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Are Government Bond Yields Bounded or Quasi-bounded at the Zero? – Credibility of Central Banks' Commitments

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Abstract

This paper develops a model based on a target-zone approach in which the dynamics of government bond yields follow a quasi-bounded process, such that the zero lower bound (ZLB) can be breached if the probability leakage condition of the dynamics is met. A one-sided U-shaped bond yield distribution illustrates accumulation of probability at the ZLB. Allowing the expected return and variance of the market's return proportional to the square of the state variable governing changes in production and investment opportunities over time suggests the state variable following an asymmetric mean-reverting process with strong counteracting force at the ZLB representing the credibility of the bound committed by a central bank. Empirical calibrations of the proposed process for the US and French government bond yields show that the process can adequately describe their dynamics. While the yields were bounded above the ZLB during most of the time, as indicated by their dynamics, the conditions for breaching the bound were met in January 2013 for the French government bond and March 2020 for the US Treasury using only information until those points. The economic and financial condition uncertainties are negatively co-integrated with the mean reversion in the dynamics, suggesting increased likelihood of the yield breaching the ZLB and erosion of the credibility of the bound amid higher uncertainties.

Keywords: Zero lower bound, negative interest rates, government bonds, stochastic process
JEL classification: E43, E58, G12

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1. Introduction

Since the 2009 financial crisis, central banks in many advanced economies have set policy rates close to or below zero. The financial market reaction to the introduction of negative rates is that government bonds in those economies are trading at negative yields. While the US policy rate has remained above the zero lower bound (ZLB), widespread Covid-19 in the US since March 2020 has triggered deep economic downturn of uncertain duration. The Federal Reserve has cut its fed funds target rate by a total of 1.5% bringing it down to a range of 0% to 0.25%. The rate affects longer-term borrowing rates including the US Treasury bond yields. The Fed has also offered forward guidance on the future path of its policy rate, saying that rates will remain low while the economy is on track to achieve its maximum employment and price stability goals. Such forward guidance along with purchasing massive amounts of long-term securities (quantitative easing) put downward pressure on longer-term Treasury yields. Against this background, the Fed committed to not considering negative interest rates. Instead, the Fed focused on other "liquidity tools" to keep credit flowing and financial markets operating properly.[1] However, when markets do not expect that such monetary policy measures can bring inflation to target, in particular quantitative easing becomes less effective as bond yields approach their ZLB, negative interest rates may be a useful option as argued by Lilley and Rogoff (2019), as well as Rogoff (2015, 2016, 2017). They consider that in a low inflation rate world, central banks need scope for being able to set significantly negative interest rates in deep recessions and crises.

When some central banks in developed countries lowered their policy rates below zero during 2012-2016, their corresponding government bond yields dropped between 10 and 30 basis points subsequently [see Ubide (2017) and Christensen (2019)]. Gagnon and Jeanne (2020) show that if a central bank communicates a credible commitment to keeping its policy

[1] See https://www.cnbc.com/2020/03/15/powell-says-the-fed-doesnt-see-negative-rates-as-appropriate-policy-for-the-united-states.html

rate above a given bound (i.e., zero) under all circumstances, then government bond yields must be higher than that bound. We consider such commitment as an analogy to a foreign exchange target zone in which the central bank commits its currency value to lie within a band. To illustrate how this analogy works, suppose that the bond yield falls close to zero. If the central bank which commits to positive bond yield policy is credible, it will not allow the yield to fall further and the yield will either remain above the ZLB or rise. Market participants thus expect the yield to rise and will not buy the bond with no expected capital gain, inducing stabilising effect on the yield. However, if the economy is too weak, bond holders will start to doubt the central bank's commitment if the yield continues to fall. Market participants will then expect a capital gain if they buy the bond. The demand for the bond and its price will therefore increase, leading to a further fall in its yield. In such scenario, the central bank will step in to sell bonds in order to prevent the yield from breaching the ZLB. Therefore, this mechanism involving both the market expectation and central bank's market operation keeps the bond yield above the ZLB, suggesting that the central bank is credible. The same mechanism works for the foreign exchange target-zone regime.[2] Conversely, if the recession is too severe and prolongs, market participants may expect the central bank to abandon the positive interest rate policy such that its credibility will evaporate.

To assess the credibility of a central bank's commitment, this paper presents a simple model to capture the stochastic process of the nominal government bond yields which are quasi-bounded at the ZLB committed by the central bank. For the purpose of analysis under the ZLB constrain, the bond yield is modelled to resemble its empirical statistical properties and ensure the bond yield to be non-negative definite but allow it to breach the ZLB when the probability leakage condition determined by its stochastic process is met. Lo et al. (2015) and Hui et al. (2016) show that the quasi-bounded process can describe the exchange rate

[2] The theory and mechanism including central banks' interventions of the foreign exchange target-zone regime are studied by Krugman (1991), Froot and Obstfeld (1991), and Bertola and Svensson (1993) for example.

dynamics and interest rate differentials of the Hong Kong dollar against the US dollar in a target zone and the Swiss franc against the euro during the target zone regime of September 2011 to January 2015 respectively. Regarding the Swiss franc exchange rate, the condition for breaching the strong-side limit was met in November 2014 using only information until that point, i.e., about two months before abandoning the limit.

Time series data of bond yields are typically confined to a band, leading to the imposition of mean reversion in the dynamics. Bond yields are constrained to lie between the ZLB and an upper bound which can be time varying in general. Motivated by forecasting with time series decomposition, this model is not only capable of capturing most stylised facts of the bond yields but also highly intuitive. Specifically, we detrend the time series of the bond yields and simply concentrate on modelling their fluctuations only. Since the proposed process has an analytically tractable probability density function, calibration of model parameters for different financial observables can be easily performed in a straightforward manner. In terms of the calibrated model parameters, we can determine whether the stochastic process is still a bounded one or not. As a result, the quasi-boundedness of the process at the ZLB can provide an indicator of likelihood of negative bond yields. This approach captures the bond yield dynamics governed by the quasi-bounded process in the target-zone framework. Therefore, it is not necessary to model the interest rate term-structure dynamics for assessing the credibility of the central bank's commitment to the ZLB. Such approach is different from the shadow rate term-structure model proposed originally by Black (1995) which is a popular candidate under the ZLB and is used to measure the effectiveness of the unconventional monetary policies [see Krippner (2013) and Wu and Xia (2016) for example].

To study how the fundamental (state variable) dynamics which capture the state of the economy and the central bank/government's interventions driving the bond yields, we assume that both the long-term and short-term interest rates can be treated as equilibrium interest rates in an economy.[3] As in Longstaff (1989) investigating the short-term interest rate dynamics, we allow the expected return and variance of the market's return proportional to the square of the state variable governing changes in production and investment opportunities over time. Under this assumption, the state variable is shown to follow an asymmetric mean-reverting process with strong counteracting force at the ZLB representing the credibility of a central bank's commitment of the bound.

Empirical calibrations of the proposed process for the 10-year US and 2-year French government bond yields shown in Figure 1 demonstrate that the process can adequately describe their dynamics. The 10-year US Treasury is the benchmark in the long-term bond market and its yield represents the responses of the long-term interest rates to the Fed's monetary policy. We choose the French government bond for the euro area because the French government bond market is the largest in the euro area. In addition, it is representative of the general euro area in that it is less plagued by the risk premiums due to fiscal problems in the peripheral countries and the flight-to-safety effects observed in the German bund market. The 2-year French government bond is commonly used to gauge the expectation for the ECB adopting the negative interest rate policy [see Christensen (2019)]. While the yields were bounded above the ZLB during most of the time, as indicated by their dynamics, the conditions for breaching the bound were met in January 2013 for the French government bond and March 2020 for the US Treasury using only information until those points.

Given that financial market and economic uncertainties affect central banks' monetary policy that influences the bond yield dynamics, we use a co-integration analysis to test the dynamic relationships of the 10-year US Treasury yield with the financial market and economic uncertainties. The market uncertainty is measured by the VIX which gauges the

[3] See Ruge-Murcia (2006) for a study of the effect of the ZLB for long-term interest rates under the expectations theory.

level of market participants' risk aversion to put capital into the market and is primarily designed to measure market expectations of uncertainty in the equity market. At the macro level, economic policy uncertainty innovations foreshadow uncertainties and declines in investment, output, and employment in the US. The Federal Open Market Committee (2009) and the IMF (2012, 2013) suggest that uncertainty about fiscal, regulatory, and monetary policies contributed to a steep economic decline in the 2009 global financial crisis and slow recoveries afterwards. In view of the close relationship between economic policy uncertainty and economic uncertainty, we use the index of economic policy uncertainty developed by Baker et al. (2015) based on newspaper coverage frequency, which proxies for movements in policy-related economic uncertainty, as the measure of economic uncertainty. The financial market and economic uncertainties are found to be negatively co-integrated with the mean reversion in the dynamics, suggesting increased likelihood of the yield breaching the ZLB and erosion of the credibility of the bound amid higher uncertainties.

The paper is organised as follows. We develop the model for the bond yield dynamics incorporating the ZLB constraint in the following section. The corresponding probability density function and state variable dynamics are derived and discussed. The calibrations of the bond yield dynamics and its probability leakage condition are presented in section 3. The bond yield dynamics' relationship with the financial market and economic uncertainties are studied by a co-integration analysis in section 4. The final section is the conclusion.

2. Government bond yield dynamics with zero lower (quasi)-bound

2.1 Quasi-bounded process at ZLB

Given a lower bound imposed by a central bank, we consider a bond whose nominal yield is a function of time t and is constrained to lie within the interval $[S_L, S_U]$ for $0 \leq S_L < S_U$. In general, both S_L and S_U can be time-varying. Given the ZLB, S_L is set to be zero here.

Regarding a time series of the bond yield S, i.e., $\{\ldots, S_{i-2}, S_{i-1}, S_i, S_{i+1}, S_{i+2}, \ldots\}$, we construct another time series for the moving average $\bar{S}$, which is defined by

$$\bar{S_i} = \frac{1}{M}\sum_{j=i-M}^{i} S_j \quad , \tag{1}$$

for some natural number M. It is well known that the moving average gives the trend component of the time series of S. In terms of these two variables we introduce the dimensionless normalised value $V \equiv S/\bar{S_i}$ of the bond yield, which lies between two boundaries, namely $V_L \equiv S_L/\bar{S} < 1$ and $V_U \equiv S_U/\bar{S} > 1$, where S_L and S_U are lower and upper boundaries for S. Then, the fractional deviation D of S from $\bar{S}$ is given by

$$D \equiv V - 1 = \frac{S-\bar{S}}{\bar{S}} \tag{2}$$

and its value stays within the interval $[D_L, D_U]$, where $-1 < D_L \equiv V_L - 1 < 0$ and $D_U \equiv V_U - 1 > 1$. It is clear the time series of $S - \bar{S}$ corresponds to the detrended series under the assumption of an additive decomposition of a time series. Moreover, we can define a normalised fractional deviation R of S from $\bar{S}$ as follows:

$$R \equiv \frac{D_U - D}{D_U - D_L} = \frac{V_U - V}{V_U - V_L} = \frac{S_U - S}{S_U - S_L} \ , \tag{3}$$

such that $R \in [0,1]$. With $S_L = 0$, when $S = 0$, $R = 1$.

Following the quasi-bounded process of the normalised exchange rate in the target-zone model developed in Lo et al. (2015) and Hui et al. (2016), the normalised bond yield x, which is equal to the logarithm of the stochastic variable R, is governed by the mean-reverting square-root (MRSR) process:

$$dx = \kappa(\theta - x)dt + \sigma_x \sqrt{x} dZ \ , \tag{4}$$

where $x \equiv -\ln R \in [0, \infty]$ such at $x = 0$, when $S = 0$. dZ is a Wiener process with $\mathrm{E}[dZ] = 0$ and $\mathrm{E}[dZ^2] = dt$. Here $\sigma_x^2 x$ is the variance that depends upon the level of x, κ determines the speed of the mean-reverting drift towards the long-term mean θ. The variance of interest rates is commonly modelled by the square-root process [see Cox-Ingersoll-Ross (CIR)

(1985a and 1985b); Longstaff (1989 and 1992); Black (1995); Piazzesi (2010)]. The ratio of the variance of the change in the interest rate to the rate does not depend on the rate, such that the volatility of the change in the rate is proportional to the square root of the rate at a given time. With mean reversion being put into the process, the MRSR process is commonly known as the CIR process.

Based on Feller's classification of boundary points, it can be inferred that there is a non-attractive natural boundary at infinity (i.e., inaccessible) and the one at the origin is a boundary of no probability leakage for $(\sigma_x^2/4\kappa\theta) < 1$, and it is not otherwise.[4] In other words, the dynamics of x is a quasi-bounded process. The no-leakage condition ensures the bond yield will not breach the lower bound (the origin of $S = 0$ and $x = 0$) and there is no negative yield; otherwise, the yield may pass through the ZLB, i.e., the bond yield is quasi-bounded at the origin.[5] While the variance $\sigma_x^2 x$ declines towards the boundary at $x = 0$, yet x could breach the boundary under the particular condition: $\sigma_x^2/4\kappa\theta > 1$. This quasi-boundedness of the boundary at $x = 0$ can thus provide an indicator of possible negative bond yields.

Eq.(4) shows that the normalised bond yield x follows the MRSR process by using the normalisation of Eq.(3) and the implication is that the random fluctuation of the physical bond yield S is described by this stochastic process.[6] The normalised bond yield dynamics thus contains information of the physical volatility of S. This suggests that a restoring force in the normalised bond yield dynamics will pull the random variable x away from the ZLB towards the long-term mean. Leakage through the bound occurs only when the fluctuations accompanied with a plunge of the bond yield (in rare situations) shoot up drastically or the restoring force diminishes sharply. This is consistent with the observations in which volatility

[4] For boundary condition definitions, see Karlin and Taylor (1981). The probability leakage is different from the probability of passing a boundary.
[5] Such a property is similar to the bounded exchange rate dynamics in Ingersoll (1996) and Larsen and Sørensen (2007). In their models the exchange rate is completely bounded under all circumstances.
[6] The CIR process is commonly used in stochastic volatility modes such as the Heston (1993) model.

goes up or mean reversion representing an error-correction process drops during bad economic periods (see Bates (2012)).

In empirical analyses the exact stochastic dynamics of the bond yield is unknown. Making use of the data of the bond yield and its moving averages as well as defining the normalised bond yield according to Eq.(3), we can simply concentrate on modelling the fluctuations of the bond yield only. The above analysis shows that the fluctuations of the bond yield in the original measure contain the crucial information of whether the zero is bounded, and the leakage condition of the normalised bond yield dynamics following the MRSR process plays the role of signalling any possible negative bond yield.

The probability density function (PDF) of x under the CIR process is given by:

$$
\begin{aligned}
G(x,t;x',t') = {} & \frac{2}{\sigma_x^2 C_1(t-t')}\left(\frac{x}{x'}\right)^{\omega/2} \exp\left[-\frac{\omega+2}{2}C_2(t-t')\right]\times \\
& \exp\left\{-\frac{2x'+2x\exp[-C_2(t-t')]}{\sigma_x^2 C_1(t-t')}\right\}\times \\
& I_\omega\left\{\frac{4x^{1/2}x'^{1/2}\exp[-C_2(t-t')/2]}{\sigma_x^2 C_1(t-t')}\right\}
\end{aligned}
\quad , \qquad (5)
$$

where $\omega = 2\kappa\theta/\sigma_x^2 - 1$, $C_1(\tau) = [\exp(\kappa\tau)-1]/\kappa$, $C_2(\tau) = -\kappa\tau$, I_ω is the modified Bessel function of the first kind of order ω. The associated asymptotic PDF will eventually approach the steady-state exchange rate distribution, which is:

$$
K(x,t\to\infty;x',t') = \frac{2x^{\omega+1/2}}{\Gamma(\omega+1)}\left(\frac{2\kappa}{\sigma_x^2}\right)^{\omega+1}\exp\left(-\frac{2\kappa}{\sigma_x^2}x\right), \qquad (6)
$$

where Γ is the gamma function. Given the PDF in Eq.(5), the parameters of the CIR process for the bond yield dynamics are calibrated in section 3 using market bond yield data.

Figure 2 shows the steady-state bond yield distributions in S based on Eq.(6) with three values of the long-term mean θ of 0.25, 0.75 and 1.5 corresponding to S = 1.11%, 2.64% and 3.89% respectively in each panel. We use the model parameters consistent with the calibration of model parameters of the US and French government bonds in section 3, the lower boundary $S_L = 0$ and upper boundary $S_U = 5\%$ for the distributions. In Panel A with κ = 0.072 and σ_x = 0.14, the distribution with θ = 0.25 (S = 1.11%) near the ZLB has its peak at the left, showing that the PDF decays slower than a Gaussian distribution at the right. The probability of the yield accumulates near the ZLB consistent with the low bond yields in recent years as shown in Figure 1. The skewness of the distribution illustrates an environment characterised by "low-for-long" interest rates, suggesting a prolonged period of low bond yields near the ZLB. Yields are generally expected to rise only slowly over the near to medium term, and to eventually stabilise at low levels.[7] The right fat-tail effect suggests the probability of outlier negative returns of holding bonds given the constrain at the ZLB. On the other hand, the distribution with θ = 1.5 (S = 3.89%) has its peak at the right and the probability of outlier positive returns, indicating that the downside of the bond price is limited when the mean level of the yield is at a high level. The distribution with θ = 0.75 (S = 2.64%) has a relatively symmetric shape which represents the upside and downside probabilities of the yield being more or less the same.

In Panel B where κ decreases from 0.072 to 0.03, the tails of the distributions become much fatter and hump shaped, and their skewness is sensitive to a reduction in the mean reversion parameter κ. Regarding θ = 0.25 (S = 1.11%), the weaker mean-reverting force in the yield dynamics increases the likelihood of breaching the ZLB, which is reflected by the one-sided U-shaped distribution with accumulation of probability near the bound. This demonstrates that a weaker mean reversion changes the shape of the distributions substantially when the mean θ is close to the lower boundary. The changes in the distributions in Figure 2 with different model parameters demonstrate that the leakage condition ($\sigma_x^2/4\kappa\theta$) of the quasi-bounded process of the bond yield dynamics derived from

[7] See Bank for International Settlements (2018) for the low-for-long scenario.

the model is consistent with the distributions for bond yields with likelihood of breaching the ZLB.

2.2 Dynamics of the state variable incorporating central banks' credibility

Based on the theory behind modelling the term structure of (short-term) interest rates proposed by Vasicek (1977), CIR (1985a, b) and Longstaff (1989, 1992) in a simple one-state-variable production economy, there is a finite number of constant stochastic returns to scale production technologies that produce a single good that can be allocated to either consumption or investment. It assumes that the representative investor has logarithmic preferences, and that technological change is governed by a single state of variable ν. Following Longstaff (1989), ν is assumed to follow a process that behaves locally as a random walk with a drift. These state-variable dynamics are intuitively reasonable and consistent with the behaviour of a variety of economic variables. The nonlinearity broadens the class of technologies available and has the effect of inducing mean reversion in the equilibrium interest rate as shown by Sundaresan (1984). Therefore, ν is assumed to be governed by the following stochastic process with a drift μ_ν which can be a function of ν and instantaneous standard deviation σ_ν:

$$d\nu = \mu_\nu dt + \sigma_\nu dZ. \tag{7}$$

In CIR (1985b), the means and variance of production returns are linearly proportional. However, we follow Longstaff (1989) to allow them to be proportional to the nonlinear term ν^2.[8] Assuming that the long-term can be treated as equilibrium interest rates in an economy and the expected return and variance of the market are proportional to the square of the state variable, we can write the bond yield x as:

[8] The nonlinearity broadens the class of technologies available and has the effect of inducing mean reversion in the equilibrium interest rate as shown by Sundaresan (1984).

$$x(\nu) = c\nu^2 \tag{8}$$

where c is a positive constant.

By applying Ito's lemma to Eq.(4) with Eqs.(7) and (8), ν follows an asymmetric mean-reverting process with the following specification:

$$d\nu = \left(\frac{A_{-1}}{\nu} + A_1\nu\right)dt + \sigma_\nu dZ\ , \tag{9}$$

where

$$A_1 = -\frac{\kappa}{2} < 1, \tag{10}$$

$$A_{-1} = \frac{1}{2c}\left(\kappa\theta - \frac{\sigma_x^2}{4}\right) > 0, \tag{11}$$

$$\sigma_\nu = \frac{\sigma_x}{2\sqrt{c}}\,, \tag{12}$$

and $-\infty < \nu \leq 0$. As shown in Lo et al. (2015) and Hui et al. (2016) for modelling the exchange rate dynamics in target zones, this asymmetric mean-reverting fundamental dynamics incorporates the features of intervention and realignment in the target zones of the Hong Kong dollar and Swiss franc. The corresponding stochastic process that takes both marginal and intra-marginal interventions into account is commonly known as the Rayleigh process.[9] It is clear that the special case of vanishing A_1 and A_{-1} implies the absence of intra-marginal intervention.

To understand and visualise the asymmetric mean-reverting fundamental shock, Lo et al. (2019) obtain a "potential well" $U(\nu)$ by integrating the drift term in Eq.(9), in a negative form, with respect to ν:

$$U(\nu) = -\int\left(\frac{A_{-1}}{\nu} + A_1\nu\right)d\nu = -A_{-1}\ln|\nu| - \frac{1}{2}A_1\nu^2\ , \tag{13}$$

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[9] The asymmetric mean-reverting fundamental dynamics is described by the generalised Rayleigh process which is a diffusion process with mean reversion, of which some stochastic processes such as the Ornstein-Uhlenbeck process are special cases. It has been considered in the context of the path-dependent option pricing models used in economics and stochastic finance studies (see Davydov and Linetsky (2001)).

in which the state variable v is similar to a ball moving in a well, as shown in Figure 3 by plotting Eq.(13) with different values of A_{-1} and A_1. The ZLB for the bond yield corresponds to $\nu = 0$. The shapes of the potential well indicate the economic condition and the credibility of a central bank's commitment to keeping its policy rate above the ZLB under all circumstances. Decreasing the magnitude of A_1 will give an extremely flat potential well such that the Brownian random force will dominate the motion of the state variable. The state variable can then move more randomly subject to a weak mean-reverting force. Likewise, the state variable can move towards the lower boundary at $\nu = 0$ more easily together with a higher probability of ν breaching the bound when A_{-1} decreases. This illustrates that the mean-reverting force in the state variable dynamics determines whether the bond yield will fall below zero. The figure also shows the mean-reverting force in Eq.(13) is not symmetric. The restoring force (an increase in the bond yield) given by the second term in the mean-reverting drift with v close to zero is stronger than the force (a decrease in the bond yield) provided by the first term. This is consistent with the intuition that when the economy is extremely weak, such that the bond yield falls significantly, the central bank will communicate effectively about the commitment to keeping its policy rate above zero with the use of its monetary tools to boost the economy. The strength of the mean reversion in the state variable dynamics determines the likelihood of negative bond yields and the credibility of the bound. The asymmetric mean-reverting dynamics of the state variable are similar to asymmetric country-specific and global shocks in the context of contributions to violations of uncovered interest rate parity (Backus et al. (2001)) and exchange rate option (Bakshi et al. (2008); Jurek and Xu (2014)).

3. Calibration of government bond yield dynamics

In this section, we examine whether the bond yield dynamics can be characterised by the proposed model. By normalising the bond yield with the lower and upper bounds, it allows bond yields to fluctuate within a range over time. If the bond yield breaches the lower bound which is set to be zero in this study, this leads to a discrete drop in the bond yield with a magnitude that reflects the extent of uncertainty. Historical bond yields can be used as a guide to set a band confining the bond yield. The historical trend of the bond yield can be measured by a moving average S_{At} of the current and past bond yields. The moving average can be scaled by a parameter η_U, with $\eta_U > 1$, such that $\eta_U S_{At}$ forms an upper bound for the bond yield movement. If the bond yield is assumed to be normally distributed, $\eta_U S_{At}$ corresponds to the number of standard deviations from its moving average. The particular way in which past bond yields are brought into play does not affect the qualitative results of our analysis. With no loss of generality, the normalised log bond yield *x* is defined by:

$$x = -\ln\left(\frac{\eta_U S_{At} - S}{\eta_U S_{At}}\right) . \tag{14}$$

By using the log-likelihood function based on the PDF of Eq.(5), we calibrate the model parameters of the process specified in Eq.(4) by applying the maximum likelihood estimation (MLE) with a 3-year rolling window.[10]

3.1 Calibration of 10-year US Treasury yield

Regarding the sample period, we use daily 10-year Treasury yield data from 1 December 2004 to 30 September 2020. Panel A of Figure 1 shows the Treasury yield in *S* and the associated lower boundary at zero and moving upper boundary with the parameters of $\eta_U = 1.5$ on the 30-day moving average, and the transformed yield in *x* of the time series.[11]

[10] The bond yield data are from Bloomberg.
[11] The upper boundary corresponds to about 2 standard deviations.

Figure 4 reports statistically significant estimates of the drift term κ (Panel A) with the respective z-statistic maintaining above the value of 1.96 (i.e., at the 5% significance level). κ rises from 0.06 to 0.08 from 2007 to 2008, suggesting a greater restoring force for the 10-year Treasury yield towards its long-term means. Then, κ drops sharply to the 0.04 level in late-2008 when the Treasury yield dropped (from 3.85% to 2.08%) as the sub-prime crisis emerged. During the global financial crisis (2009-2016) and the period of the rise in the policy rate (2017-2018), κ oscillates in the range of 0.05 and 0.08. As the Federal Reserve lowered the policy rate again in 2019 with the dropping Treasury yield, κ edges lower to 0.04 reflecting weakened mean revision in the bond yield dynamics. When Covid-19 started to become widespread in the US in March 2020, the yield fell sharply from 1.63% to 0.54% with the fear of forthcoming severe recession. Such fall in the yield causes κ to be insignificant and not different from zero for a very short period of time, suggesting that the mean reversion weakens substantially and the yield may breach the ZLB. Subsequently, the estimation rebounds to the 0.06 level with the z-statistic higher than 1.96 when the Federal Reserve committed to not considering negative interest rates and the fiscal stimulus package was in place.[12]

Panel B of Figure 4 shows a steady estimated mean θ with the values at the level of 1.1 and the corresponding z-statistic staying above the 1.96 level. Similar to the movement of κ in March 2020, θ is insignificant and not different from zero in a very short period of time. Both the estimations of the parameters κ and θ suggest that the mean reversion in the Treasury yield dynamics diminished with the yield falling towards the ZLB when the Covid-19 pandemic emerged.

[12] See https://www.cnbc.com/2020/05/13/powell-says-the-federal-reserve-is-not-looking-at-negative-interest-rates.html.

The volatility σ_x, which is displayed in Panel C of Figure 4, is estimated to take the value between 0.02 and 0.06. The corresponding z-statistic is much higher than 1.96, indicating that the estimated σ_x is highly significant. The results suggest that the estimation of the square-root-process part of the quasi-bounded dynamics is robust. The volatility doubled from 0.03 to 0.06 after March 2010, consistent with the uncertainty in the economic outlook caused by the pandemic which increased the volatility of the Treasury yield.

As the condition of the measure ($\sigma_x^2 / 4\kappa\theta$) indicates the probability of leakage at the ZLB, Panel D of Figure 4 shows the likelihood of the yield falling below zero. The measure was substantially below 1.0 from 2005 to February 2020, suggesting there was no concern over the probability of leakage and the yield was well bounded above the ZLB. With the diminished mean reversion and the rise in the volatility of the bond yield dynamics in March 2020 due to the emerging pandemic, the measure rises sharply from 0.05 to higher than 1.0, indicating that the yield may breach the ZLB anytime. The condition reflects that the economy could enter a severe recession of uncertain duration or even a financial crisis such that the Fed may adopt the negative-interest-rate policy and demand for US Treasuries as safe assets would increase substantially during the financial turmoil. In response to the wide spread of the coronavirus, both the US government and the Fed stepped in with a broad array of actions to limit the economic damage from the pandemic.[13] The 10-year Treasury yield has then been staying in the range of 0.5% – 1% with the measure dropping back to the level of 0.14.

3.2 French government bond yield

The model calibration is conducted by applying the MLE to daily 2-year French government bond yield data from 3 January 2005 to 29 August 2014, the day before the yield

[13] See the Brookings Report “What’s the Fed doing in response to the COVID-19 crisis? What more could it do?” for the Fed’s actions at https://www.brookings.edu/research/fed-response-to-covid19/.

fell below the ZLB. Panel B of Figure 1 shows the French government bond yield in S and the associated moving upper boundaries with the parameters of $\eta_U = 2.875$ on the 30-day moving average, and the transformed yield in x of the time series.[14]

Figure 5 reports the estimates of the drift term κ (Panel A) with the z-statistic maintaining above the value of 1.96 during most of the time. κ increases from 0.02 in December 2008 to the level of 0.06 in 2010. It then stays in the range between 0.04 and 0.06 with the bond yield decreasing from about 2% in 2011 to near 0.05% in July 2012 when the European sovereign debt crisis intensified. The estimated κ becomes insignificant in January 2013 and is not different from zero for a short period time, when the yield dropped to the 0.02% level. Subsequently, κ increases to the level of 0.08 with the rebound of the yield. Panel B shows the estimated mean θ with the values ranging between 0.4 and 0.5 and the corresponding z-statistic staying above the 1.96 level during most of the time. Consistent with the estimation of κ, θ is insignificant and not different from zero in January 2013 for a short period of time, indicating the diminished mean reversion.

The volatility σ_x shown in Panel C is estimated in the range between 0.01 and 0.13. The corresponding z-statistic is much higher than 1.96, indicating that the estimated σ_x is highly significant, and the square-root-process part of the quasi-bounded dynamics is robust. The volatility increases sharply from 0.04 to 0.11 in January 2013 when the mean reversion diminishes, and then stays at the 0.13 level.

Panel D shows the measure of the probability leakage condition to identify periods with the condition greater than 1. The measure is close to zero during 2007 to 2011, indicating that the bond yield is well bounded above the ZLB. When the yield fell from 1% to 0.05% in July 2012, the measure increases to 0.1 with the weakened mean reversion and increased volatility as shown in Panels A-C. In January 2013, the measure surges beyond 1.0

[14] The upper boundary corresponds to about 7.5 standard deviations.

with the existence of the leakage condition after the French government bond yield had stayed at the near zero level for six months. Using only information until that point, the dynamics of the bond yield indicates the yield is quasi-bounded at the ZLB. Then the measure reduces to the 0.1 level.

3.3 Discussion of model calibration results

The empirical evidence shows that the quasi-bounded process adequately describes the US Treasury yield and French government bond yield dynamics by using the MLE estimation.[15] The mean-reverting force, which is represented by the parameters κ and θ, is estimated to be present during the estimation period. The diminishing mean-reverting force in the US Treasury and French government bond yields and the existences of the leakage condition (March 2020 for the US Treasury; January 2013 for the French government bond) demonstrate that the bond yields are quasi-bounded at the ZLB. The leakage condition was met with a sharp fall in the Treasury yields when widespread Covid-19 in the US in March 2020 triggered the fear of a severe economic downturn. The Federal Reserve responded with cutting its target for the federal funds rate by a total of 1.5 percentage points, bringing it down to a range of 0% to 0.25%, and has resumed purchasing massive amounts of securities. In addition, the US government introduced extremely aggressive fiscal policy. These measures along with the Fed's commitment to keeping its policy rate at or above zero under foreseeable circumstances have been able to prevent the Treasury yield falling below zero.

The European sovereign debt crisis flared up in the second quarter of 2012. Increased political disagreement prompted fears that Greece might soon leave the eurozone in a disorderly fashion. In addition to the troubled sovereigns, problems in the European banking system also worsened the situation. Deposits continued to flow out from the banks in the

[15] The model calibration results of the US Treasuries and French government bonds with other tenors show that the calibrations are robust. The results are available upon request.

heavily indebted countries to safer banks in other European countries. In September 2012, the ECB calmed the market by announcing free unlimited support for all eurozone countries involved in a sovereign state bailout/precautionary programme and preparing to do whatever was necessary to save the euro. With the policy interest rate lowered to zero, the expectation the ECB would adopt ultra-accommodative monetary policy pushed the French government bond yield towards the ZLB triggering the existence of the leakage condition in January 2013. The ECB's commitment was seen as a powerful tool at truncating the tail risk of an abrupt negative event and the French government bond yield rebounded after the leakage condition was met. However, the bond yield fell below zero in September 2014 after the ECB cut the deposit rate -0.10% in June 2014 and further to -0.2% in September 2014.

The calibration results of both the US Treasury and French government bond show that the mean reversion of their bond yield dynamics is weakened by substantial negative economic shocks, i.e., the pandemic in the US in March 2020 and the intensified European sovereign debt crisis in 2012. The shocks increase the likelihood of the bond yields breaching the ZLB and diminish the credibility of a central bank's commitment to keeping the government bond yields above zero.

4. Dynamic relationships of 10-year US Treasury yield with market and economic uncertainties

Financial and economic uncertainties will impact growth and investment which influence the Federal Reserve's monetary policy and thus the Treasury yield dynamics. Bernanke (1983) points out that high uncertainty gives firms an incentive to delay investment and hiring when investment projects are costly to undo or workers are costly to fire. Studies including Born and Pfeifer, 2014; Bachmann et al., 2013; Fernandez-Villaverde et al., 2015; Basu and Bundick, 2017; and Bonciani and van Roye, 2016, investigate how uncertainty shocks could generate business cycle fluctuations. Gilchrist et al. (2014), Pastor and Veronesi (2012, 2013) and Panousi and Papanikolaou (2012) find that uncertainty including policy uncertainty has negative impacts on precautionary spending cutbacks by households, upward pressure on the cost of finance and managerial risk-aversion. Friedman (1968), Rodrik (1991), Higgs (1997) and Hassett and Metcalf (1999) consider the detrimental economic effects of monetary, fiscal, and regulatory policy uncertainty.

Given impacts of financial market and economic (policy) uncertainties on monetary policy and the Treasury yield dynamics, we test the dynamic relationships of the 10-year US Treasury yield with the market and economic uncertainties using a co-integration analysis. The market uncertainty is measured by the VIX which is the US stock-market volatility index derived from the price inputs of the S&P 500 index options and gauges the level of risk aversion anticipated by market participants.[16] Regarding economic uncertainty, we use the index of economic policy uncertainty (EPU) developed by Baker et al. (2015).[17] The index reflects the frequency of articles in 10 leading US newspapers that contain the following triple: "economic" or "economy"; "uncertain" or "uncertainty"; and one or more of "congress", "deficit", "Federal Reserve", "legislation", "regulation" or "White House".

The 10-year Treasury yield, VIX and EPU index shown in the upper and lower panels in Figure 6 exhibit that declines in the yield occurred along with the rises in the VIX and EPU index in the global financial crisis in 2009 and the wide spread of Covid-19 in March 2020. The bond yield dynamic model expects a weakened mean-reverting force, i.e. lower κ and θ, in the bond yield dynamics under such uncertainty shocks.

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[16] Data are from Bloomberg.
[17] Data are from https://fred.stlouisfed.org/series/USEPUINDXD, Baker, Scott R., Bloom, Nick and Davis, Stephen J., Economic Policy Uncertainty Index for United States [USEPUINDXD], retrieved from FRED, Federal Reserve Bank of St. Louis; https://fred.stlouisfed.org/series/USEPUINDXD, December 14, 2020..

If there exists a long-run equilibrium relationship between the model parameters and the VIX and EPU index, their short-run dynamics can be studied through the following dynamical error-correction representation:

$$\Delta y_t = a_{10} + \alpha_y (y_{t-1} - \beta_1 X_{t-1}) + \sum_k b_{1k} \Delta y_{t-k} + \sum_k c_{1k} \Delta X_{t-k} + \varepsilon_{yt} \ , \tag{15}$$

where y_t is either κ or θ at time *t*, and α_y is less than zero. X_{t-1} is either the logarithm of the VIX ln(vix) or logarithm of the EPU index ln(epu) at time *t*-1. Under this representation, the model parameters (as represented by y_t) will respond to stochastic shocks (represented by ε_{yt}) and also the long-run equilibrium deviation in previous period (i.e., $y_{t-1} - \beta_1 X_{t-1}$). The estimated speed of adjustment (i.e., α_y) should be negative and nonzero for the co-integration relationship to be validly specified by the error-correction. In terms of absolute magnitude, a larger estimated value of α_y reflects a higher sensitivity of y_t to the long-run equilibrium deviation in the previous period.

The estimation is conducted at a weekly frequency starting from 27 November 2007 to 30 September 2020. The weekly average data for the model parameters κ and θ of the mean reversion based on the calibration results in section 3.1 are used. Table 1 reports the summary statistics, correlation coefficient and the respective Augmented Dickey–Fuller (ADF) test results for the variables both in levels and first differences. The ADF test results reflect that the existence of unit root for κ, θ and logarithm values of the VIX and the EPU index in level form cannot be rejected at the 10% significance level. Nonetheless, the respective ADF tests for the first differenced variables indicate no presence of unit root at the 1% level. Thus, the above results suggest that these four variables are all co-integrated of same order I (i.e., I(1)).

We adopt the single-equation test proposed by Engle and Granger (1987) to test the co-integration relationship between κ, θ and the market uncertainty VIX (and the economic uncertainty EPU). This Engle–Granger single-equation test essentially examines whether the residuals of the linear combinations among i.) κ and VIX (EPU); and ii.) θ and VIX (EPU) are stationary. Table 2 reports the co-integration tests between the VIX (EPU) and κ, θ, with the ordinary least squared regression residuals being tested by both the ADF and Phillips-Perron tests.[18] Overall, the results favour the alternative hypothesis of the presence of at least one co-integrating vector among the VIX (EPU) and the model parameters (κ and θ) respectively, given the statistical significances for κ and θ at the 1%.

Table 3 shows that the co-integrating vectors expressed by β between the VIX (EPU) and κ (θ) are estimated to be –0.00522 (–0.00463) and –0.0168 (–0.0167) at the 1% level. The estimated negative coefficients indicate that a higher VIX and EPU index would decrease κ and θ, holding other things constant. Intuitively, the negative relationship suggests that when the measurements of uncertainty increase, the likelihood of the yield breaching the ZLB increases, which is reflected from the weakened restoring force of the Treasury yield dynamics.

The estimates of the speed of adjustment α_y for κ (θ) reported in Table 4 are –0.04585 (–0.04669) and –0.02619 (–0.02838) respectively, which are negative but greater than –1, reflecting that the model parameters will subsequently adjust to restore the long-run equilibrium. This demonstrates a valid error correction specification and the presence of a self-restoring force, which will close the gap of the link between the mean reversion parameters (κ and θ) and the VIX (EPU).

The results of the co-integration analysis show the mean reversion is negatively co-integrated with the VIX and the EPU index, suggesting that financial and economic

[18] The critical values of the tests are based on MacKinnon (1996) and the lag length is determined by the Akaike information criterion. In addition, taking into account of the possibility of a regime shift in the co-integration model, we also test the null hypothesis of no co-integration relationship with the residual-based tests derived in Gregory and Hansen (1996) for the Engle-Granger regressions in Table 2. The results from the Gregory-Hansen test also suggest the null hypothesis for no co-integration relationship among these interest variables is rejected at 5% or 10% significance level. The results are available upon request.

uncertainties weaken the mean reversion in the Treasury yield dynamics, suggesting increased likelihood of the yield breaching the ZLB and erosion of a central bank's commitment to keep government bond yields above zero. The rise in the VIX indicates that market participants are unwilling to put capital at risk in view of market uncertainty, resulting in upward pressure on firms' financing costs. Bonciani and Ricci (2020) find that a tightening in financial conditions significantly worsens real economic activity in a persistent manner. When the financing costs go up too high due to deteriorations of economic conditions, for example triggered by the pandemic in March 2020, the Fed has to adopt ultra-accommodating monetary policy to lower both the short-term and long-term interest rates. This pushes the Treasury yields lower towards the ZLB. Similarly at the macro level to limit the material harmful effects of economic uncertainty on macroeconomic performance, the Fed has to step in with a broad array of actions including lowering the Treasury yields through quantitative easing.

5. Conclusion

This paper develops a stochastic model to study the dynamics of government bond yields which are constrained to lie above the ZLB committed by a central bank. The yield dynamics derived from the model follow a quasi-bounded process. Empirical calibrations of the process for the US Treasury and French government bond yields show that the process can adequately describe their dynamics, suggesting that the bond yields are subject to a quasi-boundary condition and the ZLB can be breached. The one-sided U-shaped bond yield distribution derived from the model illustrates accumulation of probability near the ZLB. The degree of credibility of the bound committed by a central bank is incorporated into the mean reversion in the dynamics of the bond yield and corresponding state variable. While the yields were bounded above the ZLB during most of the time, as indicated by their dynamics, the conditions for breaching the bound were met in January 2013 for the French government bond and March 2020 for the US Treasury using only information until those points. The results show that a central bank's credibility could evaporate when there is a potential severe economic downturn. If the bond yield is just above the ZLB, market participants will start to doubt that the central bank's commitment of keeping interest rates above zero when the yield continues to fall. The demand for the bond and its price will therefore rise, leading to a further fall in its yield. Weakened credibility reflected in the mean reversion increases the likelihood of the yield breaching the ZLB.

As shown by the results of the co-integration analysis, the negative relation between the mean reversion in the bond yield dynamics and both the VIX and EPU index illustrates that higher uncertainties of economic and financial conditions weaken the mean-reverting force, suggesting increased likelihood of the yield breaching the ZLB and erosion of the credibility of the bound.

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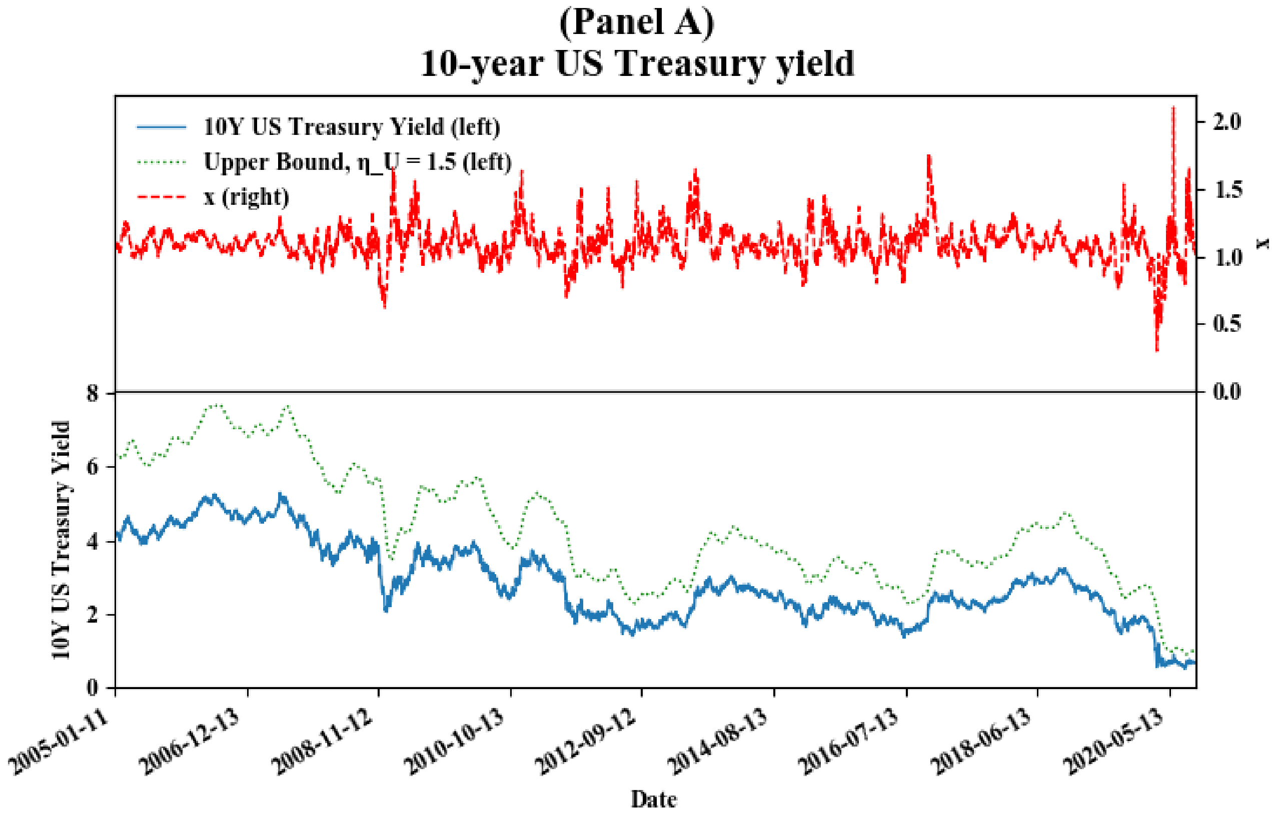
Figure 1: 10-year US Treasury (Panel A) and 2-year French government bond (Panel B) yields in S-scale (%) and x-scale, and lower and upper boundaries in S with $\eta_L = 0, \eta_U = 1.5$ for US Treasury yield and $\eta_U = 2.875$ for French government bond yield on 30-day moving average.



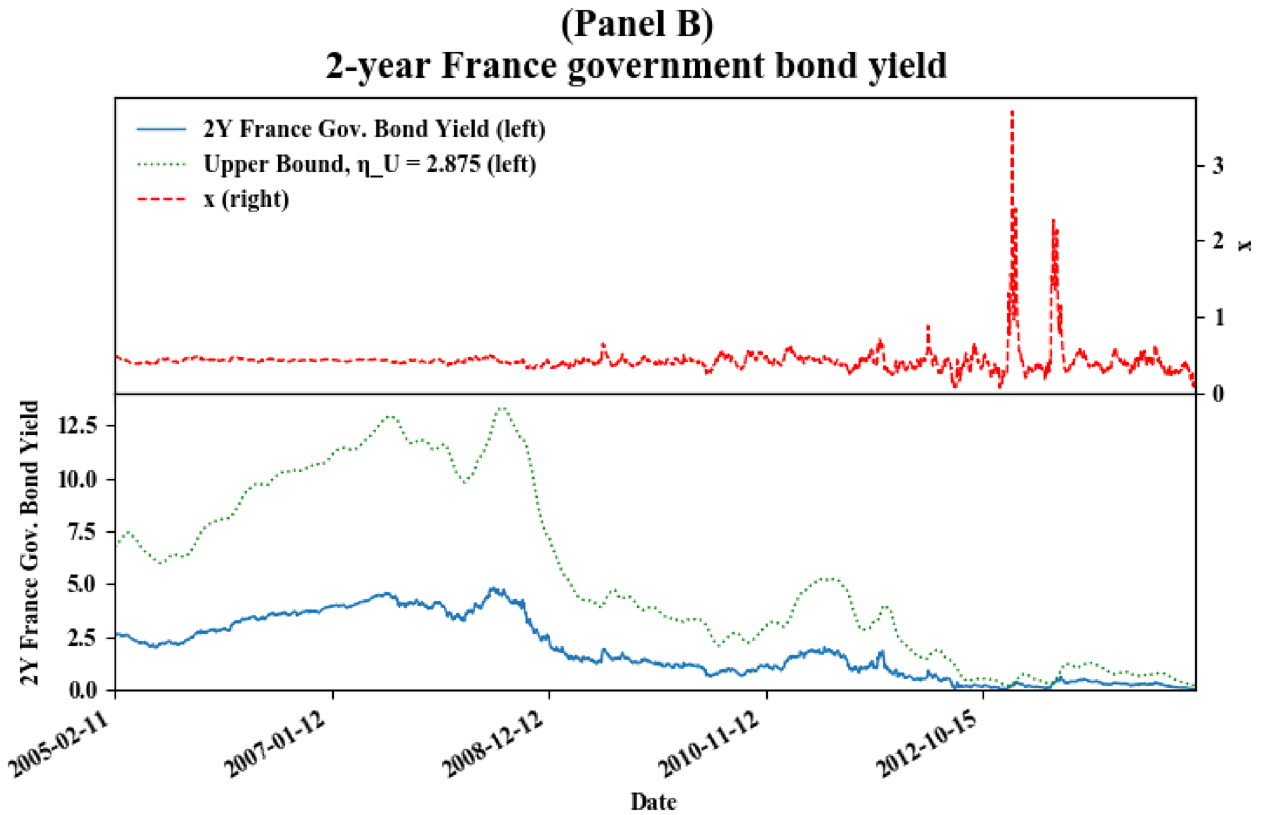


Figure 2: Bond yield (%) distributions with different values of model parameters σ_x, κ and θ under the normalisation of upper boundary $S_U = 5\%$ and lower boundary $S_L = 0\%$.

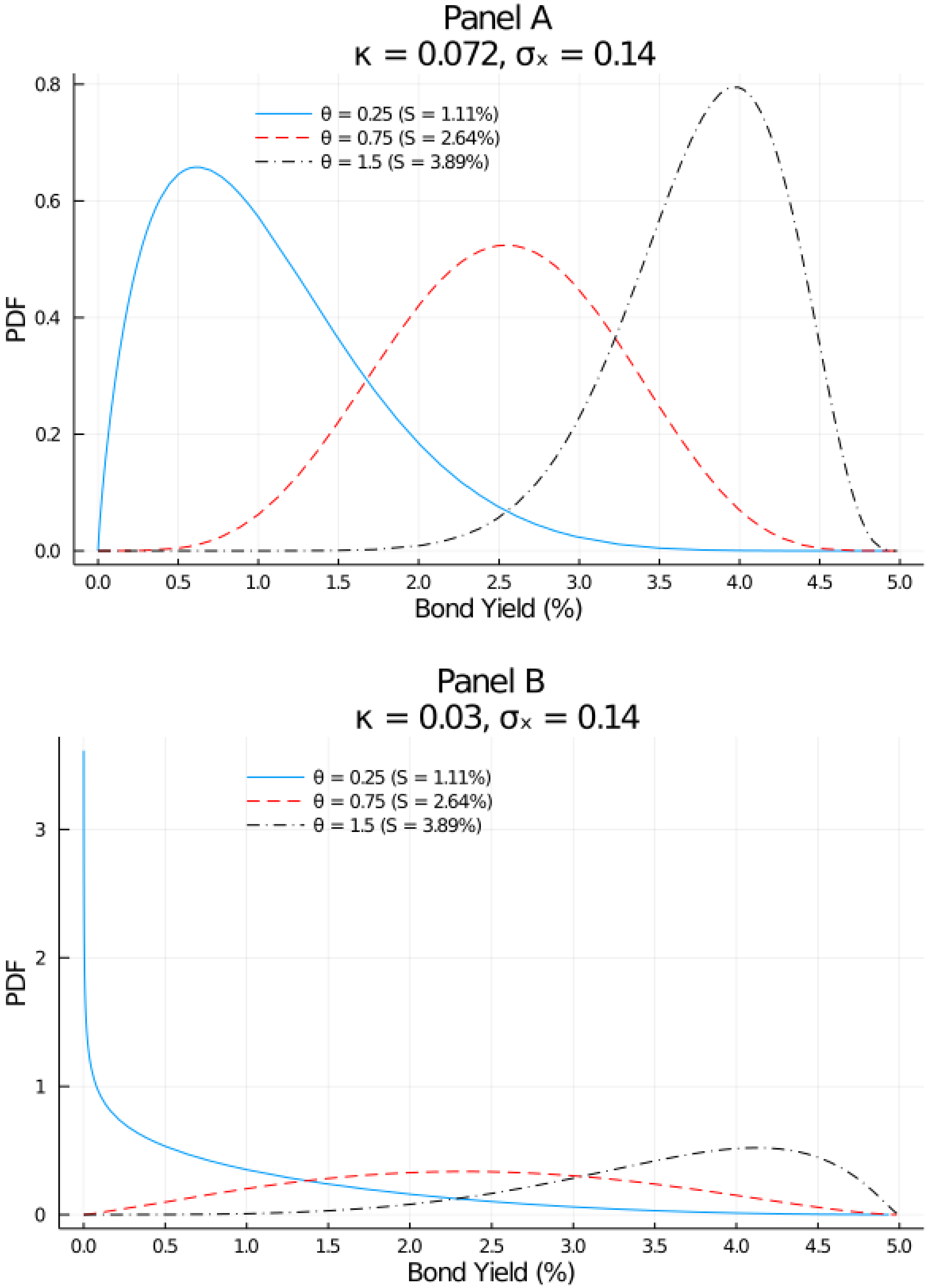


Figure 3: Eq.(13) of $U(\nu)$ by integrating drift term of state variable dynamics with different model parameters A_1 and A_{-1}.

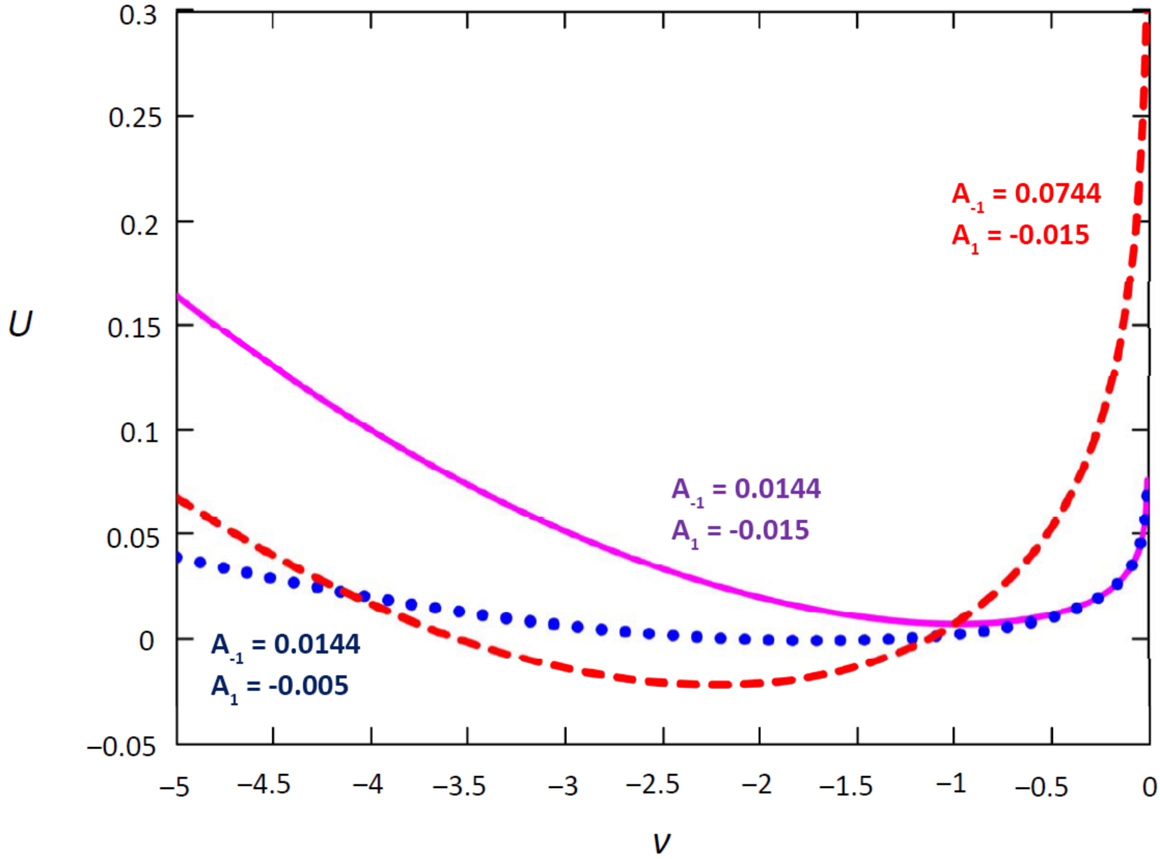


Figure 4: Estimated κ (Panel A), θ (Panel B), σ_x (Panel C), corresponding z-statistic, and leakage ratio ($\sigma_x^2 / 4\kappa\theta_x$) of 10-year US Treasury yield with 30-day moving average using 3-year rolling window.

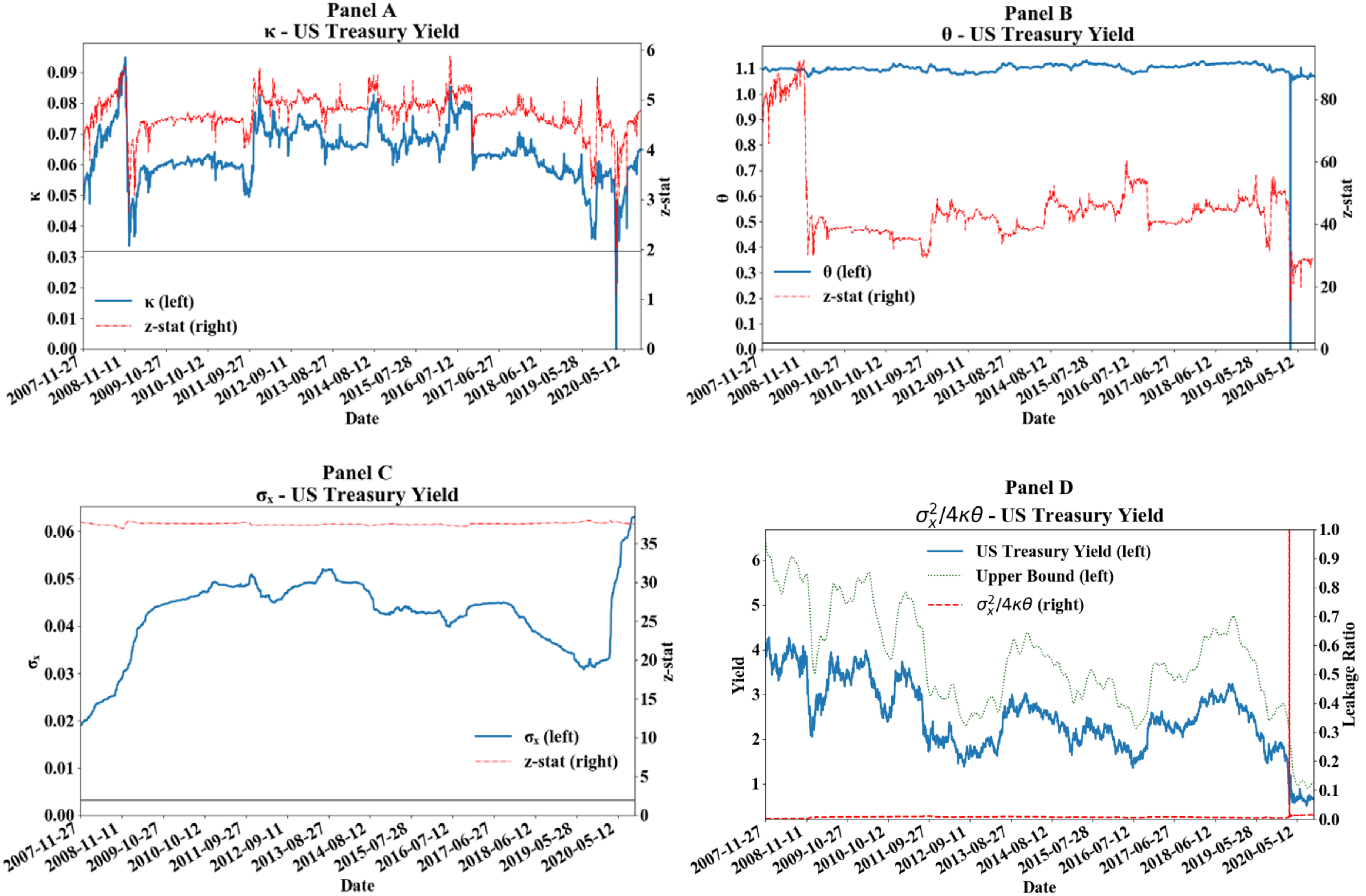


Figure 5: Estimated κ (Panel A), θ (Panel B), σ_x (Panel C), corresponding z-statistic, and leakage ratio ($\sigma_x^2 / 4\kappa\theta_x$) of 2-year France government bond yield with 30-day moving average using 3-year rolling window.

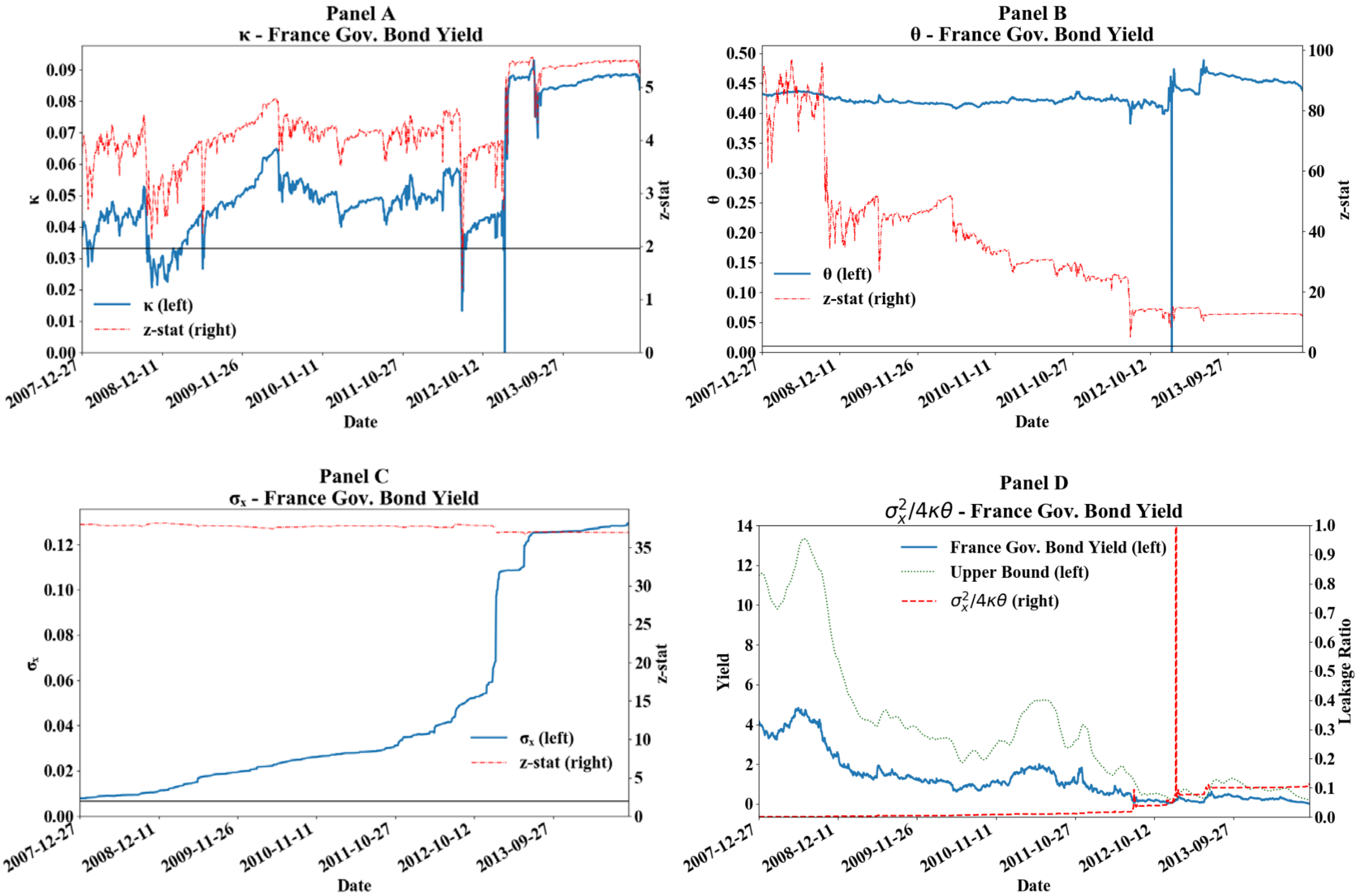


Figure 6: VIX, economic policy uncertainty (EPU) index and 10-year US Treasury yield (%).

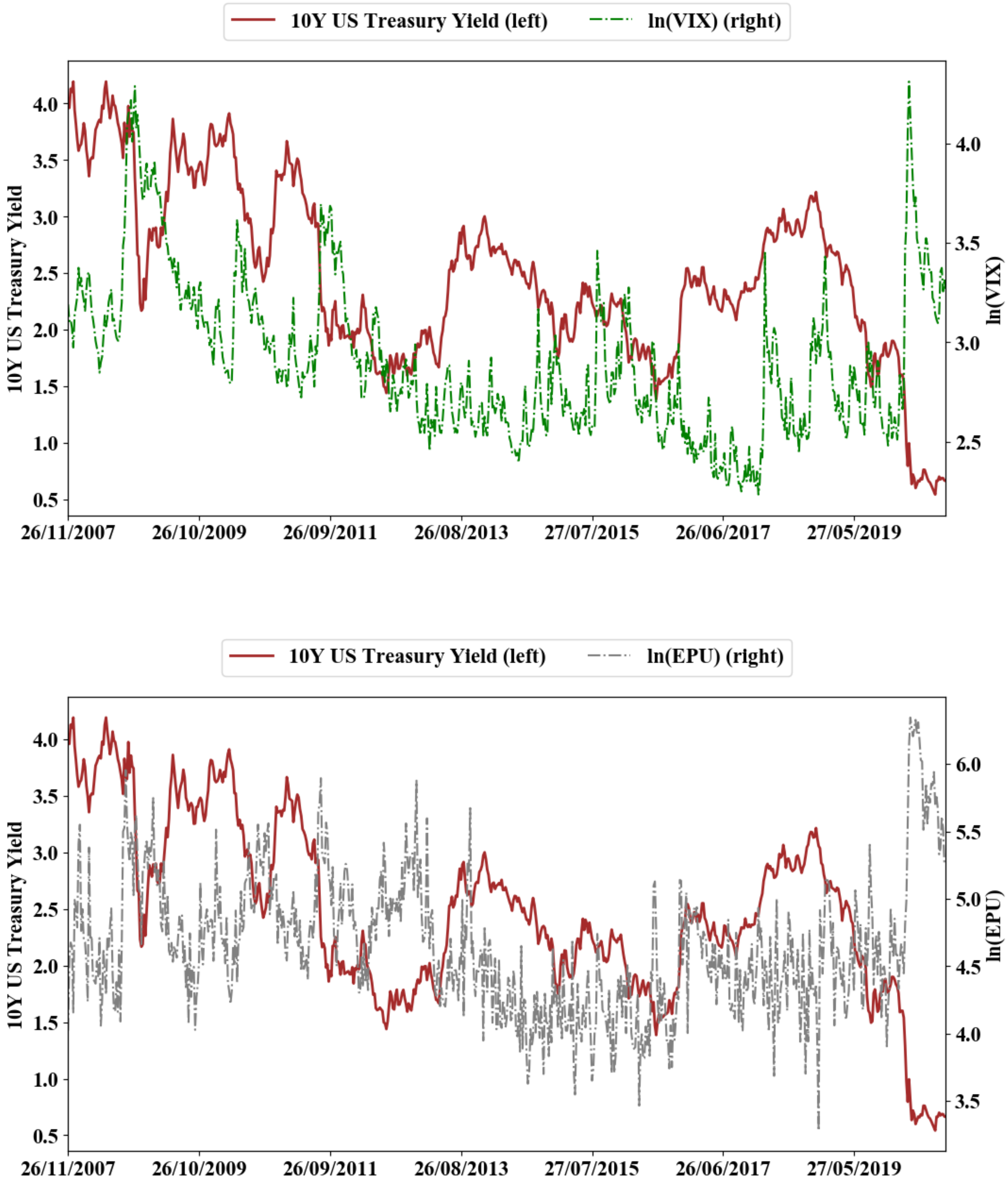


Table 1: Descriptive statistics of VIX, economic policy uncertainty index, κ and θ of US Treasury yield

Level	κ	θ	ln_vix	ln_epu
Mean	0.0638	1.1007	2.9120	4.6512
Median	0.0634	1.1015	2.8472	4.6096
Maximum	0.0934	1.1300	4.3124	6.3446
Minimum	0.0259	1.0556	2.2343	3.2811
Std. Dev.	0.0087	0.0145	0.3921	0.5099
Skewness	-0.2586	-0.4464	0.9461	0.6070
Kurtosis	4.4105	2.7667	3.7851	3.4664
Observations	671	671	671	671
ADF test statistics	-0.419	-0.404	-0.506	-0.245
Correlation with κ			-0.2075	-0.2717
Correlation with θ			-0.4399	-0.6007

Change	κ	θ	ln_vix	ln_epu
Mean	0.00002	-0.00004	0.0001	0.0019
Median	0.0001	0.00004	-0.0109	-0.0147
Maximum	0.0160	0.0132	0.7700	1.6346
Minimum	-0.0216	-0.0223	-0.3736	-1.0729
Std. Dev.	0.0022	0.0030	0.1213	0.3194
Skewness	-1.6320	-0.4163	1.1493	0.2845
Kurtosis	28.7002	10.0697	8.2479	4.5277
ADF test statistics	-19.6565 ***	-18.5352 ***	-24.9838 ***	-20.6728 ***
Correlation with $\Delta\kappa$			-0.0245	0.0363
Correlation with Δθ			-0.2039	-0.0584

1. lnvix (ln_epu) is the natural logarithm of one-week moving average of the VIX (economic policy uncertainty index). The model parameters κ and θ are based on the calibration results in section 3.1 with one-week moving average. The correlations for level of the variables are the correlations with κ and θ, and those for change are the correlation with $\Delta\kappa$ and $\Delta\theta$.
2. The ADF test checks the null hypothesis of unit root existence in the time series, assuming nonzero mean in the test equation, with lag length determined by Akaike information criterion up to maximum length of 4. *** indicates significance at the level of 1%.

Table 2: Tests for co-integration of market uncertainty (ln_vix), economic policy uncertainty (ln_epu), κ and θ.

Engle-Granger single-equation test[2]
(Null hypothesis: residual has an unit root)

On ln_vix	ADF test statistic	Phillips-Perron test statistic
Equation:		
κ	-4.7368 ***	-4.4307 ***
θ	-3.6846 ***	-3.0238 ***

On ln_epu	ADF test statistic	Phillips-Perron test statistic
Equation:		
κ	-4.4593 ***	-4.8428 ***
θ	-4.3973 ***	-6.1629 ***

Notes:

1. *** indicates significance at the 1% level.

2. The Engle-Granger single-equation test (ADF and Phillips-Perron tests) examines the null hypothesis that the residuals of the regressions of κ on ln_vix (and on ln_epu), and θ on ln_vix (and on ln_epu) respectively, given that κ, θ, ln_vix and ln_epu are non-stationary. The test assumes the existence of zero mean of the residuals in the test equation. The critical value of the test is based on MacKinnon (1996).

3. Alternatively, Gregory and Hansen (1996) derived residual-based tests for testing co-integration with regime shifts. We test the residual from the regression of θ on ln_vix and ln_equ alone based on the Gregory-Hansen co-integration test for the type of regressions with a level shift. The results based on ADF test statistics (Philip test statistic based on Zt) also indicate that the null hypothesis for no co-integration between the variables is rejected at 10% (5%) significance level (with the lag length determined by Akaike information criterion up to maximum length of 4). The identifications of break date vary across the types of shift models chosen for the test, our choice of starting date for the dummy specified as in Table 4 is quite close to the date identified by the Gregory-Hansen test based on a regime-shift-type model.

Table 3: Estimates of long-run coefficient (β) for market uncertainty (ln_vix), economic policy uncertainty (ln_epu), κ and θ.

Dependent variable:	κ_t	θ_t
ln_vix_t	-0.00522 ***	-0.0168 ***
ln_epu	-0.00463 ***	-0.0167 ***

Notes: *** indicates significance at the level of 1%. The coefficients are estimated by using the Engle-Granger single-equation and the coefficients of the short-run dynamic are in Table 4.

Table 4: Estimation results of the short-run dynamics for market uncertainty (ln_vix), economic policy uncertainty (ln_epu), κ and θ.

Dependent variable:	(With ln_vix) $\Delta\kappa_t$	(With ln_vix) $\Delta\theta_t$	(With ln_epu) $\Delta\kappa_t$	(With In_equ) $\Delta\theta_t$
Constant	1.51E-05	-3.84E-05	1.55E-05	-3.78E-05
Speed of adjustment	-0.04585 **	-0.02619 ***	-0.04669 **	-0.02838 ***
Δln_vix_{-1}	1.66E-05	-0.00051		
Δln_epu_{-1}			-0.00011	-1.60E-06
$\Delta\kappa_{t-1}$	0.2882 ***		0.2911 ***	
$\Delta\theta_{t-4}$		-0.1533 **		-0.1536 **

Notes: *** and ** indicate significance at the levels of 1% and 5% respectively. The coefficients are estimated by using the Engle-Granger single-equation and the long-run coefficients are in Table 3.