at each frequency because of control noise effects.

The performance of the naive policy rule illustrates the consequences of ignoring potential measurement error in policymaking. The noise-free optimal policy rule will not be able to reduce the system to the white noise as it would when measurement error was absent. As a special case, even when the measurement error process is white noise, $D(\omega) = 1, \forall\omega \in [-\pi, \pi]$, the system under the control of the naive policy rule is still not a white noise process; this can be verified by observing that $|1 - S(\omega)|$ is not constant over frequencies in expression (21). The second term of (21) describes the frequency-specific effects of measurement error when the policymaker uses the naive policy rule. As shown in the numerical exercises in Section 4, either in terms of the AG measure or spectral target, the naive policy rule is too active relative to the optimal policy rule that solves (15).

2.3 Optimal Policy Rule with Nonfiltered Data

Suppose now that the policymaker recognizes the presence of measurement error as well as the error-generating process. He does not filter the data, but chooses an optimal control to account for measurement error and stabilize the economy. The policymaker still restricts himself within the set of linear feedback rules as specified by (2). I will study the properties of this *optimal policy rule with nonfiltered data* by characterizing its sensitivity function $S^E(\omega)$.

The policymaker's optimization problem is posed as (15) in the frequency domain. From (8),

$$|T(\omega)|^2 = 1 + |S(\omega)|^2 - 2|S(\omega)|\cos(\theta(\omega)), \tag{22}$$

where $\theta(\omega)$ is the phase angle of $S(\omega)$ in the complex plane at frequency ω.[7] To focus on the choice of $S(\omega)$, I substitute this expression into the objective function and rewrite problem (15) as

$$\min_{|S(\omega)|,\cos(\theta(\omega))} \int_{-\pi}^{\pi} [(1 + |S(\omega)|^2 - 2|S(\omega)|\cos(\theta(\omega)))f_n(\omega) + |S(\omega)|^2 f_{x^{nc}}(\omega)]d\omega, \tag{23}$$

subject to (10). Thus, the policymaker's problem is reformulated as choosing function $|S(\omega)|$ and $\cos(\theta(\omega))$ in the frequency domain separately to minimize the loss function. This is intuitive because complex $S(\omega)$ is uniquely identified by its modulus and phase angle.

Notice that Bode's constraint is a restriction on the modulus of the sensitivity function. In equation (23), modulus $|S(\omega)|$ is subject to (10), but phase angle $\theta(\omega)$ or function $\cos(\theta(\omega))$ is not constrained. Since both $|S(\omega)|$ and $f_n(\omega)$ are nonnegative at all frequencies, the loss function (23) is nonincreasing in $\cos(\theta(\omega))$. Therefore, the optimal solution is to set $\cos(\theta(\omega)) = 1,\ \forall\omega \in [-\pi, \pi]$.[8] It then follows that for the optimal policy rule with nonfiltered data,

$$|T(\omega)|^2 = (1 - |S(\omega)|)^2. \tag{24}$$

Given $\cos(\theta(\omega)) = 1$, $\theta(\omega) = 0$, $\forall\omega \in [-\pi, \pi]$. Thus, the sensitivity function $S^E(\omega)$ associated with the optimal policy rule turns out to be a real function. This is different from the noise-free benchmark case in which $S^B(\omega)$ is not necessarily real. A sensitivity function characterizes how the feedback control transforms x_t^{nc} frequency by frequency to obtain x_t^c. In the frequency domain, "transform" means both gain and phase shifts at each frequency. A real sensitivity function implies no phase shifts or that all phase shifts are canceled out in the transform. The reason for $S^E(\omega)$ to be real is because $T^E(\omega)$ matters in this noisy control problem. As one can see from (15), a good control should have both $|S(\omega)|$ and $|T(\omega)|$ at all frequencies as small as possible. If $S(\omega)$ is not real and hence includes indelible phase shifts to x_t^{nc}, it will also induce indelible phase shifts to n_t. This will lead to a larger $|T(\omega)|$ than in the absence of such shifts. Intuitively, phase shifts also cause side noise effects, and therefore should be prevented when possible.[9]

Given equation (24), the policymaker's problem (23) reduces to

$$\min_{|S(\omega)|} \int_{-\pi}^{\pi} [(1 - |S(\omega)|)^2 f_n(\omega) + |S(\omega)|^2 f_{x^{nc}}(\omega)]d\omega, \tag{25}$$

7 Phase angle refers to the angular component of the polar coordinate representation of complex number.

8 In principle, if $f_n(\omega)$ vanishes to zero at some frequencies, then there may exist other solutions that $\cos(\theta(\omega)) \neq 1$ at these frequencies. This is not the generic case, so I withdraw from this technical issue.

9 Section A.2 of the Appendix provides a further example to illustrate the intuition for the real sensitivity function when accounting for measurement error.

subject to Bode's constraint (10). Measurement error stochastically disturbs the performance of any chosen policy rule in the time domain. However, the policy decision problem can be represented as a deterministically perturbed optimization problem in the frequency domain as (25): equation (25) is the same as benchmark (11) except that it includes the first term $(1-|S(\omega)|)^2 f_n(\omega)$, which is deterministic and captures frequency-specific costs of the control. The effects of stochastic measurement error are now characterized by a deterministic control cost function. This representation greatly simplifies calculation in the frequency domain.

To solve problem (25), let λ^E be the Lagrangian multiplier associated with Bode's constraint. The optimal solution gives the spectral density function of the controlled state variable as

$$f_{x^E}(\omega) = \lambda^E + (1-|S^E(\omega)|) f_n(\omega), \tag{26}$$

with the Lagrangian multiplier

$$\lambda^E = (|S^E(\omega)|-1)|S^E(\omega)| f_n(\omega) + |S^E(\omega)|^2 f_{x^{nc}}(\omega). \tag{27}$$

Equation (27) is also the first order conditions for the optimization problem (25). Therefore, the controlled spectral density $f_{x^E}(\omega)$ contains one constant term λ^E and the other component that varies across frequencies. This can be interpreted as follows. First, the optimal policy rule still tries to flatten the uncontrolled spectral density by targeting the constant function λ^E. Unlike the noise-free or naive-policy target (19) which is solely determined by system disturbance attenuation, the spectral target λ^E comprehensively accounts for the tradeoffs between stabilizing control effects and side noise effects, as shown in (27). Second, even though, the optimal control comes at an additional cost, captured by the second term of (26). For example, when the control shifts the uncontrolled density $f_{x^{nc}}(\omega)$ down at some frequency ω, *i.e.* $|S^E(\omega)| < 1$, the stronger the control is the more side noise $(1-|S^E(\omega)|) f_n(\omega)$ it brings into the system. When it lifts $f_{x^{nc}}(\omega)$ up, *i.e.* $|S^E(\omega)| > 1$, part of the control effect is offset by a downward force due to the fact that $(1-|S^E(\omega)|) f_n(\omega) < 0$ in this case. Only when the control exerts no impacts on $f_{x^{nc}}(\omega)$, *i.e.* $|S^E(\omega)| = 1$, this additional cost goes to zero. In other words, with noisy data, the effectiveness of the control is reduced in the sense that the control fails to reach its spectral target; at frequencies where the control is more forceful, the larger the deviations will be from the target.

To complete the characterization of the solution, I solve for $|S^E(\omega)|$ from (27),

$$|S^E(\omega)| = \frac{f_n(\omega) + \sqrt{f_n(\omega)^2 + 4\lambda^E (f_{x^{nc}}(\omega) + f_n(\omega))}}{2(f_{x^{nc}}(\omega) + f_n(\omega))}, \tag{28}$$

and determine λ^E by substituting (28) into Bode's constraint (10),

$$\int_{-\pi}^{\pi} \ln\left[\frac{f_n(\omega) + \sqrt{f_n(\omega)^2 + 4\lambda^E(f_{x^{nc}}(\omega) + f_n(\omega))}}{2(f_{x^{nc}}(\omega) + f_n(\omega))}\right]^2 d\omega = K_s. \tag{29}$$

The Lagrangian multiplier λ^E, which is also the spectral target, is important for understanding the optimal policy rule with nonfiltered data. Several observations follow. First, $\lambda^E > 0$.[10] This is guaranteed by Bode's constraint; if $\lambda^E < 0$, then (27) implies that $|S^E(\omega)| < 1$ at all frequencies, which contradicts the fact that $K_s \geq 0$. Second, as shown in the numerical exercises of Section 4, $\lambda^E \geq \lambda^N$ in general. The optimal policy rule targets a higher constant spectral density and hence is less aggressive than the naive policy rule. Third, in the special case where measurement error is absent, $f_n(\omega) = 0$, $\forall\omega$, formula (26) and (28) will give exactly the same controlled spectral density function and sensitivity function as obtained in the noise-free benchmark, *i.e.* $f_{x^E}(\omega)$ and $|S^E(\omega)|$ reduce to $f_{x^B}(\omega)$ and $|S^B(\omega)|$.[11]

As characterized above, the optimal policy rule with nonfiltered data behaves differently from the naive policy rule in the frequency domain. A numerical comparison between them in a parameterized model will be presented in Section 4. But the main message has been clear here that there is a need to account for measurement error in the design of stabilization policies.

2.4 Optimal Policy Rule with Filtered Data

Another natural method to address measurement error is to filter the data so that any chosen feedback rule can work more accurately. Suppose that the policymaker still sticks to the policy rule which is optimal for the benchmark noise-free system, but feeds back to the filtered data instead. I label this type of policies as *optimal policy rule with filtered data*.

Assume that the policymaker uses a linear filter $M(L)$,

$$\hat{x}_t = M(L)x_t^*. \tag{30}$$

He then applies the benchmark control $F^B(L)$ to the filtered data $\hat{x}_t$,

$$u_{t-1} = -F^B(L)M(L)x_{t-1}^*. \tag{31}$$

Therefore, the optimal policy rule with filtered data is

[10] Since λ^E is positive, the other root of (27) is negative and therefore is not a reasonable solution.

[11] Section A.3 of the Appendix shows that $f_{x^E}(\omega)$ and $|S^E(\omega)|$ reduce to $f_{x^B}(\omega)$ and $|S^B(\omega)|$ when there is no measurement error in the data.

$$F^C(L) = F^B(L)M(L). \tag{32}$$

The associated sensitivity function is the Fourier transform of

$$S^C(L) = [(1 - M(L))(1 - A(L)L) + M(L)W(L)]^{-1}(1 - A(L)L), \tag{33}$$

which has

$$|S^C(\omega)| = \left|\frac{1 - A(\omega)e^{-i\omega}}{(1 - M(\omega))(1 - A(\omega)e^{-i\omega}) + M(\omega)W(\omega)}\right|. \tag{34}$$

The spectral density function for the system under the control of $F^C(L)$ is

$$f_{x^C}(\omega) = \frac{\sigma_\varepsilon^2|W(\omega)|^2 + \sigma_\eta^2|D(\omega)|^2|M(\omega)|^2|1 - W(\omega) - A(\omega)e^{-i\omega}|^2}{2\pi|(1 - M(\omega))(1 - A(\omega)e^{-i\omega}) + M(\omega)W(\omega)|^2}. \tag{35}$$

Applying the optimal policy rule with filtered data $F^C(L)$, as shown in (35), the effects of the linear filter $M(L)$ are twofold: it affects the way the benchmark control $F^B(L)$ transforms the spectral density functions of the uncontrolled state variable x_t^{nc} and measurement error n_t, and it also changes the way control noise enters into the system. The former is captured by $M(\omega)$ in the denominator of (35), and the latter by the second term of (35).

It is clear that the filter $M(L)$ plays the central role in this policy scenario. If the filter is set to $M(L) = 1$, *i.e.* no filtering, then the above results immediately reduce to be the same as those of the naive policy rule. For the rest of this subsection, I will discuss the choice of the optimal filter $M(L)$. I argue that the Wiener filter is the most natural choice for the policymaker both because of its efficiency at filtering out measurement error and for its simple formula that is convenient for policy analysis in the frequency domain.

Since the data x_t^* is observed only up to time t, the Wiener filter employed by the policymaker must be *causal* in the sense that it is restricted to the one-sided form, [12]

$$M(L) = \sum_{j=0}^{\infty} m_j L^j. \tag{36}$$

[12] If the Wiener filter is in the two-sided form, $M(L) = \sum_{j=-\infty}^{+\infty} m_j L^j$, then it is called non-causal; if defined with finite order of lag polynomial $M(L)$, then called a finite impulse response (FIR) Wiener filter. Especially, the non-causal Wiener filter is such that in our case,

$$M(\omega) = \frac{f_x(\omega)}{f_{x^*}(\omega)} = \frac{f_{x^{nc}}(\omega)}{f_{x^{nc}}(\omega) + f_n(\omega)} = \frac{\sigma_\varepsilon^2|W(\omega)|^2}{\sigma_\varepsilon^2|W(\omega)|^2 + \sigma_\eta^2|D(\omega)|^2|1 - A(\omega)e^{-i\omega}|^2}.$$

The Wiener filter is efficient by the minimum mean-square error (MMSE) criterion, *i.e.*, it is such that $MMSE \equiv \mathbb{E}[(x_t - \hat{x}_t)^2]$ is minimized. The proof may be found in Priestley (1981, Chapter 10). Further, to give an explicit formula of the causal Wiener filter in the context of this paper, suppose that the spectral density function of observational x^*, $f_{x^*}(\omega) = f_{x^{nc}}(\omega) + f_n(\omega)$, satisfies the condition $\int_{-\pi}^{\pi} \ln(f_{x^*}(\omega))d\omega > -\infty$. It then admits a canonical factorization of the form

$$f_{x^*}(\omega) = \left|H_{x^*}(e^{-i\omega})\right|^2, \tag{37}$$

where $H_{x^*}(e^{-i\omega})$ is a "backward transform", a one-sided Fourier series involving only positive powers of $\{e^{-i\omega}\}$. Thus, the z-polynomial $H_{x^*}(z) = \sum_{j=0}^{\infty} h_j z^j$ has no zeros inside the unit circle. The causal Wiener filter is given by the Wiener-Kolmogorov formula[13]

$$M(\omega) = \frac{\left[f_{x^{nc}}(\omega)/H_{x^*}^*(\omega)\right]_+}{H_{x^*}(\omega)} \tag{38}$$

where $[\ ]_+$ is the annihilation operator and superscript $*$ again stands for the complex conjugate.

Notice that the spectral density function of the filtered variable $\hat{x}$ is

$$f_{\hat{x}}(\omega) = |M(\omega)|^2 f_{x^*}(\omega). \tag{39}$$

The Wiener filter places weights on the spectrum of x_t^* frequency by frequency to adjust for the spectral density function of $\hat{x}$; $\hat{x}$ is the best predictor of the unobserved state variable x_t. It is easy to verify that $|M(\omega)| \leq 1$ at all frequencies and that $|M(\omega)|$ is small at frequencies where the noise is strong. Intuitively, to approximate the spectral density function of the unobservable x_t, the filter pushes the spectral density of the observable x_t^* down since the policymaker knows that measurement error n_t contributes variance to x_t^* at all frequencies; and, at frequencies where data noise is stronger the filter needs to remove more. However, even the optimal Wiener filter is not able to completely remove the noise. Thus, when the benchmark policy rule $F^B(L)$ is applied to the filtered data, it still introduces undesirable side noise into the system.

Optimal policy rule with nonfiltered data $S^E(\omega)$ and that with filtered data $S^C(\omega)$ represent two distinct ways of dealing with measurement error. One is to adjust the policy design, while the other is to process the data. Although it is hard to analytically characterize the differences of their performance in the current abstract model, numerical exercises developed in Section 4 will show their relative performance in the frequency domain.

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[13] A equivalent representation in terms of z-polynomials is that $M(z) = \frac{1}{H_{x^*}(z)}[f_{x^{nc}}(z)/H_{x^*}^*(z)]_+$.

3. Robust Policy Rule

Now let me introduce model uncertainty. In the current framework, there are two potential types of model uncertainty: uncertainty about structure of the state variable x_t and uncertainty about that of measurement error n_t. Although models for unobservable x_t and n_t are based on the same information set x_t^*, the two types of model uncertainty are conceptually different. In this section, I will primarily consider model uncertainty with respect to the state variable x_t because it represents a major limit in the policymaker's knowledge about the structure of the economy. However, the analysis presented here can be immediately adopted to deal with uncertainty about the model of measurement error.

Specifically, the policymaker does not know the true model of x_t but knows that it is close to a baseline model. Assume model uncertainty is the following regarding the spectral density function of the uncontrolled system,

$$\int_{-\pi}^{\pi} \left[f_{x^{nc}}(\omega) - \bar{f}_{x^{nc}}(\omega)\right]^2 d\omega \leq \mu^2, \tag{40}$$

where $f_{x^{nc}}(\omega)$ is the unknown true model, $\bar{f}_{x^{nc}}(\omega)$ is the baseline model that the policymaker knows, and parameter $\mu > 0$ stands for the level of model uncertainty.[14] This specification allows for various sources of model uncertainty such as uncertainty in parameter values, dynamic structure, and lag orders.

Following Hansen and Sargent (2008), robustness analysis is considered as a two-player zero-sum game. An adversarial agent is introduced, who chooses a model $f_{x^{nc}}(\omega,\mu)$ from the feasible neighborhood around the baseline $\bar{f}_{x^{nc}}(\omega)$ as specified by (40) with uncertainty level μ to maximize the loss function $\mathbb{E}[x_t^2]$, given the policymaker's strategies. A primary agent, the policymaker, chooses a feedback control rule, represented by its sensitivity function $S(\omega,\mu)$, to stabilize the system given the potentially worst model chosen by the adversarial agent. In this sense, the *robust policy rule* is based on the worst-case analysis. However, playing against the adversarial agent the policymaker still uses the optimal control rule with nonfiltered data $S^E(\omega,\mu)$ to account for measurement error. Brock and Durlauf (2005) have shown that Nash and Stackelberg equilibria are approximately equivalent for the robustness games. I will hence use Nash equilibrium $\left(f^R_{x^{nc}}(\omega,\mu), S^E(\omega,\mu)\right)$ as the solution concept, where μ is included in the equilibrium strategies to indicate the level of model uncertainty.

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[14] This specification is similar to Hansen and Sargent's (2008, Chapter 8) formulation about the spectral density function of the innovations $W(L)\varepsilon_t$. If one is interested in uncertainty respect to measurement error model, an analogous specification would be
$\int_{-\pi}^{\pi} \left[f_n(\omega) - \bar{f}_n(\omega)\right]^2 d\omega \leq \mu^2,$
still with parameter μ measuring the degree of model uncertainty.

Assume that μ is sufficiently small, and therefore all feasible models are local. Following Brock and Durlauf's (2005) approach, I will approximate the equilibrium $\left(f^R_{x^{nc}}(\omega,\mu), S^E(\omega,\mu)\right)$ for the robust game around the baseline solution $\left(\bar{f}_{x^{nc}}(\omega), S^E(\omega,0)\right)$. This exercise is of particular interest in the frequency domain, since it allows us to explicitly characterize the policymaker's frequency-specific marginal reactions to model uncertainty. Therefore, I will be able to explore how the presence of measurement error influences the policymaker's reactions to model uncertainty across frequencies and changes the robustness properties of the optimal policy rule with nonfiltered data.

In the equilibrium, given the primary agent's choice $S^E(\omega,\mu)$, the adversarial agent solves

$$\max_{f_{x^{nc}(\omega,\mu)}} \int_{-\pi}^{\pi} [|1-S^E(\omega,\mu)|^2 f_n(\omega) + |S^E(\omega,\mu)|^2 f_{x^{nc}}(\omega,\mu)]d\omega, \tag{41}$$

subject to constraint (40). Let γ be the Lagrangian multiplier associated with the constraint. The first order conditions regarding the choice of $f_{x^{nc}}(\omega,\mu)$ are

$$|S^E(\omega,\mu)|^2 = 2\gamma\left[f^R_{x^{nc}}(\omega,\mu) - \bar{f}_{x^{nc}}(\omega)\right]. \tag{42}$$

The objective function (41) is increasing in $f_{x^{nc}}(\omega,\mu)$ at each frequency. Thus, constraint (40) is binding in the solution and $f^R_{x^{nc}}(\omega,\mu) \geq \bar{f}_{x^{nc}}(\omega)$. The decision of the adversarial agent in the Nash equilibrium then follows as

$$f^R_{x^{nc}}(\omega,\mu) = \bar{f}_{x^{nc}}(\omega) + \mu r(\omega,\mu), \qquad \text{with } r(\omega,\mu) \equiv \frac{|S^E(\omega,\mu)|^2}{\sqrt{\int_{-\pi}^{\pi}|S^E(\omega,\mu)|^4 d\omega}}. \tag{43}$$

Approximate $f^R_{x^{nc}}(\omega,\mu)$ around the baseline $\bar{f}_{x^{nc}}(\omega)$,

$$f^R_{x^{nc}}(\omega,\mu) = \bar{f}_{x^{nc}}(\omega) + \mu r(\omega,0) + o(\mu), \tag{44}$$

with

$$r(\omega,0) \equiv \frac{|S^E(\omega,0)|^2}{\sqrt{\int_{-\pi}^{\pi}|S^E(\omega,0)|^4 d\omega}}, \tag{45}$$

where $S^E(\omega,0)$ is just the optimal policy rule with nonfiltered data for the baseline model $\bar{f}_{x^{nc}}(\omega)$ given by (28) and (29).

Notice that $r(\omega,0)$ is the adversarial agent's marginal reaction to the level of model uncertainty μ at frequency ω. It is related to how much control used by the policymaker at frequency ω relative to the total amount used over the whole interval $[\pi, \pi]$ under the baseline model. The adversarial agent's choice is to deviate more from the baseline model at frequencies where the policymaker uses control more. In this way, the adversarial agent disturbs the policymaker's stabilization strategy as much as possible. From the policymaker's perspective, $f^{R}_{x^{nc}}(\omega, \mu)$ specifies the worst situation he will face when model uncertainty is of level μ.

Now turn to the equilibrium strategy $S^E(\omega, \mu)$ of the policymaker. Recall that the first order conditions for his optimization problem (25) under model $f^{R}_{x^{nc}}(\omega, \mu)$ are

$$(|S^E(\omega,\mu)| - 1)|S^E(\omega,\mu)| f_n(\omega) + |S^E(\omega,\mu)|^2 f^{R}_{x^{nc}}(\omega,\mu) = \lambda^E(\mu). \tag{46}$$

Differentiate both sides with respect to μ to solve for $\frac{\partial |S^E(\omega,\mu)|}{\partial \mu}$ and evaluate at $\mu = 0$:

$$\frac{\partial |S^E(\omega,0)|}{\partial \mu} = \frac{\frac{\partial \lambda^E(0)}{\partial \mu} - |S^E(\omega,0)|^2 r(\omega,0)}{(2|S^E(\omega,0)| - 1) f_n(\omega) + 2|S^E(\omega,0)| \bar{f}_{x^{nc}}(\omega)}, \tag{47}$$

where I use the fact that $f^{R}_{x^{nc}}(\omega, 0) = \bar{f}_{x^{nc}}(\omega)$ and that $\frac{\partial f^{R}_{x^{nc}}(\omega,0)}{\partial \mu} = r(\omega, 0)$ implied by (49). To pin down $\frac{\partial |S^E(\omega,0)|}{\partial \mu}$, $\frac{\partial \lambda^E(0)}{\partial \mu}$ in (47) has yet to be determined. Differentiate Bode's constraint with respect to μ and substitute $\frac{\partial |S^E(\omega,\mu)|}{\partial \mu}$ with (47). Then, I can solve for $\frac{\partial \lambda^E(\mu)}{\partial \mu}$ and evaluate it at $\mu = 0$:

$$\frac{\partial \lambda^E(0)}{\partial \mu} = \frac{\int_{-\pi}^{\pi} \left[\frac{|S^E(\omega,0)|^2 r(\omega,0)}{|S^E(\omega,0)| \left[(2|S^E(\omega,0)| - 1) f_n(\omega) + 2|S^E(\omega,0)| \bar{f}_{x^{nc}}(\omega) \right]} \right] d\omega}{\int_{-\pi}^{\pi} \left[\frac{1}{|S^E(\omega,0)| \left[(2|S^E(\omega,0)| - 1) f_n(\omega) + 2|S^E(\omega,0)| \bar{f}_{x^{nc}}(\omega) \right]} \right] d\omega}. \tag{48}$$

Therefore, in the Nash equilibrium the policymaker's strategy can be characterized as

$$|S^E(\omega,\mu)| = |S^E(\omega,0)| + \mu \frac{\partial |S^E(\omega,0)|}{\partial \mu} + o(\mu), \tag{49}$$

with $\frac{\partial |S^E(\omega,0)|}{\partial \mu}$ determined by (47) and (48).

The policymaker's equilibrium choice $S^E(\omega, \mu)$ is just the robust policy rule. Several observations follow. First, by linearization, the term $\frac{\partial |S^E(\omega,0)|}{\partial \mu}$ in (47) and (49) describes the policymaker's marginal reaction by frequency to model uncertainty. This reaction at frequency ω is driven by two forces. On one hand, including model uncertainty increases the constant density level that the optimal policy rule targets, and hence reduces control aggressiveness at all frequencies. This is captured by $\frac{\partial \lambda^E(0)}{\partial \mu}$ which is positive and constant over ω. On the other hand, the robust policy rule $S^E(\omega,\mu)$ also reacts to the adversarial agent's adjustment on the uncontrolled spectral density. Since the adversarial agent tends to increase the uncontrolled spectral density by rate $r(\omega,0)$, the policymaker will change his control in the opposite direction by $|S^E(\omega,0)|^2 r(\omega,0)$; the more the original control $|S^E(\omega,0)|$ is used, the more the marginal reaction should be. In the end, both parts of the reaction are adjusted by the denominator $(2|S^E(\omega,0)|-1)f_n(\omega)+2|S^E(\omega,0)|\bar{f}_{x^{nc}}(\omega)$ to satisfy Bode's constraint. Second, the analysis provided here is based on the policymaker using the optimal policy rule with nonfiltered data, but it can be extended to the cases using the naive policy rule and the optimal policy rule with filtered data with some complications. This indicates that the way of dealing with measurement error matters for both optimal and robust policies. Finally, the characterization of the robust policy rule as (49) illustrates the interaction between model uncertainty and measurement error in the design of stabilization policy rules. This is absent in, for example, Brock and Durlauf (2005) and new to the literature. In equation (49), $\frac{\partial |S^E(\omega,0)|}{\partial \mu}$ is marginal reaction by frequency to model uncertainty of level μ. However, it is clear from (47) and (48) that this reaction rate depends on the behavior of measurement error $f_n(\omega)$ in the frequency domain via several different channels. In this fashion, the interplay between the concerns over these two sources of uncertainty is characterized frequency by frequency. The numerical exercises in the next section will picture such effects more intuitively.

4. Some Applications: Monetary Policy Evaluation

In this section, I apply the general theory developed in the previous sections to evaluate monetary policy rules. First, I embed the conventional two-equation Keynesian monetary model into the scalar AR(1) framework so that I can work with the analytic devices constructed above directly. Then, based on the parameterized AR(1) model, I perform numerical experiments to assess several aspects of the measurement error effects on monetary policy design.

Typically, models employed in the literature of monetary policy evaluation contain two structural equations. One specifies the Phillips curve relating the output gap and inflation, and the other specifies the IS curve relating the real interest rate to the output gap. A simple version of the well-known Rudebusch and Svensson (1999) model which is widely used in previous studies can be written in the form of a purely backwards-looking system:

$$\pi_t = \rho\pi_{t-1} + by_{t-1} + \varepsilon_t \tag{50}$$

$$y_t = \theta y_{t-1} - \gamma(i_t - \pi_t) + e_t \tag{51}$$

where y_t stands for the gap between output and potential output, π_t is inflation and i_t is nominal interest rate. A Taylor (1993) type rule sets interest rate as $i_t = a_\pi \pi_t + a_y y_t$, or more generally as

$$i_t = a_\pi(L)\pi_t + a_y(L)y_t. \tag{52}$$

This linear feedback rule of interest rate (52), together with the IS curve (51), implies the output gap y_t as

$$y_t = -\beta(L)\pi_t + \phi(L)e_t, \tag{53}$$

with $\beta(L)$ and $\phi(L)$ determined during the rearrangement. Then, the economic environment can be considered as a one-state-variable-one-control system, with the system specified by the Phillips curve (50) and the control by (53).

In this paper, however, I still need to tailor the model slightly to have the interpretation of monetary policymaking as a feedback control problem in the presence of noisy information; I do this by restricting the control so that

$$y_{t-1} = -\beta(L)\pi^*_{t-1}, \tag{54}$$

$$\pi^*_t = \pi_t + n_t, \tag{55}$$

where the error term $n_t \equiv -\frac{\phi(L)e_t}{\beta(L)}$ can be simply understood as a process of measurement error in this context. To simplify the analysis, assume that $\rho < 1$ in (50) so that the system is stationary and that $\beta(L)$ is a one-sided lag polynomial. Further, let measurement error n_t be white noise with variance σ_n^2, in light of Orphanides' (2003) evidence that the inflation noise can be adequately modeled as a serially uncorrelated process. Fundamental innovations ε_t have variance σ_ε^2. Then, ratio $v \equiv \sigma_n^2/\sigma_\varepsilon^2$ represents the relative strength of measurement error.

To focus on the performance of policy rules in inflation stabilization, consider the loss function

$$V(\pi) = \mathbb{E}[\pi_t^2]. \tag{56}$$

This preference means that the policymaker is, in King's (1997) words, an "inflation nutter", who does not care about output stabilization. It can also be justified as inflation targeting in practice. The policy decision is then to design a feedback rule $\beta(L)$ of (54) in the presence of measurement error n_t to minimize $V(\pi)$, given the AR(1) economic model (50). And I will focus on the study of $\beta(L)$ and its sensitivity function in the frequency domain.

Admittedly, the model outlined here is highly stylized. Abstract from many practical issues, it highlights the importance to account for measurement error properly in monetary policy evaluation. Measurement error n_t can be broadly interpreted as an inaccurate control in this framework, which may have emerged in various ways. For example, n_t may arise directly from the mismeasurements of aggregate price levels and price changes, but can also result from misspecifications of the IS curve due to the lack of exact knowledge about monetary transmission mechanism, as implied by the appearance of IS error e_t. The numerical exercises below convey the idea how to quantitatively and intuitively assess the frequency-specific effects of this inaccurate control.

4.1 Policy Rules without Lagged Terms

A natural starting point is to consider the case in which policy rules depend only on the current observation of the state variable; lagged terms are excluded. This is interesting because the effects of measurement error can be characterized by one single coefficient parameter in this simple case. Also, simple instrument rules without lagged terms [*e.g.*, Taylor (1993)] or with only a few lags [*e.g.*, Onatski and Williams (2003)] have received much attention in the related literature. The results developed below are not trivial but rather relevant to policy practice.

Ignoring all pervious information, control (54) takes the form of

$$y_t = -\beta \pi^*_{t-1}. \tag{57}$$

The solution to this case can be worked out explicitly. If the policymaker is not aware of measurement error n_t, he will use the naive policy rule $y_{t-1} = -\beta^N \pi^*_{t-1}$ with

$$\beta^N = \frac{\rho}{b}. \tag{58}$$

Otherwise, integrating his knowledge of n_t into policy decision, the policymaker's optimal policy rule with nonfiltered data in this case turns out to be $y_{t-1} = -\beta^E \pi^*_{t-1}$ with

$$\beta^E = -\frac{1}{2\rho b \sigma_n^2}\left[(1-\rho^2)\sigma_n^2 + \sigma_\varepsilon^2 - \sqrt{[(1-\rho^2)\sigma_n^2 + \sigma_\varepsilon^2]^2 + 4\rho^2\sigma_n^2\sigma_\varepsilon^2}\right]. \tag{59}$$

One important observation can be made here. The optimal policy rule β^E adjusted for the measurement turns out to be less responsive than the naive policy rule β^N that assumes measurement is exact. This is because

$$|\beta^E| \leq |\beta^N|. \tag{60}$$

Details of the derivation and comparison of these policy rules can be found in Section A.4 of the Appendix. But the implication of this result is straightforward; if the policymaker is not confident in the accuracy of his data, excess activeness is harmful.

Figure 1 draws the naive policy rule β^N and optimal policy rule β^E when v varies from 0 to 10 and ρ from 0 to 1. Here, b is calibrated to be 0.1. Consistent with intuition, Figure 1 shows that whether to account for measurement error or not is irrelevant, $\beta^N = \beta^E$, only when measurement error is trivially absent, $\sigma_n^2 = 0$ (*i.e.*, $v = 0$), or when the controlled system is a white noise process itself and hence no control should be used anyway, $\rho = 0$. Otherwise, the more serious measurement error is (*i.e.*, the larger v is) or the more persistent the uncontrolled system is (*i.e.*, the larger ρ is), the less aggressive the optimal policy rule β^E will be when compared with the naive rule β^N. The role of noise level v is straightforward; when control induces side noise, the policymaker will use control more cautiously when he is aware of the stronger side effects. However, the role of system persistence ρ needs some explanation: A persistent uncontrolled system requires a very active feedback to the current observation so that the volatility inherited from previous periods can be canceled out; in other words, the stabilizing control effects of any policy action are weak relative to side noise effects, and thus, the policy works better by using a less responsive feedback rule to restrain the side noise effects.

In addition, in Figure 1 the optimal policy rule β^E is very steep when ρ is high and v is small. In persistent model, even a small level of measurement noise may cause a large change in the optimal policy rule design. This illustrates the interaction between model uncertainty and measurement noise in affecting optimal policymaking. Here, model uncertainty is about the different values of ρ. Under different models, the policymaker adjusts policy aggressiveness for measurement error in rather different degrees. I will explore this point more generally in the frequency domain later.

It is also interesting to investigate the spectral properties of the naive rule β^N and the optimal rule β^E in this non-lagged-term case. The sensitivity function associated with β^N is

$$|S^N(\omega)| = \sqrt{1 - 2\rho\cos(\omega) + \rho^2}, \tag{61}$$

and the sensitivity function associated with β^E is

$$|S^E(\omega)| = \frac{\sqrt{1 - 2\rho\cos(\omega) + \rho^2}}{\sqrt{1 + 2(b\beta^E - \rho)\cos(\omega) + (b\beta^E - \rho)^2}}, \tag{62}$$

At frequencies where $\cos(\omega) > \frac{1}{2}(\rho - b\beta^E)$, $|S^E(\omega)| \geq |S^N(\omega)|$, while at frequencies where $\cos(\omega) < \frac{1}{2}(\rho - b\beta^E)$, $|S^E(\omega)| < |S^N(\omega)|$, when $\rho > b\beta^E$; and *vice versa* when $b\beta^E \geq \rho$. The optimal rule is less sensitive than the naive rule, $|\beta^E| \leq |\beta^N|$, in general as shown above, but over a range of frequencies the former can actually be more aggressive than the latter. It is right in this sense that spectral analysis is informative when one is interested in how measurement error changes the allocation of control effects across frequencies.

4.2 Optimal Policy Rule with Nonfiltered Data

In the general case without the restriction on the order of lag polynomial, a monetary policy rule has sensitivity function and complementary sensitivity function as

$$S(\omega) = \frac{1-\rho e^{-i\omega}}{1-(\rho - b\beta(\omega))e^{-i\omega}} \text{ and } T(\omega) = \frac{b\beta(\omega)e^{-i\omega}}{1-(\rho - b\beta(\omega))e^{-i\omega}}. \tag{63}$$

The easurement error process n_t and uncontrolled inflation π_t^{nc} have spectral density functions as

$$f_n(\omega) = \frac{\sigma_n^2}{2\pi}, \qquad \text{and } f_{\pi^{nc}}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi(1-2\rho\cos(\omega)+\rho^2)}. \tag{64}$$

As established in Section 2, the optimal policy rule with nonfiltered data is then pinned down by its sensitivity function

$$|S^E(\omega)| = \frac{\frac{\sigma_n^2}{2\pi} + \sqrt{\left(\frac{\sigma_n^2}{2\pi}\right)^2 + 4\lambda^E\left(\frac{\sigma_n^2}{2\pi} + \frac{\sigma_\varepsilon^2}{2\pi(1-2\rho\cos(\omega)+\rho^2)}\right)}}{2\left(\frac{\sigma_n^2}{2\pi} + \frac{\sigma_\varepsilon^2}{2\pi(1-2\rho\cos(\omega)+\rho^2)}\right)} \tag{65}$$

where the constant spectral target λ^E is determined by numerically solving from condition (29), *i.e.* $\int_{-\pi}^{\pi}\ln[|S^E(\omega)|^2]d\omega = 0$, with $f_n(\omega)$ and $f_{\pi^{nc}}(\omega)$ given above.

I employ the following parameterizations. Following Brock *et al* (2007), I calibrate $\rho = 0.9$ and $b = 0.1$. I also specify different levels of noise strength as: $v = 0$, measurement error is absent; $v = 0.4402$, measurement error is modest and it is derived from Orphanides' (2003) estimates $\sigma_\varepsilon = 1.04$ and $\sigma_n = 0.69$; $v = 1$, measurement error has the same standard deviation as fundamental innovations ε_t; and $v = 4$, measurement error is serious and with the standard deviation twice as that of ε_t.

Figure 2 presents the sensitivity functions $S^E(\omega)$ associated with the optimal policy rules with nonfiltered data when exposed to the different levels of data noise. The central message is that policy inertia is increasing in noise strength. To flatten spectral density, an optimal policy rule is to push the uncontrolled $f_{\pi^{nc}}(\omega)$ down (*i.e.*, $|S^E(\omega)| < 1$) at low frequencies and to lift it up (*i.e.*, $|S^E(\omega)| > 1$) at high frequencies in this AR(1) framework. When measurement error is present in the data, the policymaker has to make tradeoffs between stabilizing control effects and side noise effects in applying feedback control rules as explained in Section 2. In Figure 2, the optimal policy rule pushes $f_{\pi^{nc}}(\omega)$ down less at low frequencies and also lifts it up less at high frequencies as measurement error becomes stronger. Therefore, the total effect of measurement error is to make the optimal monetary policy rule less aggressive.

However, the policymaker does not reduce aggressiveness equally across frequencies. In Figure 2, the sensitivity function moves close to constant level 1 much more quickly at high frequencies than at low frequencies as the strength of measurement error strength increases. This essentially means that measurement error induces more cautiousness in the use of control at high frequencies than at low frequencies. Especially over a small range of medium frequencies, measurement error actually leads to more active policy control. The intuition behind as follows. In this AR(1) framework, the variance of the uncontrolled state variable concentrates at low frequencies, where system volatility is so high that the policymaker has to use control somewhat forcefully even if it brings some side effects. Even when measurement error is strong, these side noise effects are still small relative to the system volatility at these frequencies. In contrast, side noise effects quickly dominate stabilizing control effects at high frequencies, so the policymaker must be much more cautious as data noise increases. Finally, at medium frequencies, the policymaker may have to be more aggressive because of the design limits – when the spectral density is pushed down at low frequencies less than the amount lifted up at high frequencies due to the unbalanced frequency-specific reactions, the density function has to popup at some medium frequencies to meet Bode's constraint.

Figure 3 shows the spectral density functions of inflation π_t^E under the control of the optimal policy rules at the different levels of measurement error. Without data noise the optimal control completely flattens the density function, as expected. In the presence of measurement error, the weaker measurement error is, the flatter the controlled density function will be. The total variance of the controlled state variable, *i.e.* the area under the densities, is increasing in the strength of measurement error. Therefore, the optimal policy rule is more effective when data noise is lower. Across frequencies, the "waterbed effects" – the reduction of variance at some frequencies results in increases of variance at other frequencies – implied by Bode's constraint are also evident from the Figure. On one hand, when measurement error is stronger, the controlled density function has a higher peak at low frequencies. This comes in two ways. First, the control is less aggressive so that the uncontrolled peak is pushed down less. Second, it brings stronger side noise effects when data noise is stronger. On the other hand, the optimal policy rule performs well at high frequencies in the sense that the controlled density is low and flat, and this does not change much when measurement error increases.

Figure 4 presents the assessment of aggressiveness of the optimal policy rule over different levels of measurement error or model persistence. Both notions of the AG measure and spectral target λ^E are used. Given model persistence $\rho = 0.9$, panel (a) shows that the spectral target λ^E is increasing in the strength of measurement error v, and panel (b) shows that the AG measure is decreasing in v. Both mean that when the measurement is noisier the optimal policy rule is less aggressive. It is also worth noting that both λ^E and AG change at a decreasing rate over v. Panel (c) displays the spectral target λ^E over varying model persistence ρ in the cases with and without measurement error. In the absence of measurement error, the optimal policy rule always targets a constant spectral level. When modest measurement error is present, the target λ^E increases in ρ. Thus, the target gap between the two cases is also increasing. Intuitively, when model persistence ρ increases, the stabilizing control effect of any given policy rule becomes weaker, and hence the policymaker has to increase its spectral target when measurement error is present and can cause side noise. In panel (d), even without measurement error the AG aggressiveness is still increasing ρ. This is because the uncontrolled density has a more precipitous peak with a larger ρ. To target the same low spectral level essentially means a more aggressive control when the model becomes more persistent. Panel (d) also shows that the AG aggressiveness is always smaller when measurement error is present than when it is absent; again the gap is increasing in model persistence ρ. Thus, the optimal policy rule reacts to the same level of measurement error more in reducing aggressiveness when facing a more persistent model. Interestingly, AG is almost linear in ρ for both cases.

4.3 Optimal Policy Rule with Filtered Data

For the optimal policy rule with filtered data, the most important step is to filter the data properly. Within the AR(1) framework, I first work out the Wiener filter explicitly, following Priestley (1981, Chapter 10), and then use it for numerical implementation.

Given that the unobservable π_t is an AR(1) process and that measurement error n_t is white noise, the spectral density function of observed π_t^* is

$$f_{\pi^*}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi|1-\rho e^{-i\omega}|^2} + \frac{\sigma_n^2}{2\pi}. \tag{66}$$

Its canonical factorization gives

$$f_{\pi^*}(z) = \frac{k(1-\phi z)(1-\phi z^{-1})}{2\pi(1-\rho z)(1-\rho z^{-1})}, \tag{67}$$

where $k = \rho\sigma_n^2/\phi$ and ϕ is the root inside the unit circle of z-polynomial $f_{\pi^*}(z)$. In this case,

$$H_{\pi^*}(z) = \sqrt{\frac{k}{2\pi}}\left(\frac{1-\phi z}{1-\rho z}\right). \tag{68}$$

As shown in Section A.5 of the Appendix, this implies that

$$\left[\frac{f_\pi(z)}{H_{\pi^*}^*(z)}\right]_+ = \frac{\sigma_\varepsilon^2}{(1-\rho\phi)(1-\rho z)\sqrt{2\pi k}}. \tag{69}$$

Therefore, the causal Wiener filter (38) in the frequency domain can be written as

$$|M(\omega)|^2 = \left[\frac{\phi\sigma_\varepsilon^2}{(1-\rho\phi)\rho\sigma_n^2}\right]^2 \frac{1}{1+\phi^2-2\phi\cos(\omega)}. \tag{70}$$

Recall that the spectral density of the filtered variable $\hat{\pi}$ is simply

$$f_{\hat{\pi}}(\omega) = |M(\omega)|^2 f_{\pi^*}(\omega), \tag{71}$$

so $|M(\omega)|^2$ are just weights placed on $f_{\pi^*}(\omega)$ across frequencies to adjust for $f_{\hat{\pi}}(\omega)$.

Panel (a) in Figure 5 presents the adjustment weights placed by the Wiener filter on $f_{\pi^*}(\omega)$. First, the Wiener weights are lower at all frequencies with weak data noise than with strong noise. For example, when measurement error is almost absent, *i.e.* $v \to 0$, the weights are almost equal to 1 everywhere, whereas the case of $v = 4$ has the lowest weighting curve. Second, the weights are high at low frequencies and low at high frequencies. This is because the AR(1) unobservable π_t amounts to a high fraction at low frequencies and a low fraction at high frequencies of volatility of the observable π_t^* when measurement error is white noise. Third, the weighting curve is not shifted down by the equal distance across frequencies when measurement error becomes stronger. For example, when v goes from 0.4402 to 1, the weighting line moves more down at high frequencies than at low frequencies, because measurement error has stronger effects at high frequencies than at low frequencies. Panel (b) shows the filtered results at frequencies $[\pi/2, \pi]$. The spectral density function of the filtered $\hat{\pi}_t$ is very close to that of the unobservable π_t, so the Wiener filter works very well. After the policymaker applies the optimal policy rule for the noise-free model to the filtered data, the associated sensitivity functions and policy performance are then shown in panels (c) and (d) of Figure 5. One can see from panel (c) that measurement error reduces the aggressiveness of policy control in general, and reduces it more at high frequencies than at low frequencies. Panel (d) indicates that the extent of the side noise effects brought into the system by the optimal policy rules with filtered data still increases in the strength measurement error.

4.4 Robust Policy Rule

To study the robustness properties of monetary policy rules, I need to specify the level of model uncertainty. Since this section mainly serves to communicate the method to track the interplay of measurement error and model uncertainty in policymaking, I would withdraw from the involvement in the data-based calibration of μ but simply set $\mu = 1$.[15] I assume that the true model is local to the AR(1) baseline model as studied above. Thus,

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[15] For more rigorous empirical implementation, see Hansen and Sargent (2001, 2008) for details on how the level of model uncertainty μ can be calibrated quantitatively from historical data.

$$\bar{f}_{\pi^{nc}}(\omega)=\frac{1}{2\pi(1-2\rho\cos(\omega)+\rho^2)}\ \text{ and }\ \int_{-\pi}^{\pi}\left[f_{\pi^{nc}}(\omega)-\bar{f}_{\pi^{nc}}(\omega)\right]^2 d\omega\leq 1. \tag{72}$$

I maintain the assumption that measurement error is white noise with the different levels of strength v.

In Figure 6, panel (a) shows the adversarial agent's marginal reactions $r(\omega,0)$ across frequencies to model uncertainty. The adversarial agent knows that the policymaker will push the uncontrolled density down very forcefully and hence it is less effective to disturb at low frequencies. In contrast, at high frequencies the policymaker allows the uncontrolled density to move up, and then the adversarial agent's disturbing actions will not be canceled out. Therefore, the adversarial agent puts most disturbing power at high frequencies. This renders the marginal reaction curves a similar shape as the sensitivity functions associated with the optimal rules shown in Figure 2.

Panel (b) shows how the policymaker will react. Consider the noise-free case first. There is only model uncertainty, and the policymaker knows that most uncertainty comes from high frequencies. Without much influence of the adversarial agent, the policymaker will not change his strategies very much at low frequencies. However, he will greatly reduce control used at high frequencies. To flatten the density function, the policymaker needs the uncontrolled AR(1) density to move up at high frequencies in the absence of model uncertainty. But, now the adversarial agent also lifts up the spectral density at high frequencies which helps the policymaker to reduce the control used. The consequence is that more control is moved to medium frequencies to meet Bode's constraint. This explains the "m" shape of the policymaker's marginal reaction curve.

In the presence of measurement error, the "m" shape is less obvious, although still present. A flat "m" shape marginal reaction curve around zero means that the policymaker only mildly reacts to model uncertainty at all frequencies. This is especially true for the case with strong data noise, $v=4$. At low frequencies, it is not necessary to react very much as the adversarial agent puts little uncertainty here anyway. Knowing that the policymaker tends to balance control used at medium and high frequencies by making the sensitivity function flatly close to constant 1 (see Figure 2) when measurement error is present, the adversarial agent also balances the allocation of model uncertainty at these frequencies as illustrated by panel (a) of Figure 6. The policymaker then has to reduce his reaction at both medium and high frequencies. Put differently, reacting actively to model uncertainty will increase control used at medium frequencies as shown in the noise-free case, but control comes with side noise effects when measurement error is strong. Thus, the policymaker simply chooses not to react that much. Reducing reaction at medium frequencies also leads to the reductions at high frequencies, according to Bode's constraint. The whole point is that the presence of measurement error reduces the policymaker's reaction to model uncertainty, and this is especially clear at medium and high frequencies.

Taking model uncertainty into consideration, the sensitivity functions associated with the robust policy rules are displayed in panel (d). Whether measurement error is present or not, the robust policy rules behave rather differently, especially at medium and high frequencies. The interaction between model uncertainty and measurement error leads to the performance of the robust policy rules as shown in panel (c).

In the light of Brainard's (1967) argument that model uncertainty justifies cautious policy, I may conclude based on these exercises: (1) cautiousness resulting from model uncertainty is not equal across frequencies – the policymaker reacts to model uncertainty less at low frequencies than at others, and the reaction at medium frequencies actually leads to active control in the contrast; (2) measurement error reduces the impacts of model uncertainty on policymaking – this is especially important at medium frequencies, which may overlap business cycle frequencies that monetary authorities potentially care most, in the sense that model uncertainty does not overthrow the general insights on monetary policymaking. Rather, model uncertainty should be assessed along with other forms of uncertainty such as data noise. In the environment of various uncertainty sources, the effects of model uncertainty may be less significant than those in the situation focusing on model uncertainty alone.

4.5 Comparing Policy Scenarios

As a final exercise, I compare the performance of naive policy rule, optimal policy rules with nonfiltered and filtered data, and robust policy rule in the current parameterized model. Recall that, in this setting, the naive policy rule is associated with the sensitivity function

$$|S^N(\omega)| = \sqrt{1 - 2\rho\cos(\omega) + \rho^2}, \tag{73}$$

and has the spectral density function under control as

$$f_{\pi^N}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} + \frac{\sigma_n^2}{2\pi}(1 - S^N(\omega))^2, \tag{74}$$

along with a constant level of spectral target $\lambda^N = \frac{\sigma_\varepsilon^2}{2\pi}$. The other three policy scenarios have already been examined as above.

Table 1 reports the comparison results under the different levels of measurement error. First, the optimal policy rules with nonfiltered and filtered data are less aggressive and have better performance than the naive policy rules. The conclusion on aggressiveness holds for both spectral target λ and AG measure under different levels v of measurement error. Meanwhile, the optimal policy rules reduce the total variances to lower levels than what the naive policy rules can reach. This indicates that failing to recognize measurement noise will lead to serious distortions in policy performance. Second, the optimal policy rules with filtered data outperform those with nonfiltered data. The former give lower variances than the latter over the whole frequency domain as well as over the high frequency range, while the former are less aggressive than the latter in the overall AG measure. This may be interpreted as the power of the Wiener filter in filtering out measurement noise. The optimal policy rules with nonfiltered data introduce more side noise because of more aggressive control. Third, the robust policy rules are the least aggressive in both target λ and measure AG, as expected. Interpreting these results, it is important to note that the robust policy rules are based on the least favorable model, which is different from the baseline model that the other rules are dealing with.

Figures 7 and 8 show the comparisons of the sensitivity functions and their performance across the different policy scenarios when measurement error is modest, $v = 0.4402$. With regard to the sensitivity functions, the four policy rules place similarly strong control at low frequencies, although the robust policy is a little weaker. The main differences come from medium and high frequencies. For example, the robust policy rule is the least responsive at high frequencies and the most active at medium frequencies. The sensitivity functions of the optimal policy rules have very similar shapes except that the rule with nonfiltered data has its sensitivity function below that of the rule with filtered data. This leads to the result in Figure 8 that the optimal policy rule with filtered data outperforms the optimal rule with nonfiltered data. In Figure 8, the controlled spectral density function under the robust policy is the highest since it assumes the highest uncontrolled spectral density rather than the AR(1) baseline density. Interestingly, the naive policy rule does not work well in total-variance reduction, but at least it is effective at low frequencies. Its inefficiencies arise at high frequencies. This again highlights the importance of studying frequency-specific effects of measurement error in monetary policy evaluation.

5. Concluding Remarks

This paper provides a framework for investigating how alternative feedback policy rules behave in the presence of measurement errors and with respect to frequency-specific performance. I argue the importance to recognize potential data noise in policy decisionmaking, and show various ways to integrate this information into the assessment of policies' efficiency and robustness. Applied to monetary policy evaluation, the numerical exercises draw insights on the frequency-specific adjustments in the design of policy rules to the measurement error.

Admittedly, this paper contains weak points, which I would like to explore in future research. Firstly, I only deal with the system of single input and single output in this paper. For monetary policy evaluation, I have to impose restrictions so that the usual two-equation system consisted of the IS and Phillips curves can be analyzed within the current framework. It is appealing to extend the existing results to bivariate or multivariate cases. To work out this extension, I need to cope with the transfer matrix and spectral density matrix instead of the scalar sensitivity function and spectral density function. Design limits will still show up but in the matrix form [Brock *et al* (2008b)]. Thus, some new techniques may be required.

Secondly, my analysis can be extended to a hybrid model containing both forward-looking and backwards-looking elements with some complications. As noted in Section 2, expectation is potentially another channel though which measurement error influences policy decision. The nature of design limits also depends on the way in which forward-looking elements determine current macroeconomic state variables; specifically, relative to a purely passive policy baseline, a feedback from expectations of future state variables to current state variables is necessary for the existence of a stabilization rule that reduces variance at all frequencies. This suggests, in considering the interplay between model uncertainty and measurement error, it is meaningful to include forward-looking elements, maybe rational expectation based, into the model.

Finally and maybe most importantly, this paper is silent to a fundamental question: where does measurement error come from? I simply assume that measurement error is there, attached to the true measure, as most previous literature on the subject does. However, an observation is the aggregation of at least three elements: true value, measurement error, and the error due to model uncertainty. They are not mutually independent. For example, measurement error and model uncertainty are generically connected because we measure variables and form expectations based on models. Model uncertainty has already taken effect in the set of observations before we base our robustness analysis on the observations. This paper makes progress towards but does not complete the full theory of interaction between measurement error and model uncertainty. There are also other endogenous sources of measurement error such as demographic change and technological development. A structural model on the origins of measurement error will be necessary for a more complete understanding of the policy implications of its presence.

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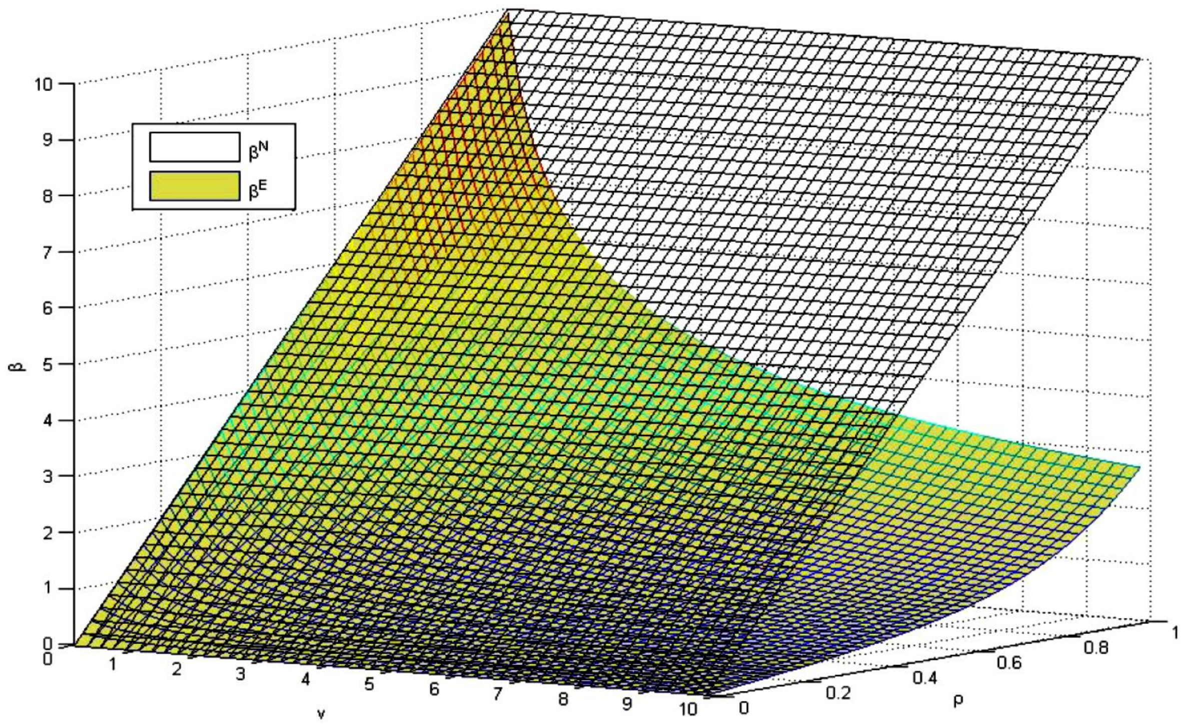
Table 1. Comparison of Policy Scenarios under Different Levels of Measurement Error

Level of Data Noise	Naive Policy	Optimal Policy, Nonfiltered	Optimal Policy, Filtered	Robust Policy
Measurement Error : $v = 0$ (No Noise)				
density level target λ	0.1592	0.1592	-	0.3843
aggressiveness *AG*	1.5411	1.5411	1.5407	1.3373
variance under control σ_c^2	1.0000	1.0000	0.9999	2.4145
high frequency variance contribution $[\frac{\pi}{2}, \pi]$	0.2500	0.2500	0.2500	0.6037
Measurement Error : $v = 0.4402$ (Orphanides Estimate)				
density level target λ	0.1592	0.1919	-	0.3802
aggressiveness *AG*	1.5411	1.2608	1.0038	0.9132
variance under control σ_c^2	1.1664	1.1336	0.9315	2.3302
high frequency variance contribution $[\frac{\pi}{2}, \pi]$	0.3087	0.2411	0.1724	0.5679
Measurement Error : $v = 1$				
density level target λ	0.1592	0.2212	-	0.3993
aggressiveness *AG*	1.5411	1.0981	0.8956	0.8347
variance under control σ_c^2	1.3780	1.2560	0.9917	2.3957
high frequency variance contribution $[\frac{\pi}{2}, \pi]$	0.3834	0.2346	0.1500	0.5594
Measurement Error : $v = 4$				
density level target λ	0.1592	0.2597	-	0.4313
aggressiveness *AG*	1.5411	0.9464	0.8357	0.7712
variance under control σ_c^2	1.7560	1.4199	1.1226	2.5216
high frequency variance contribution $[\frac{\pi}{2}, \pi]$	0.5168	0.2274	0.1352	0.5495

Note: In the case of $v = 0$, optimal control rule is calculated by setting $v = 0.0001$ to use the Wiener filter.

Figure 1. Naive Policy Rules and Optimal Policy Rules without Lagged Terms



Note: $v \equiv \sigma_n^2/\sigma_\varepsilon^2$ represents the relative strength of measurement noise, ρ is the parameter of AR(1) Phillips curve, and $b = 0.1$. β^N is the response of the naive policy rule to the current observation and β^E is that of the optimal policy rule.

Figure 2. Optimal Policy Rules with Nonfiltered Data under Different Noise Levels

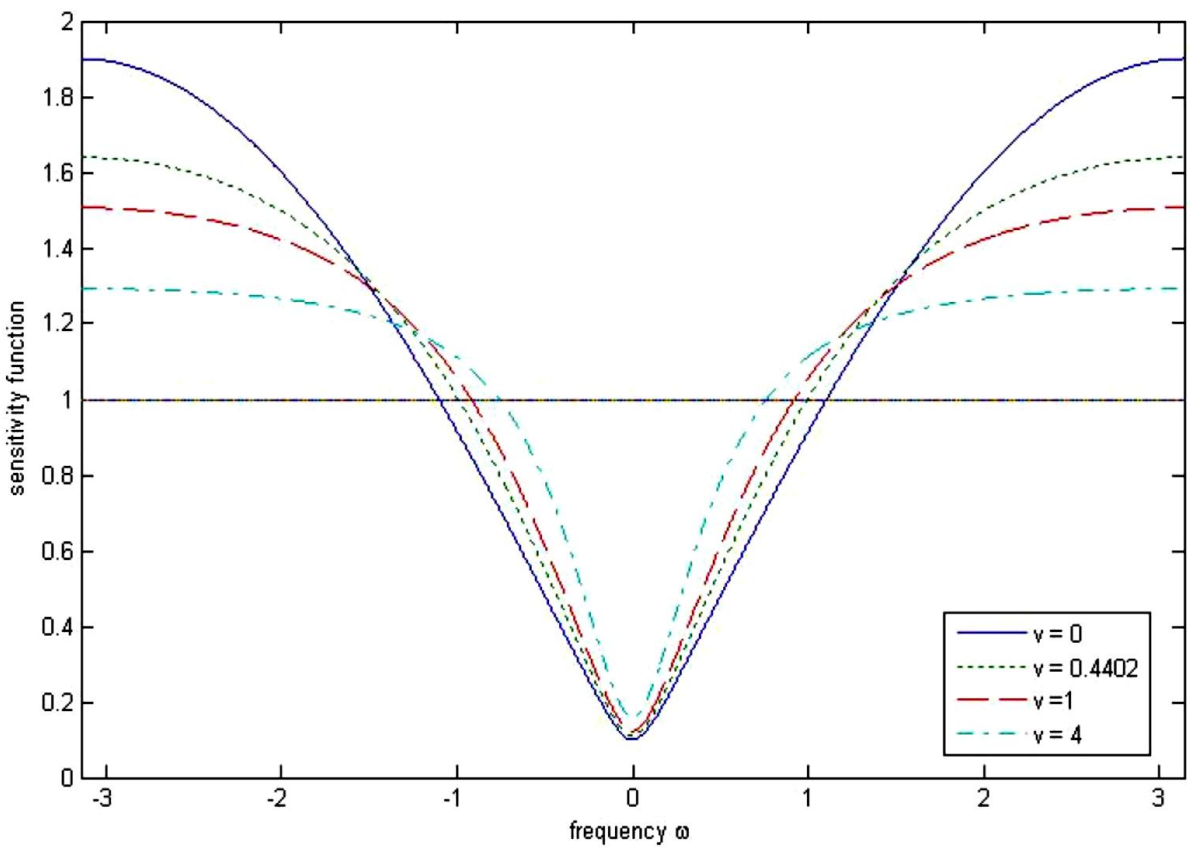


Figure 3. Spectral Density Functions of Controlled Inflation π_t^E under Different Noise Levels

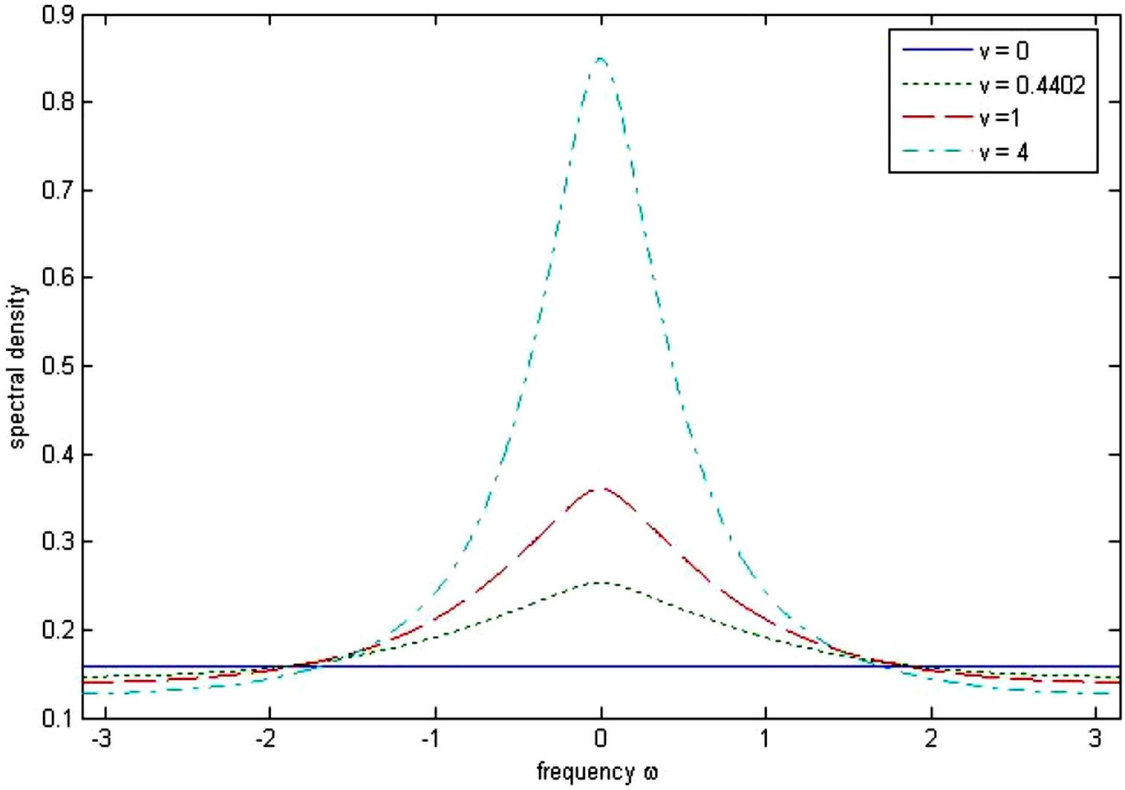
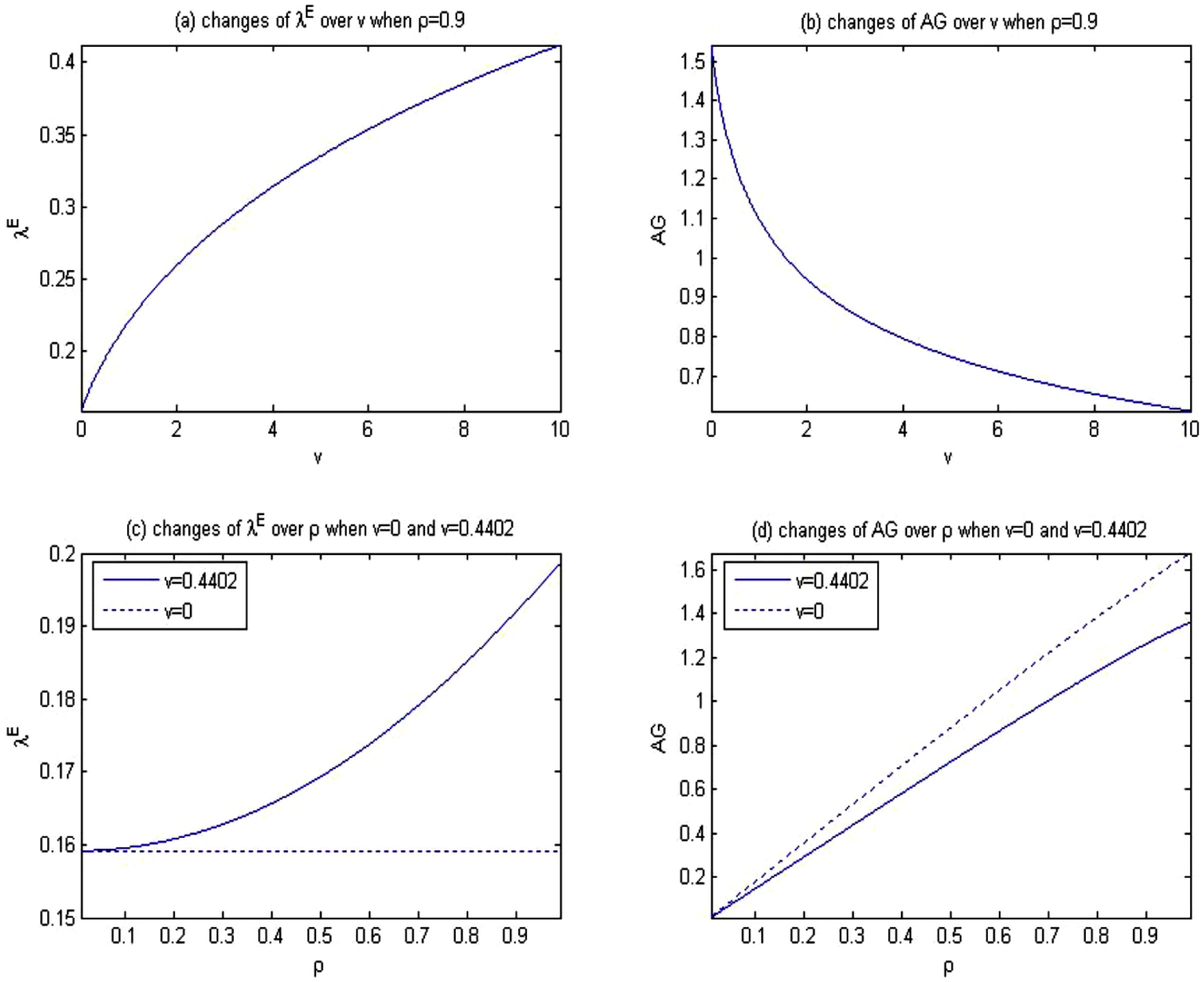
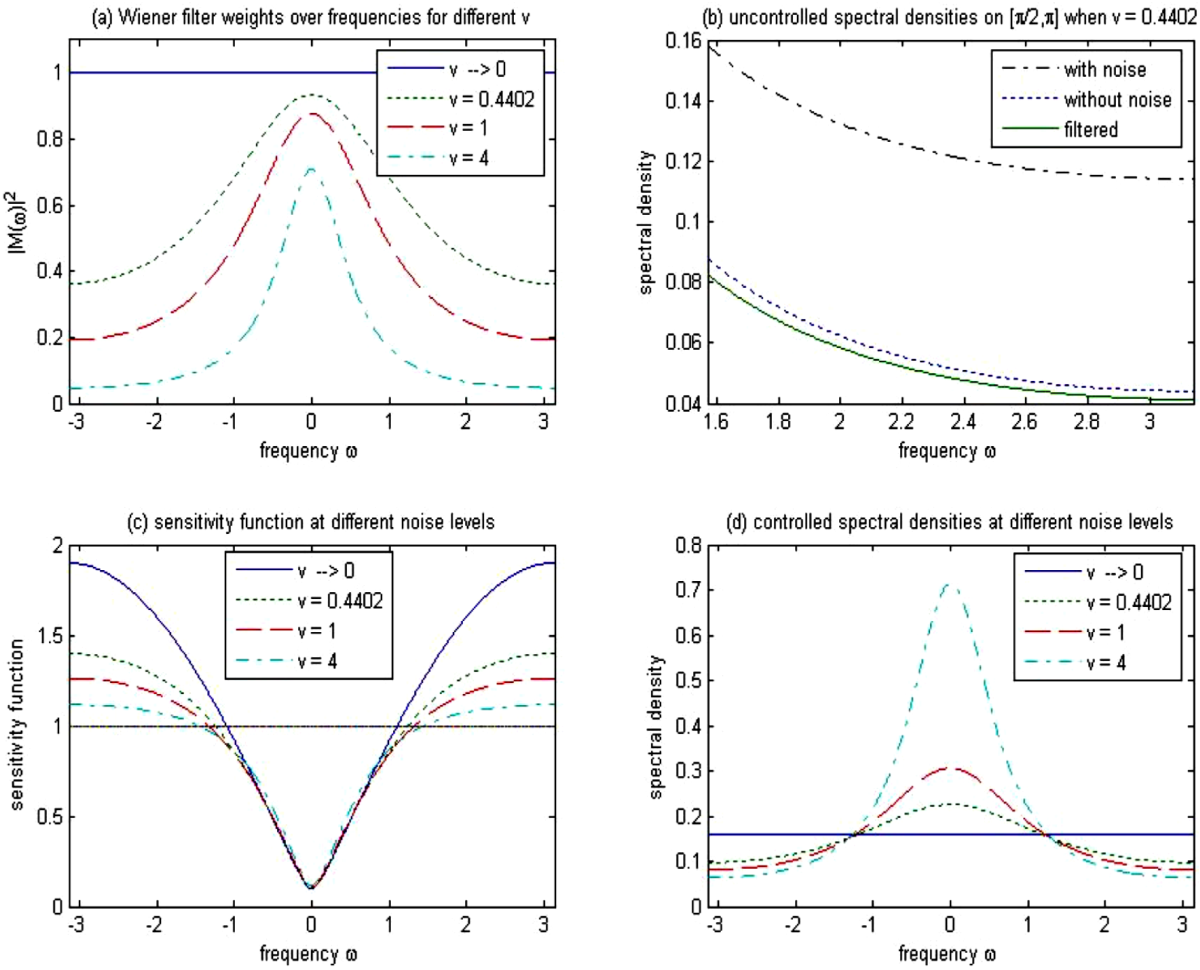


Figure 4. Policy Aggressiveness under Different Levels of Data Noise and Model Persistence



Note: AG is defined in L^2 norm $AG = \left[\int_{-\pi}^{\pi} |1 - |S^E(\omega)||^2 d\omega\right]^{\frac{1}{2}}$ and λ^E is the constant spectrum target.

Figure 5. Optimal Policy Rules with Filtered Data under Different Levels of Measurement Error



Note: $v \to 0$ is approximated by setting $v = 0.0001$. It captures the case when noise is almost absent.

Figure 6. Performance of Robust Policy Rules under Different Levels of Measurement Noise

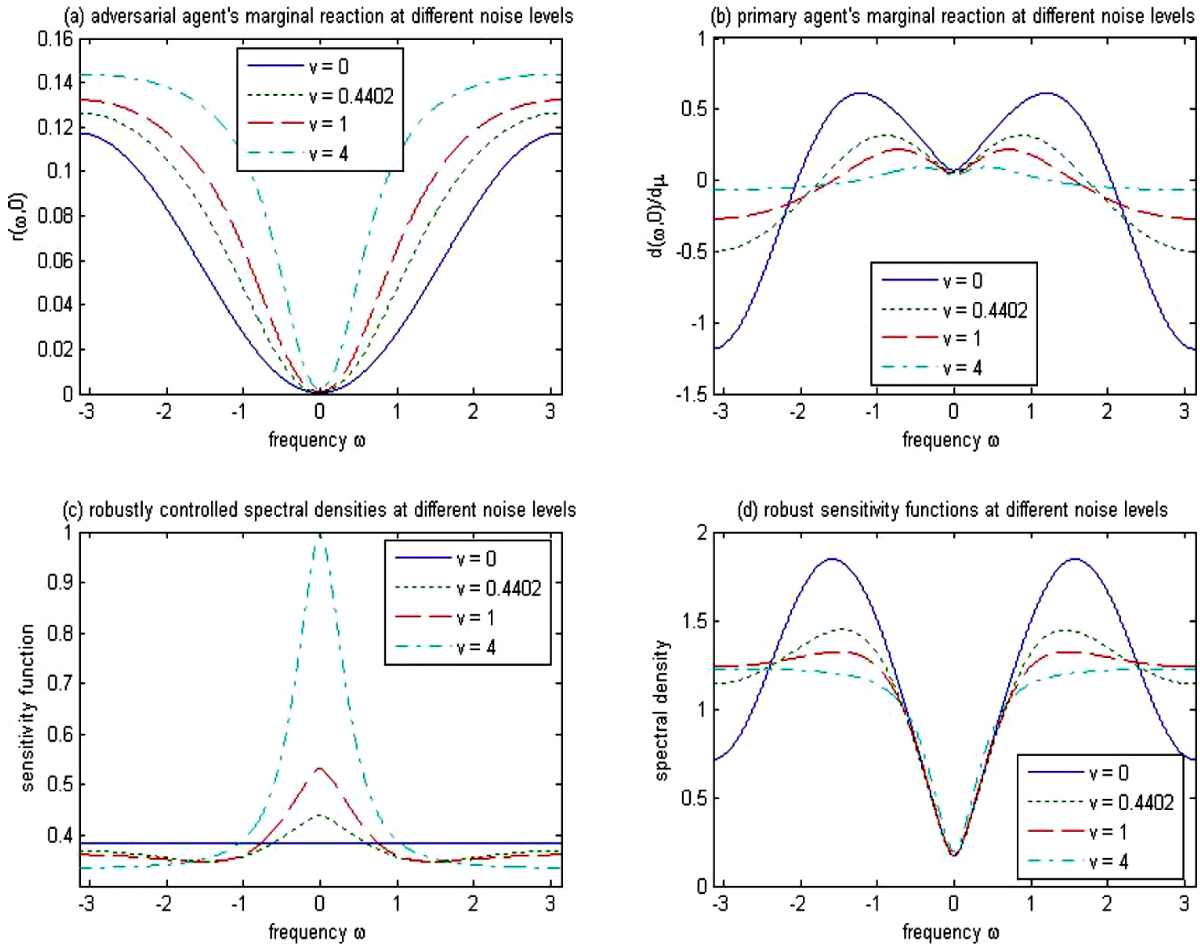


Figure 7. Sensitivity Functions Associated with Different Policy Rules, $\mathrm{v} = 0.4402$

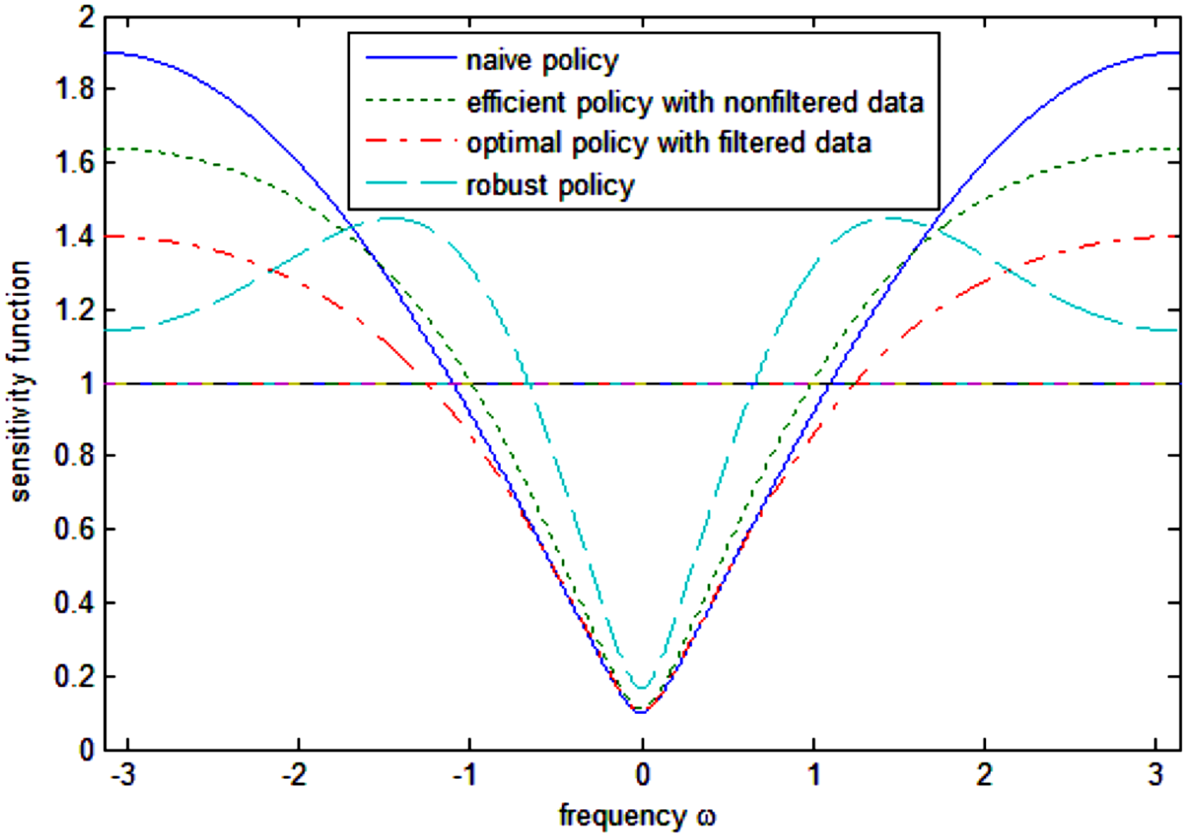
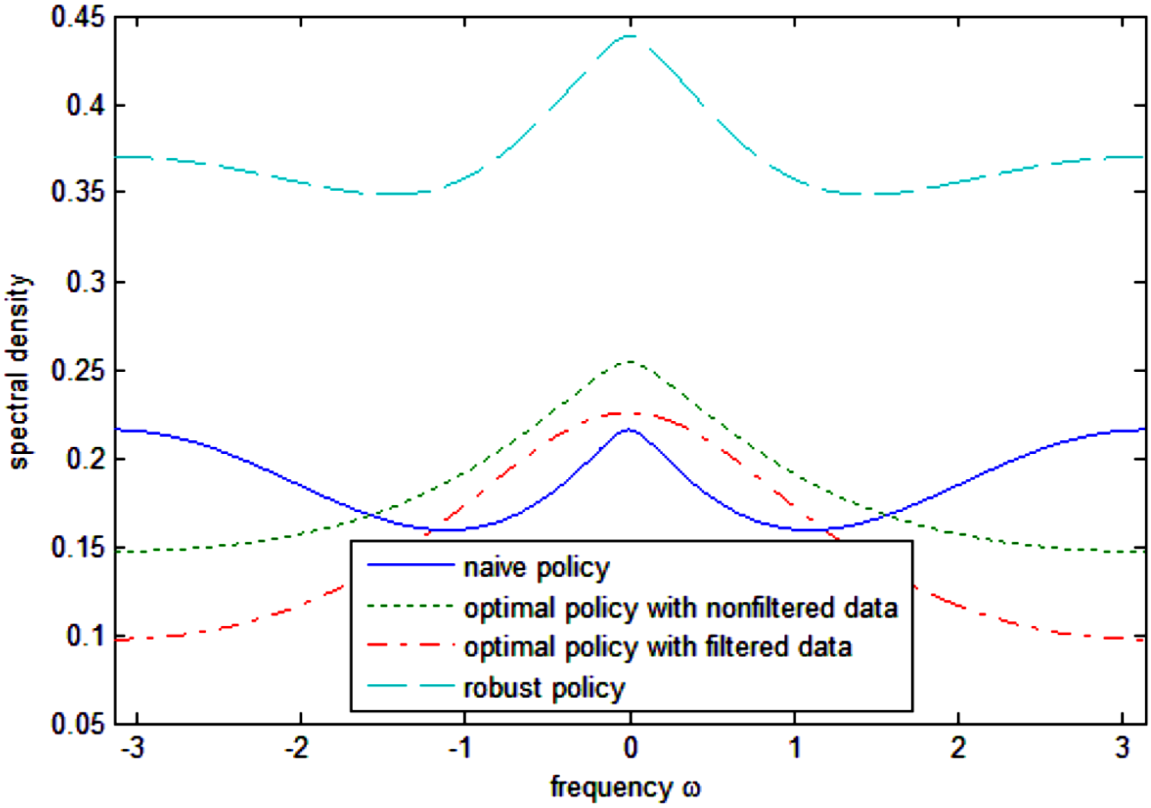


Figure 8. Spectral Density Functions under Different Policy Rules, $v = 0.4402$



Appendix

A.1 Proof of Bode's Constraint (10)

This proof of Bode's constraint (10) is an application of Wu and Jonchheere's (1992) lemma, which states that

$$\int_{-\pi}^{\pi} \ln\left|e^{i\omega} - r\right|^2 d\omega = \begin{cases} 0 & \text{if } |r| \leq 1, \\ 2\pi \ln|r|^2 & \text{if } |r| > 1. \end{cases}$$

Factorize both the numerator and denominator of the sensitivity function defined in (7) by the fundamental theorem of algebra,

$$|S(\omega)| = \left|\frac{1 - A(\omega)e^{-i\omega}}{1 - A(\omega)e^{-i\omega} + B(\omega)F(\omega)e^{-i\omega}}\right| = \frac{\prod_{j=1}^{m} \left|1 - \lambda_j e^{-i\omega}\right|}{\prod_{j=1}^{n} \left|1 - \mu_j e^{-i\omega}\right|},$$

where λ_j and μ_j are open-loop and closed-loop poles of the system. Then,

$$\begin{aligned} \int_{-\pi}^{\pi} \ln(|S(\omega)|^2) d\omega \quad &= \sum_{j=1}^{m} \int_{-\pi}^{\pi} \ln\left|e^{i\omega} - \lambda_j\right|^2 d\omega - \sum_{j=1}^{n} \int_{-\pi}^{\pi} \ln\left|e^{i\omega} - \mu_j\right|^2 d\omega \\ &= 4\pi \sum_{j:|\lambda_j|>1} \int_{-\pi}^{\pi} \ln\left|\lambda_j\right| d\omega \\ &\geq 0. \end{aligned}$$

The second equality uses Wu and Jonchheere's lemma, and it follows because all closed-loop poles μ_j are inside the unit disk in the complex plane; otherwise, the controlled system is not stationary and hence has not been stabilized by the control. The last inequality binds when all λ_j are inside the unit disk, *i.e.* when the uncontrolled system is stationary, and is strict when some λ_j are outside the unit disk. Therefore, Bode's constraint (10) holds.

A.2 Real Sensitivity Function of Optimal Policy Rule with Nonfiltered Data

This section provides some further intuitions for the real sensitivity function of the optimal policy rule with nonfiltered data developed in Section 2.3. As evident from expression (9), a good control should make the modulus of both $S(\omega)$ and $T(\omega)$ as small as possible at each ω.

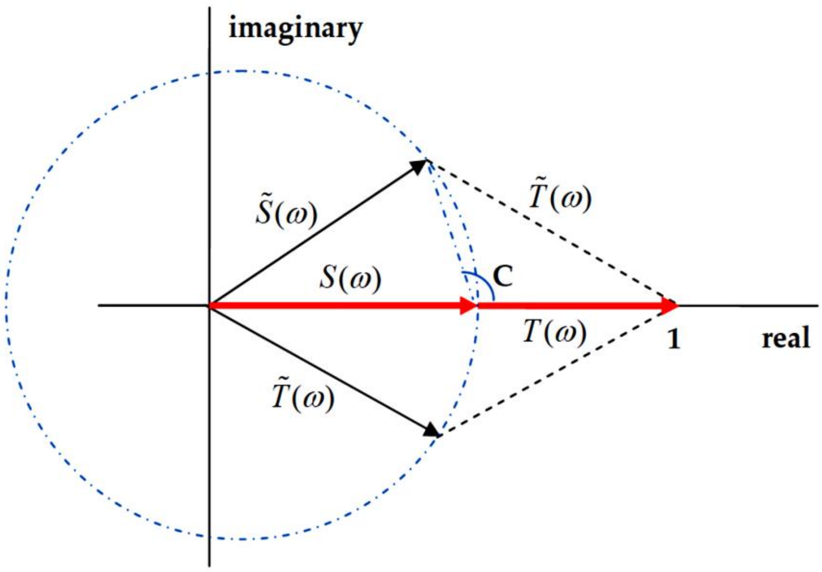
Consider two sensitivity functions, $\tilde{S}(\omega)$ and $S(\omega)$, and focus on frequency ω, where $\tilde{S}(\omega)$ is complex and $S(\omega)$ is real, but the modulus of $\tilde{S}(\omega)$ is the same as $S(\omega)$. As shown in Figure (9), $S(\omega)$ and $\tilde{S}(\omega)$ are then on the same circle in the complex plane. Due to the complementarity principle,

$$\tilde{T}(\omega) + \tilde{S}(\omega) = 1 \text{ and } T(\omega) + S(\omega) = 1.$$

Figure (9) shows that the angle C is always obtuse, the modulus of $\tilde{T}(\omega)$ is always greater than that of $T(\omega)$. Therefore, $S(\omega)$ performs better than $\tilde{S}(\omega)$ at frequency ω.

This argument follows at all frequencies, so the optimal solution must set the sensitivity function to be real. As argued in the text, a complex sensitivity function induces indelible phase shifts to the measurement error process n_t, which causes side noise effects in the presence of data noise.

Figure 9. Complex versus Real Sensitivity Functions, at Frequency ω



Note: At frequency ω, $S(\omega)$ is real, $\tilde{S}(\omega)$ is complex, $|S(\omega)| = |\tilde{S}(\omega)|$, $S(\omega) + T(\omega) = 1$, and $\tilde{S}(\omega) + \tilde{T}(\omega) = 1$.

A.3 Optimal Policy Rule with Nonfiltered Data in the Absence of Data Noise

The optimal policy rule $S^E(\omega)$ with nonfiltered data is characterized by (28) with spectral target λ^E determined by (29). For general case, I have to numerically parameterize the model and seek a quantitative solution. I use the bisection method for numerical implementation.[16]

In the case absent of data noise, however, I can show that the solution $S^E(\omega)$ reduces to the benchmark policy $S^B(\omega)$. Notice that $f_n(\omega) = 0$, $\forall\omega$ in this case and hence equation (29) gives

$$\ln\lambda^E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(f_{x^{nc}}(\omega))d\omega.$$

[16] Brock and Durlauf (2005) used Jensen's inequality to recover the representation of the feedback rule in the absence of measurement errors.

Given (5), we have $f_{x^{nc}}(\omega) = \frac{\sigma_\varepsilon^2}{2\pi}\left|\frac{W(\omega)}{1-A(\omega)e^{-i\omega}}\right|^2$. By factorizing $f_{x^{nc}}(\omega)$ and applying Wu and Jonchheere's result,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln(f_{x^{nc}}(\omega))d\omega = \ln\left(\frac{\sigma^2}{2\pi}\right).$$

Therefore, $\lambda^E = \lambda^N \equiv \frac{\sigma^2}{2\pi}$, which is same as the noise-free target. Substituting $\lambda^E = \frac{\sigma^2}{2\pi}$ into equation (28), it follows that

$$|S^E(\omega)|^2 = \frac{\sigma_\varepsilon^2}{2\pi f_{x^{nc}}(\omega)},$$

with is identical to $|S^B(\omega)|^2$ in equation (14).

A.4 Policy Rules without Lagged Terms

In the monetary model described in Section 4, the policymaker observes that the loss function (56) takes the form of

$$V(\pi) = \frac{(b\beta)^2\sigma_n^2 + \sigma_\varepsilon^2}{1-(\rho - b\beta)^2},$$

under the control of a policy rule without lagged terms as (57). To minimize the loss function, he derives the first-order conditions

$$b\beta[1-(\rho-b\beta)^2]\sigma_n^2 = [(b\beta)^2\sigma_n^2 + \sigma_\varepsilon^2](\rho - b\beta).$$

There are two roots to this equation. One is β^E as given in (59). Note that $|\rho - b\beta| < 1$ to ensure that π_t^E is stationary under the control. The other root violates this requirement since $|\rho| < 1$, and is not a solution.

Policy rule β^N in (58) follows obviously. Then,

$$\begin{aligned}
|\beta^E| \leq |\beta^N| \quad &\Leftrightarrow \frac{1}{2|\rho|\sigma_n^2}\left[\sqrt{[(1-\rho^2)\sigma_n^2+\sigma_\varepsilon^2]^2 + 4\rho^2\sigma_n^2\sigma_\varepsilon^2} - [(1-\rho^2)\sigma_n^2 + \sigma_\varepsilon^2]\right] \leq |\rho| \\
&\Leftrightarrow \sqrt{[(1-\rho^2)\sigma_n^2+\sigma_\varepsilon^2]^2 + 4\rho^2\sigma_n^2\sigma_\varepsilon^2} \leq \sigma_\varepsilon^2 + \sigma_n^2 + \rho^2\sigma_n^2 \\
&\Leftrightarrow 0 \leq 4\rho^2\sigma_n^4,
\end{aligned}$$

which proves the comparison result in (60).

A.5 Wiener Filter in the AR (1) Model

This section gives more details on the Wiener Filter in the AR (1) model that is used in Section 4.3. To start, notice that for AR (1) process π_t,

$$f_\pi(z) = \frac{\sigma_\varepsilon^2}{2\pi(1-\rho z)(1-\rho z^{-1})}.$$

Equation (66) then is based on this result.

The following shows why equation (69) holds:

$$\begin{aligned}\left[\frac{f_\pi(z)}{H^*_{\pi^*}(z)}\right]_+ &= \frac{\sigma_\varepsilon^2}{\sqrt{2\pi k}}\left[\frac{1}{(1-\rho z)(1-\phi z^{-1})}\right]_+ \\ &= \frac{\sigma_\varepsilon^2}{(1-\rho\phi)\sqrt{2\pi k}}\left[\frac{1}{1-\rho z}+\frac{\phi}{z-\phi}\right]_+ \\ &= \frac{\sigma_\varepsilon^2}{(1-\rho\phi)(1-\rho z)\sqrt{2\pi k}},\end{aligned}$$

where the last equality follows by observing that the term $\frac{\phi}{z-\phi}$ has a pole at $z=\phi$ inside the unit circle and hence is a forward transform which can be ignored.

Given (69), the causal Wiener filter is given as

$$M(z) \equiv \frac{[f_\pi(z)/H^*_{\pi^*}(z)]_+}{H_{\pi^*}(z)} = \frac{\phi\sigma_\varepsilon^2}{(1-\rho\phi)(1-\phi z)\rho\sigma_n^2},$$

which has the representation in the frequency domain as (70).