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Crude oil price dynamics with crash risk under fundamental shocks

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Abstract

Our paper presents a crude oil price model in which the price is confined in a wide moving band. A price crash occurs when the price breaches the lower boundary where a smooth-pasting condition is imposed. Using an asymmetric mean-reverting fundamental (supply/demand) shock, the solution derived from the oil price equation for the model shows the oil price follows a mean-reverting square-root process, which is quasi-bounded at the boundary. The oil price dynamics generates left-skewed price distributions consistent with empirical observations. A weakened mean-reverting force for the price increases the probability leakage for the price across the boundary and the risk of a price crash. The empirical results show the oil price dynamics can be calibrated according to the model, where the mean reversion of the price dynamics is positively co-integrated with the oil production reaction to negative demand shocks, and with the risk reversals of the commodity currencies, the Canadian dollar and the Australian dollar in currency option markets. The results are consistent with an increased price crash risk with negative demand shocks and negative risk reversals. The forecasting performance of the oil price model is better than the futures-spread models and random walk models during the crash periods. While the price of oil was above the lower boundary for most of the time, the conditions for breaching the boundary were met in 2008 and 2014 when the price fell sharply.

JEL classification: F31, G13

Keywords: Target zone; quasi-bounded process; crude oil, OPEC; oil demand shocks

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1. Introduction

Based on the intuition of OPEC's management of crude oil prices and by directly applying Krugman’s (1991) exchange rate target zone model, Hammoudeh and Madan (1995), Hammoudeh (1996) and Tang and Hammoudeh (2002) develop models to investigate oil price dynamics, in which oil price movements are manipulated by OPEC interventions and tempered by market participants' expectations of interventions by OPEC. Chapman and Khanna (2000, 2001, 2006) and Slaibi et al. (2010) develop target zone models with additional socio-political motivations for OPEC to keep the price within the zone. In their game-theoretic analyses, OPEC countries are willing to provide a steady supply of oil to western countries. If oil prices become too high and cause decline in gross national product growth in Western countries, their governments could react to high prices by establishing political or economic policies that promote substitutes for petroleum products. In addition, such policies can lead to a push for ending both the military and political support to the OPEC countries, thus increasing the probability of regime change for those nations. On the other hand, when prices are too low, this may cause national budget deficits in OPEC countries. Low oil prices also negatively impact the conservation and clean environment of Western countries. Therefore, managing oil prices within a targeted range is detrimental to OPEC and they may intervene with production levels to maintain oil prices within a zone. To test the target pricing zone hypothesis, Bharati et al. (2012) studies crude oil by examining price clustering within a zone if OPEC is able to defend the upper and lower bounds through output changes. Their results provide strong support for the underlying notion of the hypothesis, i.e., price management by OPEC. Slaibi et al. (2010) who employ a threshold autoregressive model and Monte Carlo simulations of a fitted autoregressive model find support for the hypothesis. While empirical studies have found evidence that the oil price is bounded within a zone, the Krugman-type models developed in previous studies assume a fully credible target zone with the price mean-reverting towards a central parity that is not observed empirically.

While empirical studies have found evidence that the oil price is bounded within a zone, sharp falls in the price of crude oil in 2008 and 2014, as shown in Figure 1, however demonstrate that the oil price is not always confined in a zone. This suggests Krugman-type models which assume a fully credible target zone are not able to capture the oil price dynamics with such price crashes. The 2008 global financial crisis (GFC) resulted in a major downturn in economic activity throughout the world, causing oil prices to fall by as much as 75% because of weak demand for oil amid the collapse of the global economy. For the sharp 57% fall in oil prices in 2014, Baumeister and Kilian (2016a) attribute it to a large negative flow demand shock associated with an unexpectedly slowing global economy; and a negative shock to storage demand reflecting a more positive outlook on oil production. Baumeister and Kilian (2016b) illustrate that most major oil price fluctuations dating back to 1973 are largely explained by shifts in flow (or consumption) demand for oil associated with the global business cycle (see the references therein). The empirical findings suggest that positive supply factors and a weak demand for oil played a role in both events and caused price crashes. While the crude oil price is bounded within a zone as shown empirically, it is at the same time subject to demand shocks and could thus crash.

Apart from economic aspects, after the 2008 crisis, many governments developed policies known as “Green Energy Acts” or “Renewable Energy Laws” that are intended to create an aversion to fossil-fuel, including crude oil, to reduce reliance on such fuel and meet environmental targets, in particular to keep global temperature rises well below two degrees Celsius. In addition to policy initiatives on green energy, the public consciousness on environmental issues and green energy investments could have contributed to the creation of a price responsive demand for crude oil. The surge in green energy initiatives coupled with significant budget deficits in major oil nations (OPEC, Russia, Norway) and the shale gas boom have made OPEC's management of crude oil prices difficult and led to turmoil in oil markets causing substantial oil price volatility and possible price crashes. Klevnäs et at. (2015) note that the oil price crash in 2014 has caused companies and countries to reconsider their energy choices by taking into account of the economic implications of uncertain and volatile oil prices, and of what this means for longer-term trends.

To the best of our knowledge, no previous study has been done to incorporate empirical evidence of both the oil price confined in a zone and price crashes into modelling the oil price dynamics. In view of this, we propose a model for the oil price dynamics with a crash risk, which is extended from the exchange rate target zone model developed by Lo et al. (2015) and Hui et al. (2016), which is applied to the Hong Kong dollar against the US dollar in a target zone and the Swiss franc against the euro during the target zone regime of September 2011 to January 2015 respectively. They found empirical evidence that the exchange rate process based on their model can describe the exchange rate dynamics and interest rate differentials of the currencies. The solution to the oil price derived from the equation of the proposed model is a function of the fundamentals of the oil price, including the supply and demand shocks of oil, and market participants' expectations of changes in the oil price. The model proposed in this paper is different from the Krugman-type models as it allows a price crash, and the price is mean-reverting towards a time-varying equilibrium level instead of a central parity.

In our proposed model, the oil price is bounded in a wide-moving band in which a lower boundary represents the price level indicating a price crash. Tang and Hammoudeh (2002) argue that a lower boundary is necessary to protect the national budgets of OPEC member countries. Indeed, they may intervene in production levels to maintain oil prices above the boundary. A smooth-pasting condition is imposed at the boundary for the oil price equation, suggesting an optimal boundary condition for the process, no foreseeable jump in the price and no arbitrage condition at the boundary. The smooth-pasting condition ensures that a price crash is rare. The same boundary condition is used in the model developed by Hui et al. (2018) to study currency crashes. A solution is then derived from the oil price equation for the model. We analyse the oil price model in which the boundaries are uncertain given the assumption that OPEC intervenes to keep within an assigned band that is not the current oil price level, but a moving average of the current and past prices over a time horizon. While the boundary is uncertain, it is updated over time as the oil price reaches possible values. If OPEC intervenes by reducing oil production, then uncertainty on the boundary is resolved.

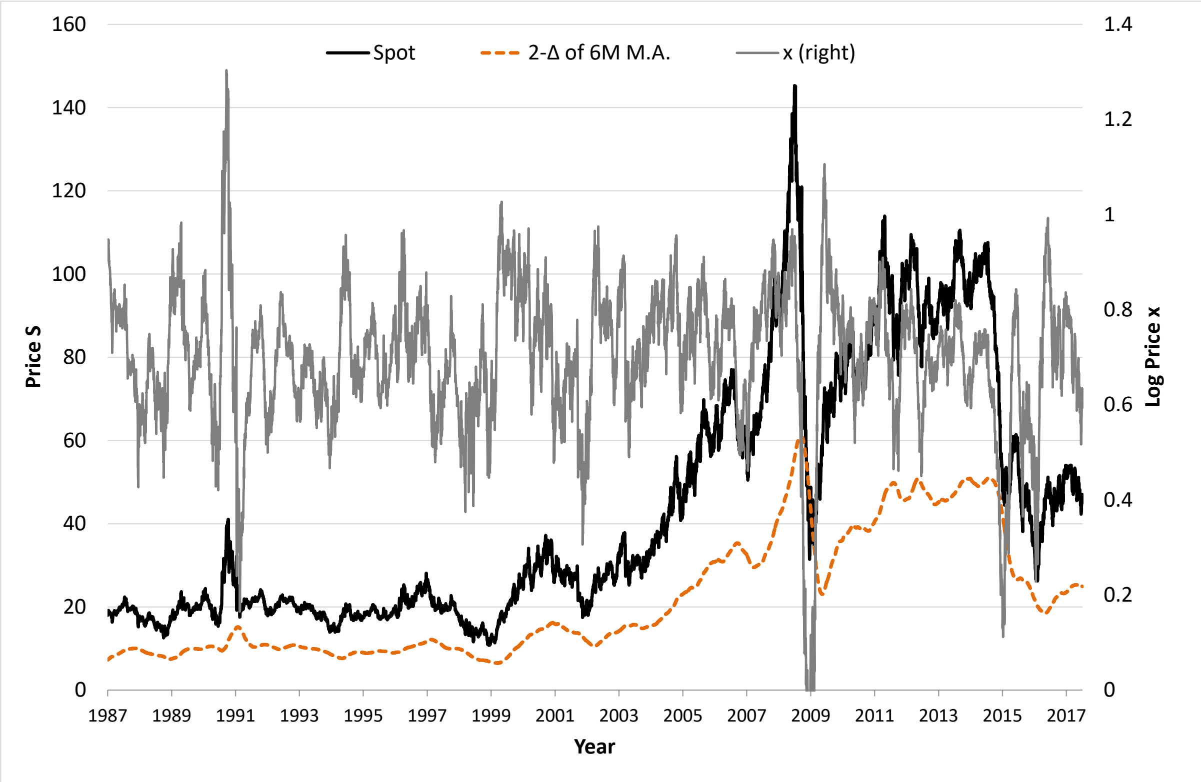
By using an asymmetric mean-reverting fundamental shock, the solution to the oil price equation shows that the log-normalised oil price follows a mean-reverting square-root process, which has a closed-form probability density function. The oil price exhibits the quasi-bounded characteristic at the lower boundary which can break down when the probability leakage condition is satisfied. The probability density function is able to generate left-skewed price distributions consistent with empirical observations. Based on the model, we can calibrate the oil price dynamics with market oil price data. The specification of the fundamental shock suggests there is a relationship between the oil price dynamics and OPEC's production. A co-integration analysis is used to test this relationship which provides empirical evidence for supporting the use of the asymmetric mean-reverting fundamental shock in the model.

The linkage between currency crashes and oil price crashes can be drawn from the finding by Chen et al. (2010) who empirically investigate the resulting dynamic relationship between commodity price movements and exchange rate fluctuations. They find that the exchange rates of commodity currencies (including the Australian dollar (AUD) and Canadian dollar (CAD)) are very useful in forecasting future commodity prices, including the oil price. With such a linkage, the mean-reverting square-root oil price dynamic derived from the model is expected to have a relationship with the observed risk reversals in the commodity currency option market. The “risk reversal”, which is the difference between the implied volatility of an out-of-the-money call option and an out-of-the-money put option at the same (absolute) delta[1], can be adopted to measure the price of currency crash risk. A co-integration analysis is used to test the relationship between the oil price dynamics and the risk reversals of the AUD and the CAD. In the oil price model, a weakening of mean-reverting force for the oil price will be joined by a rise in crash risk, indicating increased probability leakage for the price across the lower boundary increases. The risk reversals are expected to be positively related to the mean-reverting drift of the oil price dynamics. To further evaluate the performance of the proposed oil price model particularly during the oil price crash episodes, the forecasting performance of the oil price model is compared with that of the futures-spread models and random walk models.

The rest of our paper is structured as follows. In section 2, we develop the crude oil price model associated with price crashes. The corresponding price dynamics and probability density function are derived. Section 3 presents the calibration results using market data and the probability leakage condition. The oil price dynamic relationship with oil production and commodity currency risk reversals in currency option markets are studied by a co-integration analysis in section 4. Section 5 evaluates and compares the out-of-sample forecasting performance of the proposed model with random walk models and forecast models based on futures spreads, in particular during the periods of oil price crashes. The final section concludes.

[1] Taking the derivative of the option price w.r.t. the spot exchange rate gives the option delta. The price of a currency crash risk is reflected by the risk reversal, which measures the implied volatility difference between an out-of-the-money call on the currency and an out-of-the-money put at the same (absolute) delta. The risk reversal reflects asymmetric expectations on the directions of exchange rate movement.

Figure 1: Oil price in *S*-scale and *x*-scale, and lower boundary in *S* with 2 standard deviations ($\varDelta$) and 6-month moving average.



2. Crude oil price solution with crash risk

This section shows how a price crash is incorporated into a crude oil price model by imposing a smooth-pasting boundary condition at some lower price levels. The price solution is derived and analysed accordingly, showing that the oil price follows the mean-reverting square-root process and exhibits quasi-bounded feature at a lower boundary. The proposed model therefore allows the oil price to breach the lower boundary of a target zone under certain conditions in the price dynamics, suggesting a price crash. The probability leakage for the price across the boundary increases with a weakened mean-reverting force for the price dynamics, reflecting increased price crash risk. The Krugman-type target-zone models in previous studies which assume a fully credible target zone do not have such property. We demonstrate that the predicted price distributions generated from the proposed model can take several different shapes, which may correspond to possible price crashes and widely different (marginal and intra-marginal) intervention policies. The shapes of these distributions are consistent with the empirical observations of price returns, while the Krugman-type models only generate a U-shaped distribution which is not observed empirically.

2.1 Crude oil price model

We consider a basic log linear model of the oil price based on the standard flexible-price monetary model in the exchange rate target zone literature for a small open economy. The log oil price *s* at time *t* follows the equation:

$$s(t) = m + v + \alpha \frac{E[ds(t)]}{dt}, \tag{1}$$

where m is the logarithm of the constant oil supply, $v(t)$ is the logarithm of cumulative industry shock (fundamental) which could be a demand shock, a supply shock or the difference between the two, E is the expectation operator, and α is the absolute value of the semi-elasticity of the market price against its expected rate of change.[2] OPEC is prepared to change m by reducing the supply to prevent s from falling too fast or far.

The "fundamental" (v) is the source of uncertainty and is assumed to follow a stochastic process with a drift μ_v which is a function of v and instantaneous standard deviation σ_v:

$$dv = \mu_v dt + \sigma_v dZ, \tag{2}$$

where dZ is a Wiener process with $\mathrm{E}[dZ] = 0$ and $\mathrm{E}[dZ^2] = dt$. Ito's lemma is applied to Eqs.(1) and (2) to obtain,

$$\frac{E[ds(t)]}{dt} = \mu_v \frac{ds}{dv} + \frac{1}{2}\sigma_v^2 \frac{d^2 s}{dv^2}. \tag{3}$$

Then substituting Eq.(3) into Eq.(1) yields a 2nd-order linear ordinary differential equation:

$$\frac{1}{2}\alpha\sigma_v^2 \frac{d^2 s}{dv^2} + \alpha\mu_v \frac{ds}{dv} - s = -v - m \ . \tag{4}$$

If OPEC refrains from intervening to offset demand shocks to the fundamental and is expected to remain passive whatever the oil price moves, the driving process of the fundamental is relatively simple, i.e., a zero trend of $\mu_v = 0$. The oil price solution of Eq.(4) is:

$$s = v + m. \tag{5}$$

On the other hand, OPEC may intervene in the market at certain oil price levels to influence the price by altering the stochastic process governing (relative) oil-supply growth, which will alter the process driving the fundamental v. The stochastic fundamental no longer follows a zero trend μ_0 in Eq.(2), and the solution in Eq.(5) is therefore invalid. By incorporating the risk of a price crash, the oil price solution takes into account both the fundamental's dynamic process and the boundary conditions associated with the crash.

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[2] The speed of market price adjustment in Hammoudeh and Madan (1995) is absorbed in the variables m and v.

2.2 Crude oil price crashes and the smooth-pasting boundary condition

To derive the oil price solution, we consider that the value of the oil will likely fall significantly in a crash. Following Hui et al. (2018), we define a lower boundary as a tolerance limit for a distribution of the price's mean and standard deviation. Without assuming any distribution of the price *S*, the lower boundary S_L is taken to be the number (Δ) of standard deviations (Σ) from its mean $\overline{S}$: $S_L = \overline{S} - \Delta\Sigma$, suggesting that falling to the lower boundary is a crash, provided that the level of the lower boundary is adequately low.[3] The process of the price dynamics is not affected by the choice of the level of the lower boundary. Historical prices provide guidance for setting a trading band through which the price is not expected to breach. The historical trend of the price can be measured by a moving average $S_A(t)$ of the current and past price. For crude oil sustaining downward price pressure, the moving average can be scaled by a parameter η_L, with $0 < \eta_L < 1$, such that $\eta_L S_A(t)$ forms a lower boundary for the price movement. The parameter η_L tells how far OPEC can tolerate a fall in the oil price, or how much market participants expect the maximum or extreme downside risk of holding oil in terms of a fraction of the moving average value $S_A(t)$. A smaller η_L suggests OPEC tolerates, or the market expects, wider price fluctuations over short horizons. Such specification of a lower boundary assumes market participants and oil producers are mindful of oil price movements over a time interval, rather than just its current level. The derivation of the price solution and the qualitative results are not affected by the particular way to measure the historical prices. Similarly, an upper boundary can be set by using a scaling parameter η_U, with $\eta_U > 1$.

With no loss of generality, the normalised log oil price *s* is defined by:

[3] If a normal distribution is assumed, if Δ is set equal to 1.5 and 2, the cumulative normal probabilities when the price falls below the boundaries are 0.0668 and 0.0227 respectively and the corresponding percentage drops from the mean are 37.5% and 50%.

$$s = \ln\left[\frac{\eta_U S_{At} - S_t}{(\eta_U - \eta_L) S_{At}}\right], \tag{6}$$

where η_L and η_U are adjustable parameters for the lower and upper boundaries of a band respectively. By normalising the oil price with a moving boundary, which is scaled by a moving average of prices over a time horizon, the relationship between the oil price and the fundamental in the model depends upon the past history of the price. The proposed model shares a key feature of the soft exchange rate target zone model for the European Exchange Rate Mechanism proposed by Bartolinia and Pratib (1999). This model shifts the reference for intervention from the level of the exchange rate (oil price in our model) at each instant to the behavior of the exchange rate (oil price) over a time interval by featuring the central bank (oil producers) intervening to keep only a moving average of past exchange rates (oil prices) within a range. The main implication here is that it allows prices to fluctuate within a wider range over short time horizons giving oil producers more time to observe the price movements and market reactions. As such, they can postpone their decisions on reducing oil production until the negative oil demand shocks are exhausted. When the price breaches the boundary under a price shock, there is a discrete drop in the price with a magnitude that depends on the extent of the uncertainty.

To solve Eq.(4) with a crash when the price s breaches the lower boundary at $s = 0$ ($S_t = \eta_L S_{At}$), we specify the following boundary conditions at the fundamental of $\nu = 0$:

$$s(0) = 0, \tag{7}$$

$$\left.\frac{ds(v)}{dv}\right|_{v=0} = 0, \tag{8}$$

where the former condition ensures a proper normalisation of the oil price and the latter is the smooth-pasting boundary condition at $\nu = 0$ suggesting an optimal boundary condition for the process and no arbitrage. As shown by Krugman and Rotemberg (1990), the smooth-pasting condition of Eq.(8) ensures the price does not cross the boundary. If the condition breaks down, the price could jump across the boundary and crash.

Given that oil producers like OPEC will intervene when the oil price drops sharply under a negative demand shock, a simple way to model such interventions is to specify the fundamental dynamics in which a restoring force moves the fundamental towards a mean level and its magnitude is proportional to the deviation from the mean. One driving force behind the mean reversion could be attributed to the strategy of "leaning against the wind" adopted by OPEC. The mean-reverting fundamental can represent an error-correction policy on the part of OPEC by controlling its oil production, particularly under negative demand shocks. Another driving force from "stability speculation" by market participants could also restore the oil price to its long-run equilibrium whenever it drifts too far apart. OPEC can adopt both marginal and intra-marginal intervention policies. As OPEC will intervene more intensively when the oil price falls than when it rises sharply, to avoid negative impacts on the national incomes of major oil nations, the corresponding mean-reverting fundamental shock due to interventions is likely to be asymmetric, with a stronger force pushing the price away from the lower boundary. The asymmetry in the fundamental dynamics is also consistent with disaster risks, including oil price crashes, which are inherently asymmetric and one-sided events. The asymmetric fundamental shock is also similar to asymmetric country-specific and global shocks in exchange rate option prices (Bakshi et al. (2008); Jurek and Xu (2014)) and violations of uncovered interest rate parity (Backus, et al. (2001)).

Lo and Hui (2019) show that the fundamental ν is uniquely determined because of the boundary conditions in Eqs.(7) and (8), and that it follows an asymmetric mean-reverting process with the following specification:

$$d\nu = \left(\frac{A_{-1}}{\nu} + A_1\nu\right)dt + \sigma_\nu dZ, \tag{9}$$

where $A_1 < 0$, $A_{-1} > 0$, $-\infty < \nu \leq 0$.[4] To understand and visualise the asymmetric mean-reverting fundamental shock, we obtain a "potential well" $U(\nu)$ by integrating the drift term in Eq.(9), in a negative form, with respect to ν:

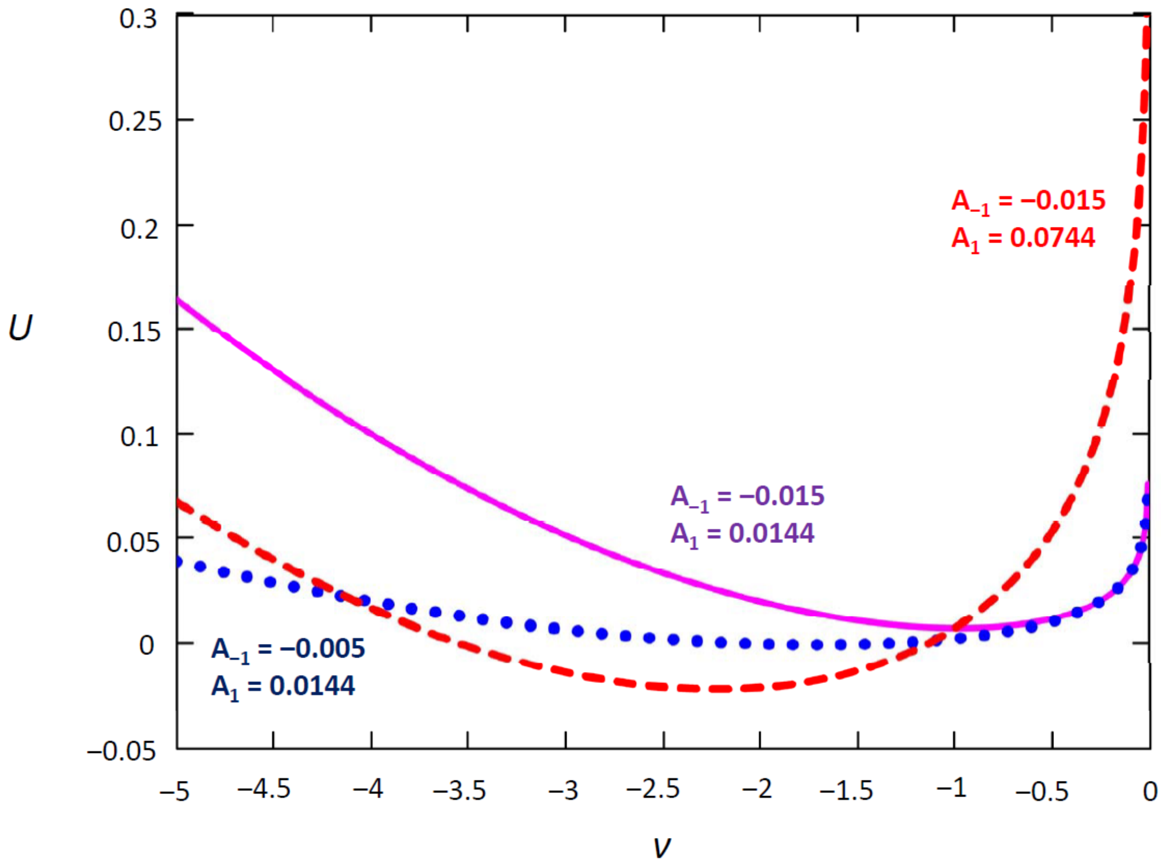
$$U(\nu) = -\int \left(\frac{A_{-1}}{\nu} + A_1 \nu\right) d\nu = -A_{-1} \ln|\nu| - \frac{1}{2} A_1 \nu^2 \,, \tag{10}$$

in which the fundamental variable ν is similar to a ball moving in a well, as shown in Figure 2 by plotting Eq.(10) with different values of A_{-1} and A_1. The shapes of the potential well indicate the capability of OPEC to intervene the oil price in the market. Decreasing the magnitude of A_1 will give an extremely flat potential well such that the Brownian force will dominate the motion of the fundamental. The fundamental can then move more randomly subject to a weak mean-reverting force. Likewise, the fundamental can move towards the lower boundary at $\nu = 0$ more easily together with a higher probability of ν breaching the boundary when A_{-1} decreases. This illustrates that the mean-reverting force in the fundamental dynamics determines the crash risk of the oil price. The figure also shows the mean-reverting force in Eq.(9) is not symmetric. The restoring force (an increase in price) given by the second term in the mean-reverting drift with v close to zero is stronger than the force (a decrease in price) provided by the first term. This is consistent with the intuition that when the demand for oil is extremely weak, such that the oil price falls significantly, the oil producers will intensively reduce the oil supply to push the price away from the lower boundary. The no leakage condition discussed in the following subsection ensures that the crude oil price will not breach the lower boundary; otherwise, the price may pass through the boundary, i.e., the price is quasi-bounded at the boundary. The no leakage condition may fail to hold at the boundary, therefore the smooth-pasting condition of Eq.(8) may break down.

[4] Lo and Hui (2019) provided a rigorous derivation of the asymmetric mean-reverting fundamental dynamics proposed by Lo et at. (2015) and Hui et al. (2016) for target-zone exchange rates, and has also shown that the proposed fundamental dynamics is indeed the unique choice and is described by the Rayleigh process.

The quasi-bounded process for the oil price is derived and discussed in the following subsection.

Figure 2: Eq.(10) of $U(v)$ by integrating drift term of fundamental dynamics with different model parameters A_1 and A_{-1}.



2.3 Oil price dynamics and probability density function

By the power series method we are able to obtain the desired solution of the ordinary differential equation in Eq.(4) with the boundary conditions specified by Eq.(7) and Eq.(8) in the form:

$$s(v) = v^2 \sum_{n=0}^{\infty} B_n v^n \qquad (11)$$

which vanishes at $v = 0$. Lo and Hui (2019) derive the coefficients in Eq.(11) as

$$B_0 = -\frac{m}{\alpha(\sigma_v^2 + 2A_{-1})}, \qquad B_1 = -\frac{1}{3\alpha(A_{-1} + \sigma_v^2)}$$

$$B_{n+2} = \frac{2[1 - \alpha(n+2)A_1]}{\alpha(n+4)[2A_{-1} + (n+3)\sigma_v^2]} B_n \qquad \text{for } n = 0,1,2, \ldots$$

and show that:

$$B_2 = \frac{1}{2}\frac{d^2 s}{dv^2}\bigg|_{v=0} = -\frac{m}{\alpha\left(\sigma_v^2 + 2A_{-1}\right)} < 0\,, \tag{12}$$

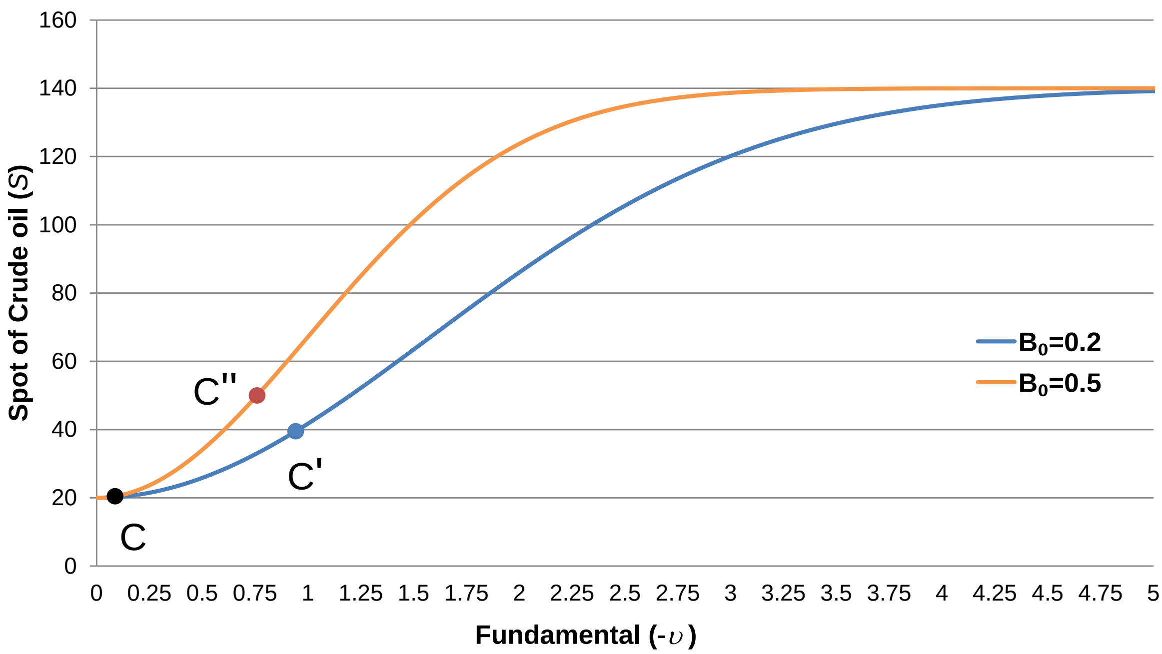
suggesting s attains its maximum at $v = 0$. The second-order linear ordinary differential equation of Eq.(4) uniquely determines the second-order derivative of s with respect to v at $v = 0$ by itself.

Lo and Hui (2019) shows the series solution is a convergent series for all v by means of the ratio test. Motivated by the rapid convergence of the series solution shown in those studies, we propose to approximate the exact solution by an optimal approximate solution of the form:

$$s(v) = B_0 v^2 = -\frac{m}{\alpha(\sigma_v^2 + 2A_{-1})} v^2 \tag{13}$$

Figure 3 plots the relationship between the oil price and the fundamental expressed in Eq.(13). It shows that changes in the price flatten with changes in the fundamental at the two boundaries. This implies that even when the fundamental changes substantially, oil price could marginally move away from the boundaries. When a negative demand shock pushes the price towards its lower boundary, there is a counteracting tendency of a mean reversion to the equilibrium level which acts as a stabilising force, as shown in Eq.(9), to limit a further fall in the price. One of the factors behind the restoring force is OPEC’s action to reduce oil production. Based on the model, as changes in oil production alter the fundamental, the oil price could move from C to C” or C’, where the paths depend on the coefficient B_0 in Eq,(13), which represents the state of the crude oil market including the oil supply (m), parameters (A_{-1}) of the asymmetric fundamental shock, and sensitivity (α) of price to its expected rate of change. A larger B_0 suggests that the oil price is more sensitive to changes in the fundamental (demand). This happens when the oil supply m is relatively ample or both A_{-1} and α are relatively small. This means the restoring force in the fundamental dynamics is weak (as shown in Figure 2) and the price is less sensitive to the expected price, generating higher oil price crash risk. It can be shown analytically that conditional on a surge in price volatility or a weakening of the mean reversion in the price dynamics, the smooth-pasting condition can break down, thus triggering a crash with the price jumping across the lower boundary.

Figure 3: Relationship between oil price (S) and fundamental (ν) based on Eq.(13) with sensitivities B_0 = 0.2 and 0.5 of oil price to its expected rate of change.



To illustrate the oil price dynamics, we use the notation $x \equiv -s$ so $0 \leq x < \infty$ with $x = 0$ corresponding to the lower boundary. By applying Ito's lemma to Eq.(9) with Eq.(13), x is shown to follow a mean-reverting square-root (MRSR) process:

$$dx = \kappa(\theta - x)dt + \sigma_x \sqrt{x} dZ \ , \tag{14}$$

where

$$\kappa = 2|A_1|, \qquad \theta = \left|\frac{B_0}{A_1}\right| \left(A_{-1} + \frac{1}{2}\sigma_\nu^2\right) \tag{15}$$

$$\sigma_x = 2\sigma_\nu \sqrt{\lceil B_0 \rceil} \qquad . \tag{16}$$

κ determines the speed of the mean-reverting drift towards the long-term mean θ. The drift stabilises the price movement and reduces the crash risk. When the price is close to the lower boundary of $\nu = 0$, the standard deviation $\sigma_x \sqrt{x}$ becomes very small with the corresponding price dynamics dominated by the mean-reverting drift which pushes the price towards the mean and away from the boundary. The time-varying equilibrium level of the long-term mean θ can be determined through changes in crude oil production and action by market participants in restoring the price towards its mean level. The size of κ suggests how strong the mean-reverting force for the oil price dynamics is. The properties of the MRSR process is also shown by the well-known Cox–Ingersoll–Ross (CIR) model (1985) for interest rate term structures.

Following Feller's classification of boundary points, it can be inferred that there is a non-attractive natural boundary at infinity (i.e. inaccessible) and that the one at the origin is a boundary of no leakage for $(\sigma_x^2 / 4\kappa\theta) < 1$ in Eq.(14), and it is not otherwise.[5] When the no-leakage condition holds, it prevents the oil price to breach the lower boundary; otherwise, the price may pass through the boundary, i.e., the price is quasi-bounded at the origin.[6,7] If the no-leakage condition does not hold at the boundary, the smooth-pasting condition of Eq.(8) may break down and a crash could occur. The boundary condition at the origin under the MRSR process is studied in CIR (1985) and Longstaff (1989, 1992).

The effectiveness of OPEC in defending the oil price near the lower boundary is reflected in the drift coefficient κ. In addition to OPEC's interventions, market participants who believe that OPEC is able to prevent the oil price breaching the boundary could engage in 'stabilising speculation', which helps to push the price towards its mean level θ. Such

[5] For boundary condition definitions, see Karlin and Taylor (1981).
[6] Hui et al. (2016) find empirical evidence that the quasi-bounded process can describe the price dynamics and interest rate differential of the Swiss franc against the euro during the target zone regimen of September 2011 to January 2015.
[7] Such a property is similar to the bounded price dynamics in Ingersoll (1996) and Larsen and Sørensen (2007) in which the price is completely bounded under all circumstances.

effects increase the size of κ, suggesting a stronger mean-reverting force for the oil price dynamics. Conversely, if market participants anticipate that OPEC is not capable of managing the oil price because of weak demand, their speculative selling of oil, which induces a substantial reduction in OPEC's oil production, will weaken the restoring force (i.e., smaller κ) towards its mean level and increase the price crash risk. The empirical results in section 4.1 below show that the parameters (κ and θ) of the mean-reverting force have a long-run positive relationship with oil production, suggesting a negative relationship between crash risk and oil production.

The probability density function (PDF) of x under the MRSR process is given by:

$$\begin{aligned} G(x,t;x',t') = {} & \frac{2}{\sigma_x^2 C_1(t-t')}\left(\frac{x}{x'}\right)^{\omega/2} \exp\left[-\frac{\omega+2}{2}C_2(t-t')\right]\times \\ & \exp\left\{-\frac{2x'+2x\exp[-C_2(t-t')]}{\sigma_x^2 C_1(t-t')}\right\}\times \\ & I_\omega\left\{\frac{4x^{1/2}x'^{1/2}\exp[-C_2(t-t')/2]}{\sigma_x^2 C_1(t-t')}\right\} \end{aligned}, \qquad (17)$$

where $\omega = 2\kappa\theta/\sigma_x^2 - 1$, $C_1(\tau) = [\exp(\kappa\tau)-1]/\kappa$, $C_2(\tau) = -\kappa\tau$, I_ω is the modified Bessel function of the first kind of order ω. Based on Eq.(17), the parameters of the MRSR process for the price dynamics are calibrated in section 3 using market price data. The associated asymptotic PDF will ultimately converge to the following steady-state distribution:

$$K(x,t\to\infty;x',t') = \frac{2x^{\omega+1/2}}{\Gamma(\omega+1)}\left(\frac{2\kappa}{\sigma_x^2}\right)^{\omega+1}\exp\left(-\frac{2\kappa}{\sigma_x^2}x\right), \qquad (18)$$

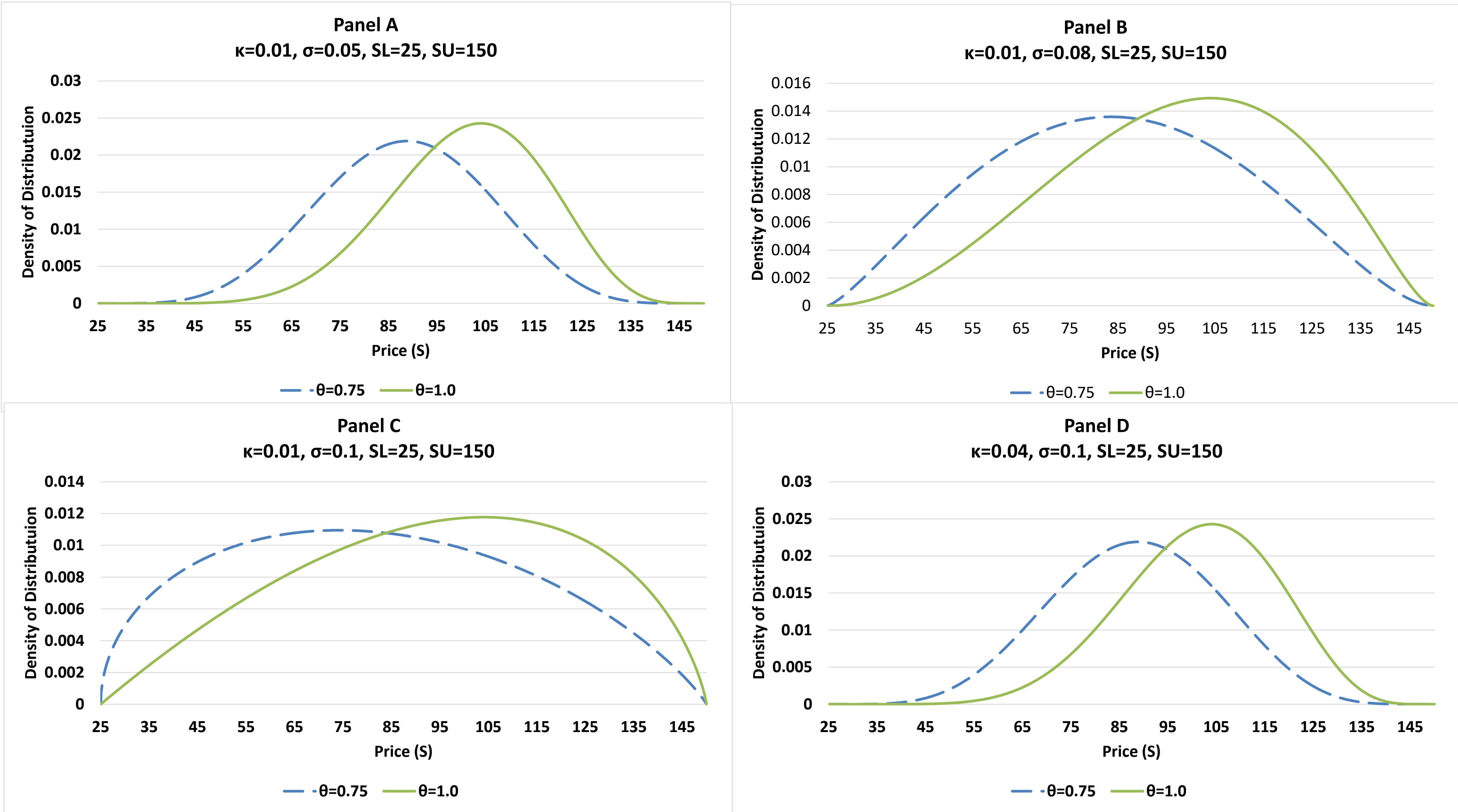
where Γ denotes the gamma function.

Figure 4 presents the steady-state price distributions in S based on Eq.(18) with two values of the long-term mean θ of 0.75 and 1.0 under different choices of model parameters respectively: the former θ (in blue dotted line) is closer to the lower boundary than the latter (in green solid line). We use the model parameters for $\sigma_x = 0.05$, 0.08 and 0.1, and κ = 0.01

and 0.04, which are consistent with the estimations in section 3, with the lower boundary $\eta_L S_{At} = S_L = 25$ and upper boundary $\eta_U S_{At} = S_U = 150$. The peaks of the distributions are located at the right side, reflecting the PDF will decay more slowly at the left side compared to a Gaussian distribution, i.e., the so-called "fat-tails" effect. This indicates the probability of outlier negative returns. This feature is consistent with the empirical observations of price returns and the left-skewed distributions in Ball and Mankiw (1995), Askari and Krichene (2008) and Wu et al. (2012).

Panels A, B and C in Figure 4 respectively display the price distributions under different values of σ_x, while keeping κ constant. It can be shown that the price distributions exhibit fatter left tails and hump shape when the value of σ_x increases from 0.05 to 0.1. This shows that their left-skewness is sensitive to an increase in the price volatility, suggesting the higher price volatility increases the likelihood of a crash reflected by the fat left tails. By holding σ_x = 0.1 and varying κ from 0.01 to 0.04 in Panel D, the price distributions have less fat left tails and the shapes of the distributions become similar to those in Panel A. This is consistent with an increase in the mean reversion in the price dynamics reduces the crash risk. When the long-run mean θ is more distanced from the lower boundary, the corresponding oil price distributions have fatter tails in all panels in Figure 4. The changes in the property of price distribution suggest that when there is an expected increase in the oil price in the near term, the probability of outlier negative returns becomes more significant. The price distributions in Figure 4 with different price parameters illustrate that the left-skewed price distributions are consistent with crash risk measured by the leakage condition of the MRSR process of the price dynamics derived from the model.

Figure 4: Price distributions with different values of model parameters σ_x, κ and θ.



3. Empirical test of crude oil price dynamics

3.1 Calibration of model parameters

In this section, we investigate whether the MRSR process derived from the oil price model can describe the oil price dynamics. We estimate the model parameters of the process specified in Eq.(14) using the maximum likelihood estimation (MLE), where the log-likelihood function is derived from the PDF of Eq.(17). Oil price data in daily frequency are used for estimation with the sample period from 16 January 1987 to 6 July 2017. Figure 1 shows the oil prices in S and the associated moving boundary with two standard deviations (Δ) and 6-month moving average, and the transformed price in x of the time series. It also shows that falls in the oil price occured in 2008 and 2014 respectively. The former fall in 2008 caused the price to breach the boundary, while the oil price stayed above the boundary in the latter event in 2014.

Figure 5, which is based on the estimation result using a 3-year rolling window, shows that the estimates of the drift term κ (reported in Panel A) was significant with the z-statistic above the 1.96 level (i.e., at the 5% significance level) during 1990 – 2008 when κ was higher than 0.01. κ increased from 0.02 in 2004 to above 0.04 in 2006, reflecting the increased restoring force pushing the oil price towards its long-term mean. However, the drift then became weaker after 2006 and dropped below 0.01 when there was a sharp fall in the price in September 2008 with the onset of the GFC. As the oil price dropped towards its lower boundary during this period, as shown in Figure 1, the mean-reverting force diminished with the price crash. It is noted that the estimation of κ became insignificant from September 2008 to the end of 2011, with κ not significantly different from zero. Subsequently, the estimation rebounded to the 0.2 level with the z-statistic higher at 1.96 before falling sharply again in the second half of 2014. The analysis of the oil crash risk is presented in the following subsection.

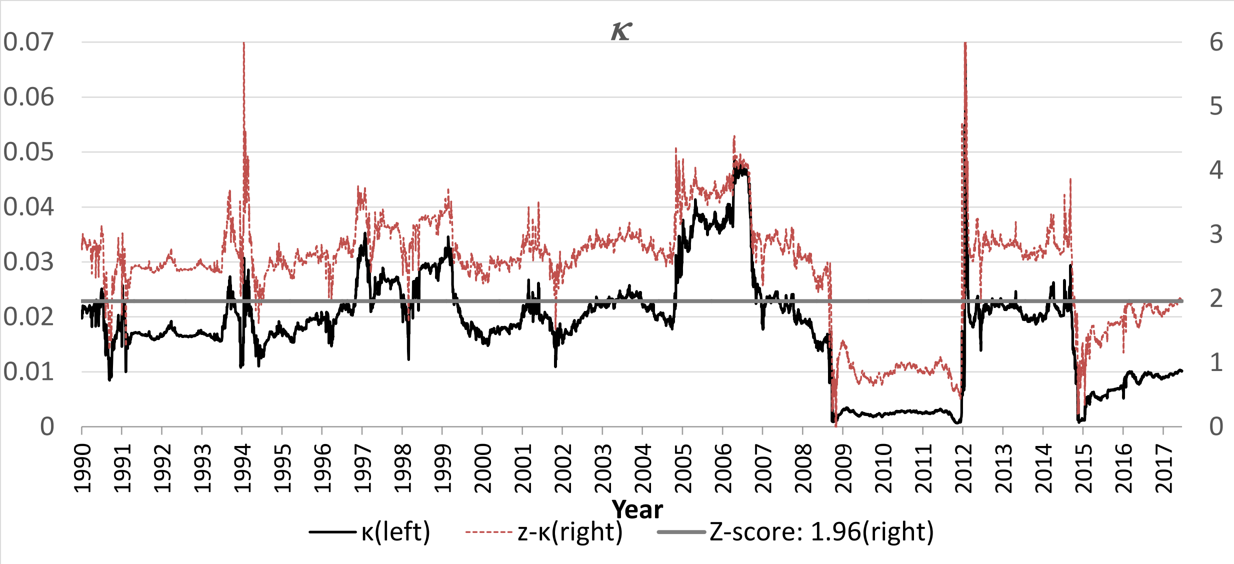
Panel B of Figure 5 shows that the estimated mean θ is steady and mainly ranges between 0.5 and 1.0 with the *z*-statistic higher than 1.96 during 1990 – 2008. It is then marginally below the 1.96 level after the 2008 GFC until end-2011. Subsequently θ dropped to 0.75 and was significant except for a short period in 2014 when the oil price fell sharply. In terms of statistical significance, the parameters κ and θ representing the mean reversion of the oil price dynamics exhibited similar patterns.

Panel C of Figure 5 shows that the estimate of the volatility σ_x ranges between 0.02 and 0.07, with corresponding *z*-statistic much higher than 1.96. The result indicates a highly significant σ_x except in a very short period in 2008 and suggests a robust estimation of the square-root process component of the quasi-bounded price dynamics. The volatility increased after the sharp falls in the oil price in 2008 and 2014 when the restoring force weakened with the decreased κ and θ.

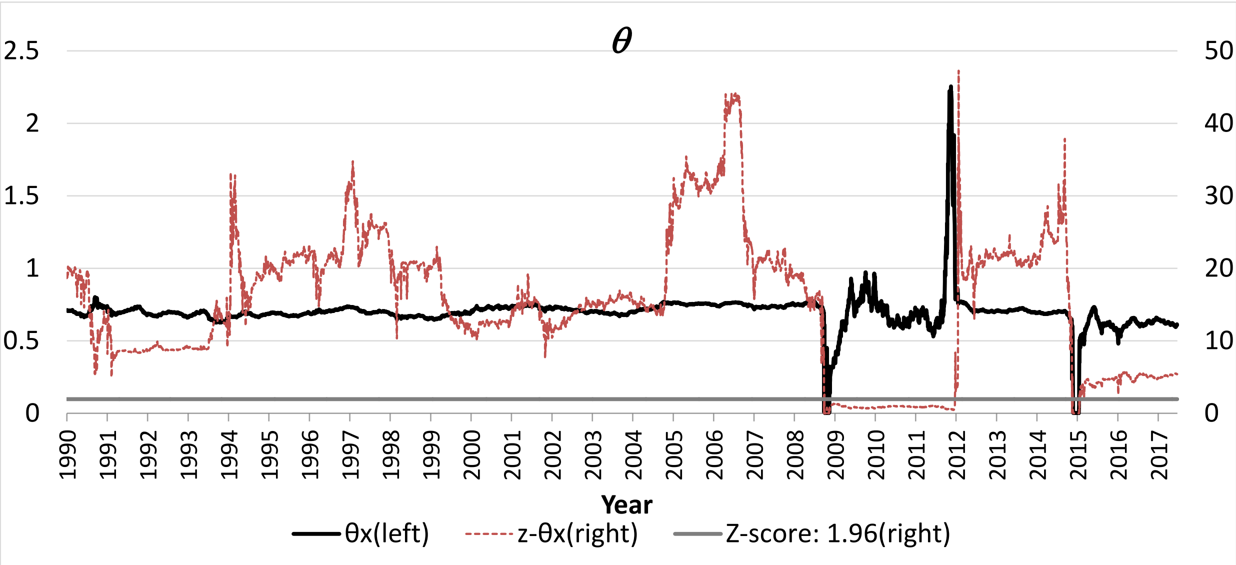
In summary, the estimation results using market data from 16 January 1987 – 6 July 2017 based on the MLE shown in Figure 5 provide evidence that the MRSR process adequately fits the data on the crude oil price with a lower boundary representing the price crash level. By using the 3-year rolling window, the short-term price dynamics are captured in such a way that the estimated model parameters are shown to be time varying. The mean-reverting force, which is represented by the parameters κ and θ, is estimated to be present during most of the estimation period, except for the periods following the sharp price falls after the 2008 GFC and in 2014. The weakening mean-reverting force and the rise in the volatility σ_x show crash risk building at the lower boundary during the price falls.

Figure 5: Estimated κ (Panel A), θ (Panel B), σ_x (Panel C) and corresponding *z*-statistic with moving boundary of 2 standard deviations (Δ) and 6-month moving average using 3-year rolling window.

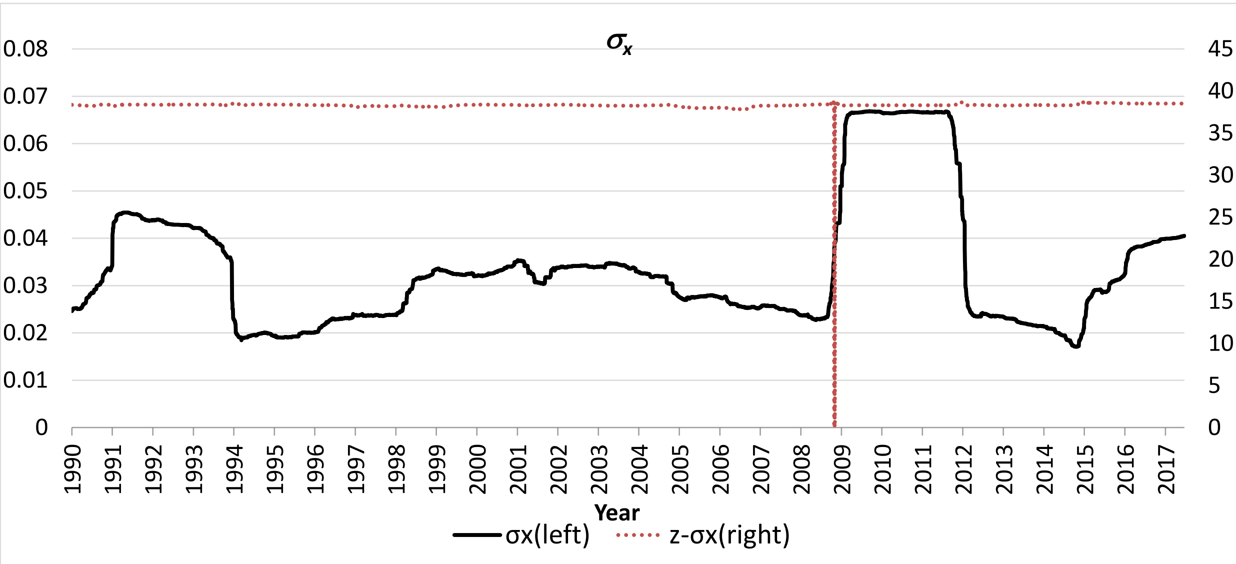
Panel A



Panel B



Panel C



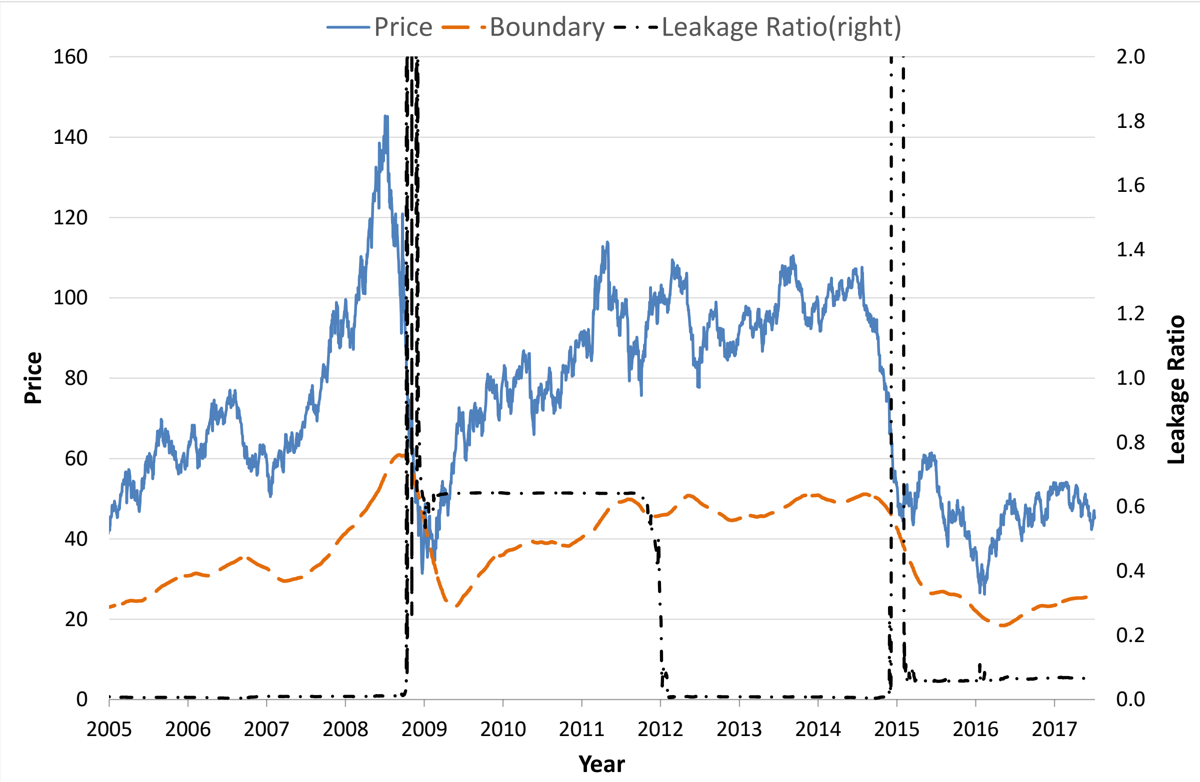
3.2 Oil price crash risk

As the condition of $(\sigma_x^2 / 4\kappa\theta) < 1$ indicates no probability of leakage at the lower boundary, Figure 6 shows the measure $(\sigma_x^2 / 4\kappa\theta)$ based on the estimations using the 3-year rolling window to study the crash risk of the crude oil price. The measure was substantially below 1 from 1989 to October 2008, suggesting there was no concern over the probability of leakage indicating that the oil price was well bounded above the lower boundary. Given that OPEC's output quota and associated production changed during this period, there was every indication OPEC's policy of adjusting the supply to meet the demand was working adequately. However, the situation changed substantially after the onset of the GFC. Conditions worsened sharply in October 2008 with the existence of the "leakage condition", i.e. $(\sigma_x^2 / 4\kappa\theta) \geq 1$, when the oil price fell drastically from \$145 per barrel on 3 July 2008 to \$85 per barrel on 10 October 2008, indicating that the price was no longer bounded at the lower boundary, which was then breached on 18 November 2008. The increase in the leakage was consistent with a further fall in the price to \$34 on 19 December 2008. This was a reaction to the expected sharp fall in the demand for crude oil due to the severe global economic slowdown. At the same time, OPEC was not able to maintain the oil price by adjusting the production rate.

The second major leakage condition came on 8 December 2014 when the price of oil dropped to \$63 from \$107 on 20 June 2014, indicating that the price might not be bounded at the lower boundary. Despite the drop, OPEC had previously (November 2014) decided to maintain production levels. On 26 January 2015, the price fell again to \$45, consistent with the unbounded price dynamics suggested by the leakage condition. Oil market participants apparently considered that OPEC had stopped acting as a swing producer given rapidly rising non-OPEC production such as shale oil.

The existence of the leakage conditions on 10 October 2008 and 8 December 2014 using only information at that time, and the subsequent price falls, indicate that the dynamics of the oil price are informative. They also suggest an erosion of OPEC's ability to maintain a targeted oil price by reducing production, particularly when the demand for oil drops substantially.

Figure 6: Estimated values of the leakage ratio $(\sigma_x^2/4\kappa\theta)$ from 2005 – 2017 with lower boundary of 2-Δ with 6-month moving average using 3-year rolling window.



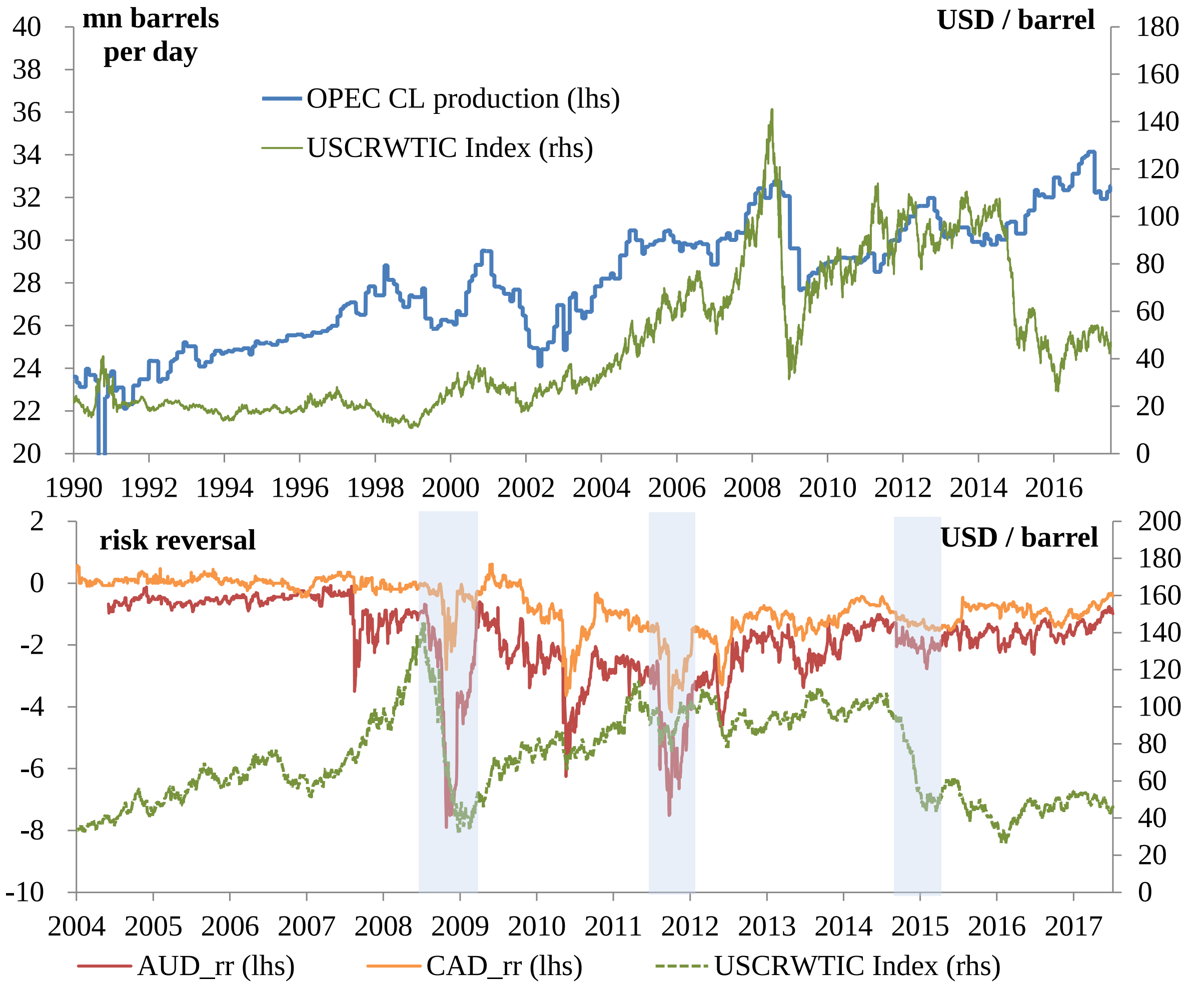
4. Dynamic relationship between oil price dynamics and market factors

4.1 The relationship between mean reversion in oil price dynamics and oil production

To maintain the price of oil, producers including OPEC, will reduce their oil production to prevent the price falling close to the lower boundary due to negative demand shocks. The upper panel in Figure 7 shows the crude oil price and OPEC's daily crude oil production respectively. By changing production such intervention can affect the mean reversion of the oil price dynamics governed by the parameters κ and θ, which reflect the intervention policy adopted by the oil producers under severe demand shocks. To examine the inter-relationship between the variables considered, the co-integration method is adopted for the analysis. We verify empirically that the calibrated model parameters in Eq.(14) of the oil price dynamics response to shocks on the oil price fundamental. Specifically, the supply shock on the fundamental is represented by changes in crude oil production volume, whereas an oil price crash identified by Baumeister and Kilian (2016a, 2016b) as a demand shock is measured by crash risk of "commodity currencies" in the following subsection. If there exists interactions between the model parameters and these market factors, this suggests the underlying fundamental dynamics as proposed in Eq.(9) and the price crash risk feature incorporated in the oil price model adequately determine the actual oil price dynamics. Since the pioneer work by Engle and Granger (1987), the co-integration analysis has been widely applied to study long-run relationships between non-stationary variables.[8] As such, this section studies the dynamic relationship between the shocks and the model parameters using the co-integration analysis.

[8] For instances, Kaufmann et al. (2004) apply this methodology to study OPEC's influence on the price of oil.

Figure 7: (Upper panel) OPEC's daily crude oil production and crude oil price (USCR WTIC Index); (lower panel) crude oil price, AUD-USD risk reversal (AUD_rr) and CAD-USD risk reversal (CAD_rr).



Provided that long-run equilibrium relationships between the model parameters and OPEC oil production exist, their short-run dynamics can be represented by a dynamical error-correction model as follows:

$$\Delta y_t = a_{10} + \alpha_y(y_{t-1} - \beta_1 X_{t-1}) + \sum_k b_{1k}\Delta y_{t-k} + \sum_k c_{1k}\Delta X_{t-k} + \varepsilon_{yt} \ , \qquad (19)$$

where y_t is either κ or θ at time t, and α_y is less than zero. X_{t-1} is a logarithm of OPEC's oil production ln(Oil) at time t-1. Under this specification, y_t will response to stochastic shocks (represented by ε_{yt}) and also the previous period's deviation from the long-run equilibrium (i.e., $y_{t-1} - \beta_1 X_{t-1}$). The estimated speed of adjustment α_y must be nonzero for the co-integration and error-correction model to be a valid specification. In terms of absolute magnitude, a larger estimated value of α_y reflects a greater response of y_t to the previous period's gap from the long-run equilibrium.

We estimate the monthly error-correction model with the sample during 19 January 1990 - 26 June 2017. The month-end estimations are used for the model parameters κ and θ based on the 3-year rolling window for the lower boundary with 2Δ and the 6-month moving average. Table 1 reports the summary statistics, correlation coefficient and the respective Augmented Dickey–Fuller (ADF) test results for the interested variables both in levels and first differences. The ADF tests suggest that the presence of a unit root for κ, θ and oil production in levels cannot be rejected at the 10% significance level. Nevertheless, the respective ADF tests for the first differenced variables are rejected at the 1% level. Thus, their levels are non-stationary while the first differenced forms are stationary. The above results suggest that κ , θ and oil production are all integrated of same order 1 (i.e. I(1)) and satisfies the requirement for being co-integrated.

We use the Engle–Granger single-equation test which is proposed by Engle and Granger (1987) to test the co-integration between κ, θ and oil production. This co-integration test in essence tests the stationarity of the residuals of the linear combination among any two potentially co-integrated variables based on the ordinary least squares method. Table 2 reports the co-integration tests between oil production and κ, θ. We employ the ADF and Philips-Perron tests to check whether the residuals of the regression of oil production and κ, θ are stationary. The results from the unit root tests, which adopt Akaike information criterion as selection criterion and critical values based on MacKinnon (1996), suggest significance for κ and θ at the 5% or 1% level. We reject the null hypothesis that oil production and κ, θ are not co-integrated in favour of the alternative hypothesis that there is at least one co-integrating vector.

Table 3 reports the estimated co-integrating vectors between oil production and κ, θ. The positive coefficients β for κ and θ are 0.033 and 0.187 respectively at the 5% level, suggesting that, other things being equal, a reduction in oil production would decrease the κ and θ. Intuitively, the positive relationship suggests that OPEC reduces its production when the crash risk of the oil price increases, which is reflected in the weakened restoring force of the oil price dynamics. When the crash risk is induced by negative demand shocks, such as that during the GFC, OPEC's reaction to such shocks is to cut its production in an attempt to stabilise the price, limit further falls and reduce the crash risk. Under OPEC's reaction, both κ and θ decrease with reduced production as shown in the empirical results. Conversely, OPEC is willing to increase its production in the event of a reduced crash risk when the mean reversion parameters κ and θ increase.

Table 4 shows that the estimates of the speed of adjustment α_y for κ and θ are -0.103 and -0.081 respectively, which are both negative but greater than -1. This suggests a self-restoring force exists to close the gap between the mean reversion parameters (κ and θ) and oil production. κ and θ will subsequently adjust to restore the long-run equilibrium.

Table 1: Descriptive statistics of oil production, κ and θ.

	ln(Oil)			κ			θ		
	Level	Change		Level	Change		Level	Change	
Mean	10.2223	0.00105		0.02111	-6.73E-06		0.70337	-0.00054	
Median	10.2190	0		0.02003	-1.39E-05		0.70235	-0.00023	
Maximum	10.4382	0.14609		0.04661	0.03138		0.85769	0.04109	
Minimum	9.88226	-0.17866		0.00683	-0.01831		0.56231	-0.09082	
Std. Dev.	0.10886	0.02022		0.00702	0.00304		0.0377	0.01152	
Skewness	-0.2042	-1.29317		1.1565	-2.6598		-0.2785	-1.4877	
Kurtosis	2.5796	32.5077		5.1199	45.3254		3.9884	15..984	
ADF test statistics	-1.8155	-13.355	***	-2.1937	-7.1041	***	-2.2746	-9.5428	***
Correlation with ln(Oil)/Δln(Oil)				0.1083	0.0799		0.0928	-0.1454	
Observations	302	299		302	299		302	299	

Notes:
1. *** indicates significance at the 1% level.
2. The ADF tests check the null hypothesis of unit root existence in the time series, assuming nonzero mean in the test equation.
3. (Oil) is the oil production. The correlations for level of the variables are the correlations with ln(Oil), and those for change are the correlation with ln(Oil).

Table 2: Tests for co-integration of oil production, κ and θ.

	Engle-Granger single-equation test[2] (Null hypothesis: residual has an unit root)	
	ADF test statistic	Phillips-Perron test statistic
Equation:		
κ	-2.571 ***	–3.698 ***
θ	–2.826 ***	–5.050 ***

Notes:

1. *** represents statistical significance at the 1% level.

2. The Engle-Granger single-equation test (ADF and Phillips-Perron tests) examines the null hypothesis that the residuals of the regressions of κ on ln(Oil) and θ on ln(Oil) respectively, given that κ, θ and ln(Oil) are non-stationary. We have assumed a zero mean in the residuals in the test equation, with the respective critical value based on MacKinnon (1996).

Table 3: Estimates of long-run coefficient (β) for oil production, κ and θ.

Dependent variable:	κ	θ
ln(Oil)(β)	0.033 **	0.187 **

Notes: ** indicates significance at a level of 5%. The coefficients are estimated by using the Engle-Granger single-equation and the coefficients of the short-run dynamic are in Table 4.

Table 4: Estimation results of the short-run dynamics for oil production, κ and θ.

Dependent variable:	$\Delta\kappa_t$	$\Delta\theta_t$
Constant	-0.033 **	-0.097
Speed of adjustment	-0.103 ***	-0.081 ***
Δln(Oil)$_{t-1}$	0.013 *	-0.068 **
$\Delta\kappa_{t-1}$	-0.102 **	
$\Delta\theta_{t-1}$		0.131 **

Notes: ***, ** and * respectively represent statistical significance at the level of 1%, 5% and 10% respectively. Dummy variables for the global financial crisis (May-Oct 2008), European sovereign debt crisis (Jan-Jun 2012) and period after the oil crash (since Oct2014) are added in the estimation of the short-run dynamics. The coefficients are estimated by using the Engle-Granger single-equation and the long-run coefficients are in Table 3. For brevity, the estimated coefficients of dummy variables are not shown in this table.

4.2 Relationship between mean reversion in oil price dynamics and commodity currency risk reversals

It can be argued that oil price crashes are rare events and can be compared to a world disaster. Disasters correspond to bad times when asset prices fall drastically despite the fact they occur with a low probability. Regarding currency crashes, Farhi et al. (2015) propose a disaster-based structural model in which investors incorporate a currency crash-risk premium into the value of the exchange rate, and calibrate the crash probability to option prices.[9] Farhi and Gabaix (2016) develop a model that when the risk reversal of a country goes up (more negative), its currency contemporaneously depreciates. The price of hedging against the currency's downside risk (crash risk) is higher than its up-side risk. Similarly, Brunnermeier et al. (2009) show the price of the currency crash risk is reflected by the price of the risk-reversal. Jurek (2014) derives a measure of crash risk from currency options and finds that exposure to a currency crash can be used to explain, at most, one-third of the portion of carry trade returns.[10]

Chen et al. (2010) show that the exchange rates of "commodity currencies", including the AUD and CAD, have surprisingly robust power in predicting global commodity prices.[11] The currency option-based indicator "risk reversal" is commonly adopted to measure the price of currency crash risk. A more negative risk reversal of a currency suggests the price of hedging against the downside risk (crash risk) of the currency is higher than its up-side risk. Based on the oil price dynamics, an increase in the oil price crash risk is accompanied by a weakening of the mean-reverting force, reflecting an increase in probability leakage for the rate across the lower boundary. The currency risk reversals are expected to be positively

[9] The quantitative importance of downside risk can be linked to the rare disasters model of Barro (2006).

[10] In a carry trade, an investor sells a currency with a relatively low interest rate and uses the funds to buy a different currency yielding a higher interest rate. This strategy attempts to capture the difference between the rates of the two currencies provided that their exchange rate is stable.

[11] Canada is one of the largest suppliers of crude oil to the US. For Australia, despite its small participation in the crude oil market, the country is the major supplier of commodities such as copper, gold, iron ore and nickel.

related to the mean-version of the oil price. The lower panel in Figure 7 shows the crude oil price and the 3-month 25-delta risk reversals of the CAD and AUD against the US dollar (USD). It is particularly noticeable that the risk reversals display strong positive relationships with the oil price when there were substantial declines in the oil price in late 2008, late 2011 and early 2015.

To test the relationship between the oil price dynamics and the risk reversals of the CAD and AUD, we test whether there is a long-run relationship between the model parameters (κ and θ) and the risk reversals. Their short-run dynamics represented as a dynamical error-correction model is the same in Eq.(19), with the currency risk reversals of the CAD (CAD_rr) and AUD (AUD_rr) respectively at time t-1 denoted by X_{t-1} and the estimation being conducted using weekly data starting from 1 June 2004 to 30 June 2017.[12]

Table 5 provides summary statistics for the time series of the four variables (κ, θ, CAD_rr and AUD_rr) in levels and changes. The ADF test fails to reject at the 10% level the presence of a unit root for these variables in level, but the hypotheses of the presence of a unit root for the first difference of them are significantly rejected at the 1% level or less. The levels of the CAD and AUD risk reversals and model parameters (κ and θ) are non-stationary, while their changes appear to be stationary. This again suggests the series are I(1) and fulfils the requirement for the variables being co-integrated.

The co-integration relationships between the κ, θ and the risk reversals are tested based on the Engle–Granger single-equation test. Table 6 reports the co-integration results for the CAD and AUD risk reversals with κ and θ in the upper and lower panels respectively. The results of the ADF and Philips-Perron tests reject that the residuals from the regressions of two risk reversals with κ and θ to have a unit root at the 1% or 5% significance level. Thus, we reject the null hypotheses corresponding to each regression in which each of the

[12] The risk reversal data are from Bloomberg.

two risk reversals and κ, θ are not co-integrated in favour of the alternative hypothesis that there is at least one co-integrating vector.

The two panels in Table 7 report the estimated co-integrating vectors between the CAD (and AUD) risk reversal and the mean reversion parameters κ and θ. The positive coefficients β for κ and θ are 0.0087 (0.0058) and 0.0287 (0.0190) respectively at the 1% or 5% significance level, suggesting that a more negative CAD (AUD) risk reversal is correlated with decreases in κ and θ. Given that the mean reversion in the oil price dynamics weakens when the oil crash risk increases, the positive relationship illustrates that market participants expect the CAD (AUD) exchange rate to co-move with the oil price with a material crash risk, and depreciate substantially against the USD. Such expectation generates strong demand for put options on the CAD and AUD to hedge against the potential loss of long CAD and AUD positions, and causes more negative CAD and AUD risk reversals. As reported in Table 8, the estimates of the speed of adjustment α_y for κ and θ are negative but greater than -1, indicating that κ and θ will subsequently adjust to restore the long-run equilibrium.

Table 5: Descriptive statistics of CAD and AUD risk reversals, κ and θ.

	κ		θ		CAD_rr		AUD_rr	
	Level	Change	Level	Change	Level	Change	Level	Change
Mean	0.0228	-1.9e-06	0.7149	-0.0003	-0.5366	0.0013	-1.3376	0.0032
Median	0.0213	-1.2e-05	0.7287	-0.0002	-0.3500	0.0000	-1.2852	0.0000
Maximum	0.0482	0.0203	0.8581	0.0245	0.3750	0.6575	-0.1000	1.6500
Minimum	0.0068	-0.0112	0.5548	-0.0850	-3.2900	-0.7475	-4.5400	-2.1750
Std. Dev.	0.0099	0.0020	0.0495	0.0066	0.6634	0.1207	0.8549	0.2653
Skewness	0.6791	4.3531	-1.0282	-4.4737	-0.8464	-0.2732	-0.9487	-1.0646
Kurtosis	2.8937	53.780	3.6731	63.998	3.6279	9.6878	3.7748	18.8224
ADF test statistics	-2.214	-16.517 ***	-1.739	-17.606 ***	-1.937	-10.658 ***	-2.236	-12.386 ***
Correlation with CAD_rr / ΔCAD_rr	0.4495	-0.0562	0.4239	0.1222				
Correlation with AUD_rr / ΔAUD_rr	0.4155	-0.0133	0.2594	0.1176				
Observations	445	441	445	441	445	441	445	441

Notes:

1. CAD_rr (AUD_rr) is the 3-month 25-delta risk reversal of CAD (AUD) against USD. The correlations for level of the variables are the correlations with CAD_rr (AUD_rr), and those for change are the correlation with ΔCAD_rr (ΔAUD_rr).

2. The weekly sample above is based on the criterion that 3-year rolling window estimated κ and θ are statistically significant at 10% level (i.e. z-statistic is higher than 1.645 level).

3. The ADF test checks the null hypothesis of unit root existence in the time series, assuming nonzero mean in the test equation, with lag length determined by Akaike information criterion up to maximum length of 12 (4 for first differencing variables). *** indicates significance at levels of 1% respectively.

Table 6: Tests for co-integration of CAD and AUD risk reversals, κ and θ.

Engle-Granger single-equation test[2]
(Null hypothesis: residual has an unit root)

On CAD_rr	ADF test statistic	Phillips-Perron test statistic
Equation:		
□κ	-2.283 **	–2.268 **
□θ	–3.360 ***	–4.757 ***

On AUD_rr	ADF test statistic	Phillips-Perron test statistic
Equation:		
□κ	–2.292 **	–2.315 **
□θ	–4.342 ***	–4.314 ***

Notes:

1. *** and ** respectively represent statistical significance at the 1% and 5% level.

2. The Engle-Granger single-equation test (ADF and Phillips-Perron tests) examines the null hypothesis that the residuals of the regressions of κ on CAD_rr (AUD_rr) and θ on CAD_rr (AUD_rr) respectively, given that κ, θ and CAD_rr (AUD_rr) are non-stationary. We have assumed a zero mean in the residuals in the test equation, with the respective critical value based on MacKinnon (1996).

Table 7: Estimates of long-run coefficient (β) for CAD and AUD risk reversals, κ and θ.

Dependent variable:	κ_t		θ_t	
CAD_rr$_t$	0.0087	***	0.0287	***
AUD_rr$_t$	0.0058	**	0.0190	***

Notes: *** and ** respectively represent statistical significance at the level of 1% and 5%. The coefficients are estimated by using the Engle-Granger single-equation and the coefficients of the short-run dynamic are in Table 8.

Table 8: Estimation results of the short-run dynamics for CAD and AUD risk reversals, κ and θ.

Dependent variable:	$\Delta\kappa_t$	$\Delta\kappa_t$	$\Delta\theta_t$	$\Delta\theta_t$
Constant	0.0020 ***	0.0022 ***	0.0071 ***	0.0639 ***
Speed of adjustment	-0.0663 ***	-0.0658 ***	-0.0957 ***	-0.0847 ***
ΔCAD_rr$_{t-1}$	0.0006		-0.0049 **	
ΔAUD_rr$_{t-1}$		-1.2E-05		-0.0021 *
$\Delta\kappa_{t-1}$	0.2443 ***	0.2477 ***		
$\Delta\theta_{t-1}$			-0.002	0.00104

Notes: ***, ** and * respectively represent statistical significance at the level of 1%, 5% and 10%. Dummy variables for the global financial crisis (May-Oct 2008), European sovereign debt crisis (Jan-Jun 2012) and period after the oil crash (since Oct 2014) are added in the estimation of the short-run dynamics. The coefficients are estimated by using the Engle-Granger single-equation and the long-run coefficients are in Table 7. For brevity, the estimated coefficients of the dummy variables are not shown in this table.

5. Forecasting performance

Following the methodology in Alquist and Kilian (2010), we further evaluate the performance of the proposed oil price model, particularly during the oil price crash episodes. We first derive and estimate the short-run (i.e., 1-month-ahead and 3-month-ahead) forecasts based on the oil price model on a monthly basis[13]. The model forecasts are compared with two types of models – the futures-spread (FS) models and random walk (RW) (or no-change) models employed in Alquist and Kilian (2010) using standard out-of-sample forecasting evaluation tools. The futures spread is defined as the percent deviation of the oil futures price from the spot price of oil. Given that the futures prices of crude oil are the market expectation of the oil price based on the current information of expected supply and demand for oil in a given time horizon, market participants commonly consider the futures prices as the reasonable forecasts. While Alquist et al. (2013) and Lang and Auer (2019) also review a wider class of oil price forecasting models which require macroeconomic variables as the input information, they do not show any out performance.

5.1 Model forecasts

We derive easy-to-apply forecasting steps from the oil price model in Eq.(1). By applying simple algebra to Eqs.(13), (15) and (16), the coefficients in Eq.(1) are related to the parameters in Eq.(14) with the following equation:

$$\frac{\alpha}{m} = \kappa\theta \, . \tag{20}$$

The 1-month-ahead forecast nominal price of oil is derived by first differencing Eq.(1):

$$\mathrm{s}_{t+1|t} - \mathrm{s}_t = 2B_{0t}\upsilon_t(E[\upsilon_{t+1|t} - \upsilon_t]). \tag{21}$$

[13] While the oil price model can be applied to any frequency, as noted in Alquist and Kilian (2010), there can be no exact matching futures with the date of delivery in h-month ahead for any particular day. Therefore, we consider only the monthly basis in this section for the purpose of comparison. The oil futures data are from Bloomberg. For more details on the source of data, see Table 9.

To obtain the time-varying coefficient B_{0t}, we first calibrate $(\kappa_t, \theta_t, \sigma_{xt})$ in the MLE based on x_t for each time point t using a 36-month rolling window under the following transformation and hypothetical target bands:

$$s_t = -\max(0.001, |\ln\left(\frac{\eta_U S_{A,t} - Spot_t}{\eta_U S_{A,t} - \eta_L S_{A,t}}\right)|) \tag{22}$$

$$\eta_U = 2.25\ ,\ \eta_L = 0.5 \quad \text{and} \quad S_{A,t} = MovAvg_{i=t-5}^{i=t}(Spot_i) \tag{23}$$

The non-linear least square fitting method is applied to the following transformed Eq.(1) under a 36-month rolling window:

$$s_t = \frac{|\alpha_t|}{\kappa_t \theta_t} + \sqrt{\frac{-s_t}{|B_{0t}|}} + |\alpha_t| E[ds_t] + \eta_t \tag{24}$$

$$s_t = \frac{\beta_0}{\kappa_t \theta_t} + \beta_1 \sqrt{-s_t} + \beta_2 MovAvg_{i=t-5}^{t}(\Delta s_i) + \eta_t\ , \tag{25}$$

subject to linear constraints $\beta_0 = \beta_2 \geq 0$ and $\eta_t \sim N(0,1)$, where $\widehat{\beta_1} = -\sqrt{\frac{1}{|\widehat{B_{0t}}|}}$, $\widehat{\beta_2} = |\widehat{\alpha_t}|$, and $\hat{\upsilon}_t = \sqrt{\frac{-s_t}{|\widehat{B_{0t}}|}}$ for each time point t.

The 1-month-ahead $\hat{s}_{t+1|t}$ and the corresponding spot oil price $\widehat{\text{Spot}}_{t+1|t}$ are derived by approximating $E(\Delta \upsilon_t)$ as the moving average of previous 3-months changes:[14]

$$\hat{s}_{t+1|t} = s_t + 2\widehat{B_{0t}}\widehat{\upsilon_t}\left(MovAvg_{i=t-2}^{t}(\Delta \upsilon_i)\right) \tag{26}$$

$$\widehat{\text{Spot}}_{t+1|t} = \eta_U S_{A,t} - \exp\left(-\hat{s}_{t+1|t}\right) * (\eta_U S_{A,t} - \eta_L S_{A,t}). \tag{27}$$

By repeating the calibration and non-linear least square fitting method with the 1-month-ahead forecast, $(\kappa_{t+1|t}, \theta_{t+1|t}, \sigma_{xt+1|t})$ and $B_{0t+1|t}$ are estimated for calculating the 2-month-ahead forecast. Likewise, the (t+1+h)-month-ahead forecasts can then be estimated iteratively through repeating the steps above. According to the above formulation, as the forecasted changes in the oil price are primary driven by the expected changes in fundamental

[14] The expectation term in the oil price model can be approximated by other choices, such as the spread between prices of futures contracts and spot, results from macro-econometrics models, etc. We consider here the results solely based on historical changes in the derived fundamentals such that the forecasting performance of the model can be considered independently from other forecasting models.

shocks which are approximated by the moving average of previous 3-months changes, forecasts based on the above setting by natures are closer to following a random walk with a drift model than the no-change forecast model.

5.2 Out-of-sample forecasting evaluation

Alquist and Kilian (2010) consider several commonly employed forecasting models to predict future nominal oil price movements, and show that these models cannot outperform the random walk model using a sample period of 1990 to 2007. We present the out-of-sample performance of the oil price model and compare it with that of those forecasting models (both RW- and FS-types) studied by Alquist and Kilian (2010). The data sources for oil spot and futures prices are reported in Table 9, whereas the details of the forecasting models can be found in Alquist and Kilian (2010). Since the oil price model incorporates the crash risk, we consider the forecast evaluation period of January 2005 – December 2018 including the relevant oil price crash episodes. Tables 10 and 11 evaluate the predictive accuracy of the forecasting models (RW models with drift (2)-(6); FS models (7)-(11); oil price model (12)) against the benchmark of a RW model (model (1)) without drift for time horizons of 1-month-ahead and 3-month-ahead. The results for the mean squared prediction error (MSPE), mean absolute prediction error (MAPE), bias and success ratio for predicting directional movements are reported for studying the forecasting performance.

Table 9: Details of prices of crude oil spot and futures contracts from Bloomberg

Variable	**Ticker**	**Name**
Spot oil price	USCRWTIC Index	Bloomberg West Texas Intermediate Cushing Crude Oil Spot Price
1-month futures	CL1 Comdty	Generic 1st 'CL' Future (The front-month Nymex crude oil contract)
3-month futures	NRGSCL3 Index	Bloomberg Nymex Crude Oil 3 Month Strip Futures Price
6-month futures	NRGSCL6 Index	Bloomberg Nymex Crude Oil 6 Month Strip Futures Price

Note: It is noted that the USCRWTIC Index is usually at parity to the front-month Nymex crude oil contract, with the exception of its three-day delivery scheduling period after the front-month contract expires. Specifically, we construct the monthly forecasting sample by identifying the spot and the prices of *h*-month oil futures contract traded as the last price at the trading day on or closest before the 25th delivery date for each month.

Table 10. 1-month-ahead recursive forecast error diagnostics

Model class	**No.**	**Model**	**MSPE ratio**	**Bias**	**MAPE ratio**	**Success ratio**
RW model	(1)	S_t	59.308	-0.010	5.583	n.a.
RW drift models	(2)	$S_t(1+\Delta\overline{S_t^1})$	1.661	-0.808	1.397	0.440
	(3)	$S_t(1+\Delta\overline{S_t^3})$	1.183	-0.687	1.115	0.488
	(4)	$S_t(1+\Delta\overline{S_t^6})$	1.168	-0.684	1.100	0.470
	(5)	$S_t(1+\Delta\overline{S_t^9})$	1.121	-0.614	1.078	0.476
	(6)	$S_t(1+\Delta\overline{S_t^{12}})$	1.107	-0.570	1.052	0.506
FS models	(7)	F_t^1	0.969	-0.357	0.990	0.530
	(8)	$S_t(1+\ln(F_t^1/S_t))$	0.969	-0.348	0.990	0.530
	(9)	$S_t(1+\hat{\alpha}+\hat{\beta}\ln(F_t^1/S_t))$	1.019	-0.769	1.016	0.554
	(10)	$S_t(1+\hat{\beta}\ln(F_t^1/S_t))$	1.004	-0.409	1.012	0.548
	(11)	$S_t(1+\hat{\alpha}+\ln(F_t^1/S_t))$	0.983	-0.710	0.995	0.542
Oil price model	(12)	oil price model	1.037	0.186	1.055	0.530

Note: The RW model, RW drift models and FS models respectively refer to the random walk without a drift model, random walk with the local drift models, and the forecast models based on futures-spot spreads. The details of these models are in Alquist and Kilian (2010). The oil price model refers to the forecast obtained based on the description in Section 5.1. The MSPE and MAPE results are presented as ratios relative to the benchmark no-change forecast model, except for the no-change forecast model. The forecast evaluation period is 2005.1-2018.12. The initial estimation windows for the rolling regressions (9)-(11) are 1990.1-2004.12. F_t^1 is the futures price with maturity of one month, and $\Delta\bar{S}_t^{(l)}$ denotes the trailing geometric average of the monthly percent change for *l* months. The success ratio is defined as the fraction of forecasts that correctly predict the sign of the change in the oil price. The bias is defined as the average amount by which S_{t+1} exceeds the prediction.

Table 11. 3-month-ahead recursive forecast error diagnostics

Model class	**No.**	**Model**	**MSPE ratio**	**Bias**	**MAPE ratio**	**Success ratio**
RW model	(1)	S_t	224.011	0.063	10.318	n.a.
RW drift models	(2)	$S_t(1+\Delta\overline{S_t^1})$	1.127	-0.853	1.155	0.482
	(3)	$S_t(1+\Delta\overline{S_t^3})$	1.044	-0.699	1.045	0.488
	(4)	$S_t(1+\Delta\overline{S_t^6})$	1.100	-0.658	1.042	0.470
	(5)	$S_t(1+\Delta\overline{S_t^9})$	1.074	-0.574	1.027	0.494
	(6)	$S_t(1+\Delta\overline{S_t^{12}})$	1.073	-0.522	1.025	0.512
FS models	(7)	F_t^3	0.963	-0.683	0.965	0.571
	(8)	$S_t(1+\ln(F_t^3/S_t))$	0.963	-0.658	0.965	0.571
	(9)	$S_t(1+\hat{\alpha}+\hat{\beta}\ln(F_t^3/S_t))$	1.018	-2.660	0.973	0.637
	(10)	$S_t(1+\hat{\beta}\ln(F_t^3/S_t))$	0.964	-1.236	0.972	0.607
	(11)	$S_t(1+\hat{\alpha}+\ln(F_t^3/S_t))$	1.008	-1.963	0.963	0.631
Oil price model	(12)	oil price model	1.142	0.978	1.185	0.524

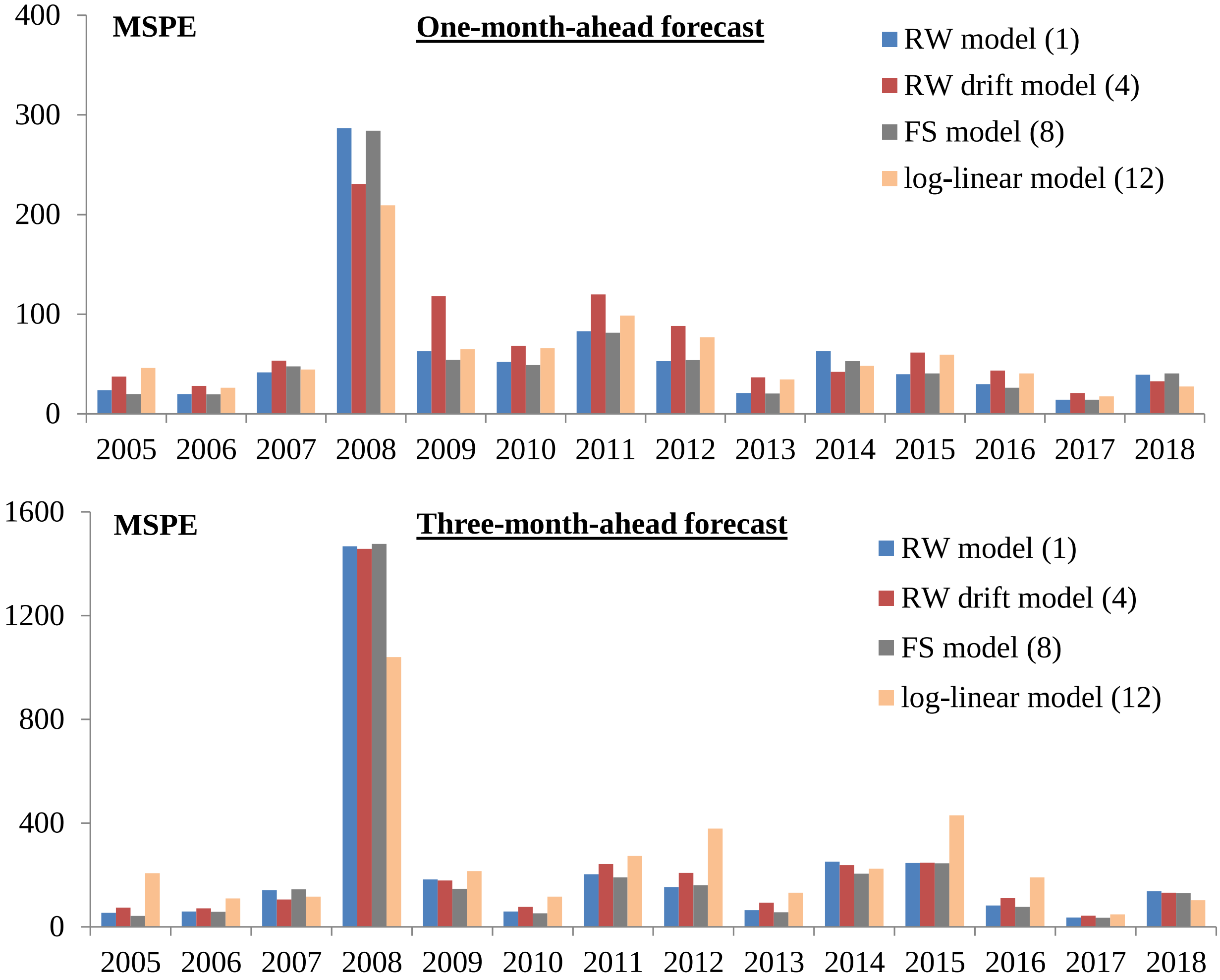
Note: See the note in Table 10. F_t^3 is the futures price with maturity of three months. The bias is defined as the average amount by which S_{t+3} exceeds the prediction.

As shown in the last row of Table 10, the oil price model in general presents smaller MSPE and MAPE ratios, higher success ratio than the RW drift (2)-(6) models on the 1-month-ahead forecast horizon, suggesting that the oil price model forecasts perform fairly well. While the oil price model shows marginally higher MSPE and MAPE ratios than the RW (1) model and FS (7)-(11) models, the short-horizon forecasts based on these models usually hover around the current oil spot price, thus the error ratios tend to be lower than the RW drift models and oil price model during the periods of low oil price volatility. Due to the error accumulating feature embedded in the model iteration steps, the error ratio of the oil price model increases with the longer (3-month) forecasting horizon. However, both MSPE and MAPE error ratios of the oil price model are not significantly higher than the other models and the success ratio remains higher than all RW drift models.

Given that the oil price model incorporates price crash risk, we split the entire evaluation period into annual subsamples to evaluate the forecasting performance of the model during

2008, 2014 and 2018 when there were sharp falls in the oil price. As shown in Figure 8, the oil price model presents lower MSPE than selected models (1), (4) and (8) during these years for both 1-month-ahead and 3-month-ahead forecasts. The better forecasting performance of the oil price model than the RW and FS models during these years reflects that the crash risk feature of the oil price model plays an important role in the oil price dynamics. When a large negative fundamental shock is anticipated, the weakened mean-revering force in the oil price dynamics could cause the oil price to breach the lower boundary of the target zone and thus price crashes occur. This explains why the oil price model preforms better during the periods with the oil price crashes. Such findings are consistent with large negative supply/demand fundamental shocks after the onset of the GFC in 2008, rising non-OPEC production such as shale oil in 2014 and a sudden anticipation of global economic slowdown at the end of 2018. Therefore, the results in this section provide further empirical evidence that the oil price model adequately explain the oil price dynamics, in line with the discussions in sections 3 and 4.

Figure 8: Selected models' MSPE for each year within the evaluation period



6. Conclusion

We present a crude oil price model and derive the corresponding log price equation in which a smooth-pasting boundary condition is imposed at a lower boundary. A price crash occurs when the price breaches the boundary. The model assumes oil producers will intervene by reducing oil production when the price falls close to a moving lower boundary. The moving lower boundary shifts the reference for intervention from the price level at each instant to the moving average of prices over a time interval. Oil producers can therefore have more time to observe price movements and market reaction giving them the option of postponing their decisions on reducing oil production until a demand shock has emerged.

By using an asymmetric mean-reverting fundamental shock, the solution derived from the oil price equation shows the oil price follows a mean-reverting square-root process. The price is quasi-bounded at the lower boundary and can breach the boundary with a weakened mean-reverting force in the price dynamics. The boundary condition ensures that a crash is rare. Indeed, the price solution generates left-skewed price distributions, which is a feature consistent with empirical observations. The empirical results using market data from 1986 – 2017 suggest that this model can describe oil price dynamics. While the price was above the boundary for most of the time, the condition for breaching the boundary was met in 2008 and 2014 when the oil price fell sharply.

The model parameters of the restoring force of the oil price dynamics are positively co-integrated with oil production, indicating that OPEC reduces oil production when the risk of an oil price crash becomes material, i.e., a weakened mean reversion in the price dynamics. A reduction in oil production reflects the impact of a demand shock which increases the crash risk. On the other hand, when the restoring force increases, OPEC will be willing to increase its production given the reduced crash risk. The empirical results are thus consistent with the proposed asymmetric mean-reverting fundamental dynamics which incorporates demand shocks and OPEC's intervention as a counteracting force.

Given the linkage between commodity currency crashes and oil price crashes, the co-integration analysis testing the relationship between the oil price dynamics and the risk reversals of the AUD and CAD shows that the risk reversals are positively related to the mean reversion of the oil price. The results are consistent with the expectation that a higher crash risk (more negative risk reversals) of the commodity currencies suggests a higher oil price crash risk where the restoring force in the oil price dynamics weakens. The better forecasting performance of the oil price model than the futures spread models and random walk models during the crash periods reflects that the crash risk feature of the oil price model plays an important role in the oil price dynamics.

One implication of the quasi-bounded process of the oil price with a crash risk is that it can affect the price of associated derivatives such as options if the oil price is restrained by the actions of oil producers under demand shocks. In view of such implication, a study of the derivatives' pricing models based on the quasi-bounded process is left for further research.

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References

Alquist, R., Kilian, L., 2010, What do we learn from the price of crude oil futures?, Journal of Applied Econometrics, 25, 539-573.

Alquist, R., Kilian, L, Vigfusson R. J.,2013. Forecasting the price of oil. Handbook of Economic Forecasting, 2, 427-507.

Askari, H., Krichene, N., 2008. Oil price dynamics (2002-2006). Energy Economics 30, 2134-2153.

Backus, D. K., Foresi, S., Telmer, C., 2001. Affine term structure models and the forward premium anomaly. Journal of Finance 56, 279-304.

Bakshi, G., Carr, P., Wu, L., 2008. Stochastic risk premiums, stochastic skewness in currency options, and stochastic discount factors in international economies. Journal of Financial Economics 87, 132-156.

Ball, L., Mankiw, N. G., 1995. Relative-price changes as aggregate supply shocks. Quarterly Journal of Economics 110, 161-193.

Barro, R. J., 2006. Rare disasters and asset markets in the twentieth century. Quarterly Journal of Economics 121 (3), 823-866.

Bartolini, L., Prati, A., 1999. Soft exchange rate bands and speculative attacks: theory, and evidence from the ERM since August 1993. Journal of International Economics 49(1), 1-29.

Baumeister, C., Kilian, L., 2016a. Understanding the decline in the price of oil since June 2014. Journal of the Association of Environmental and Resource Economists 3, 131-158.

Baumeister, C., Kilian, L., 2016b. Forty years of oil price fluctuations: Why the price of oil may still surprise us. Journal of Economic Perspectives 30, 139-60.

Bharati, R., Crain S. J., Kaminski, V., 2012. Clustering in crude oil prices and the target pricing zone hypothesis. Energy Economics 34, 1115-1123.

Chapman, D., Khanna, N., 2000. World oil: the growing case for international policy. Contemp. Economic Policy 18, 1–13.

Chapman, D., Khanna, N., 2001. An economic analysis of aspects of petroleum andmilitary security in the Persian Gulf. Contemporary Economic Policy 19, 371-381.

Chapman, D., Khanna, N., 2006. The Persian Gulf, global oil resources, and international security. Contemporary Economic Policy 24, 507-519.

Chen, Y., Rogoff K., Rossi, B., 2010. Can exchange rates forecast commodity prices? Quarterly Journal of Economics 125 (3). 1145-1194.

Cox, J. C., Ingersoll J. E., Ross S. A., 1985. A theory of the term structure of interest rates. Econometrica 53, 385-407.

Engle, R. F., Granger, C. W. J., 1987. Co-integration and error correction: Representation, estimation and testing. Econometrica 55, 251-276.

Farhi, E., Fraiberger, S. P., Gabaix, X., Ranciere, R., Verdelhan, A., 2015. Crash risk in currency markets. Working Paper, New York University.

Farhi, E., Gabaix, X., 2016. Rare disasters and exchange rates. Quarterly Journal of Economics 131 (1). 1-51.

Hammoudeh, S., Madan V., 1995. Expectations, target zones, and oil price dynamics. Journal of Policy Modeling 17, 597-613.

Hammoudeh, S., 1996. Oil price, mean reversion and zone readjustments. Southern Economic Journal 62, 916-929.

Hui, C. H., Lo, C. F., Fong, T., 2016. Swiss franc's one-sided target zone during 2011-2015. International Review of Economics and Finance 44, 54-67.

Hui, C. H., Lo, C. F., Liu C. H., 2018. Exchange rate solutions with currency crashes. Hong Kong Institute for Monetary Research Working Paper, WP No.25 / 2018.

Ingersoll, J. E., 1997. Valuing foreign price derivatives with a bounded exchange process. Review of Derivatives Research 1, 159-81.

Jurek, J. W., 2014. Crash-neutral currency carry trades. Journal of Financial Economics 113, 325-347.

Jurek, J. W., Xu, Z., 2014. Option-implied currency risk premia. Working Paper, University of Pennsylvania.

Karlin, S., Taylor, H. M., 1981. A Second Course in Stochastic Processes. Academic Press, New York.

Kaufmann, R., Dées, S., Karadeloglou, P., Sánchez, M., 2004. Does OPEC matter? An econometric analysis of oil prices. Energy Journal 25, 67–90.

Klevnäs, P., Stern, N., Frejova, J., 2015. Oil prices and the new climate economy. Global Commission on the Economy and Climate briefing paper, May 2015. Available at http://newclimateeconomy.report.

Krugman, P., Rotemberg, J., 1990. Target zones with limited reserves. NBER Working Paper No. 3418.

Krugman, P., 1991. Target zones and price dynamics. Quarterly Journal of Economics 106, 669-82.

Lang, K., Auer B. R., 2019. The economic and financial properties of crude oil: A review. North American Journal of Economics and Finance, forthcoming.

Larsen, K. S., Sørensen, M., 2007. A diffusion model for prices in a target zone. Mathematical Finance 17, 285-306.

Lo, C. F., Hui, C. H., Fong, T., Chu, S. W., 2015. A quasi-bounded target zone model – Theory and application to Hong Kong dollar. International Review of Economics and Finance 37, 1-17.

Lo, C. H., Hui, C. H., 2019. Unique Fundamental Dynamics of Target-zone Exchange Rates. Working Paper, The Chinese University of Hong Kong. Available at SSRN: http://dx.doi.org/10.2139/ssrn.3394637.

Longstaff, F., 1989. A non-linear general equilibrium model of the term structure of interest rates. Journal of Financial Economics 23, 195-224.

Longstaff, F., 1992. Multiple equilibria and term structure models. Journal of Financial Economics 32, 333-344.

MacKinnon, J. G., 1996. Numerical distribution functions for unit root and cointegration test. Journal of Applied Econometrics 11, 601-618.

Slaibi, A., Chapman, D., Daouk, H., 2010. A geopolitical theory of oil price behaviour: An econometric evaluation. Applied Economics 42, 2783-2800.

Tang, L., Hammoudeh, S., (2002). An empirical exploration of the world oil price under the target zone model. Energy Economics 24.577-596.

Wu, C. C., Chung, H., Chang, Y. H., 2012. The economic value of co-movement between oil price and exchange rate using copula-based GARCH models. Energy Economics 34, 270-282.