Central Bank Policy Impacts on the Distribution of Future Interest Rates

By

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Original version: May 15, 2013
Current version: March 16, 2016

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We thank Robert Merton, Robert Litterman, David Shimko, Stephen Ross, Stephen Figlewski, Adriano Rampini, Campbell Harvey and seminar participants at MIT, Duke, NYU and the Federal Reserve Banks of New York and Minneapolis, the Bank of England, the Bank of Italy and the Bank of France for helpful comments and discussions. We thank Lina Ren at Amundi Smith Breeden , B.J. Whisler at Harrington Bank, Layla Zhu at MIT and Matthew Heitz, Yeonkyo Choi, Tingyan Jia, Chloe Peng, and Xingchen Ling at Duke for excellent research assistance. Of course, all remaining errors are our own.

Abstract

The century low, near-zero and even negative short-term interest rates in the USA, Euro Area, the UK and Japan after the Great Recession of 2008/2009 and the European Sovereign Debt Crisis of 2010-2013 make the non-normality and non-lognormality of short-term interest rates quite clear. To uncover the changing implicit state prices and risk-neutral densities for future short-term interest rates, we use the prices of interest rate caps and floors with various strike rates and maturities from 2 to 10 years. We show that butterfly spreads of time spreads of cap and floor prices give sensible implied risk-neutral densities and state prices that reflect key moves made by the Federal Reserve, and the European Central Bank. The state prices and risk-neutral densities computed are largely distribution-free, preference-free and model-free results, building from the arbitrage-based computations of state prices from option prices that were presented in Breeden and Litzenberger (1978). In addition to showing positive skewness since 2008, the distributions are often bimodal as markets price in both scenarios of normalization and recessionary scenarios, with low pricing for middle rate scenarios.

I. Introduction.

In the "Great Recession" of 2008/2009 (a modern financial panic) and the aftermath, the U.S. Federal Reserve Board policy reduced short-term interest rates to near zero in December 2008 in an attempt to provide liquidity and stimulate the economy. While the unemployment rate dropped from over 10% to below 5%, it remained historically high over seven years later in 2016. In response, the Fed kept rates near zero for over 7 years, before finally "lifting off" with a first rate increase in December 2015, after strong job market growth and an unemployment rate down to 4.9%. Similarly, the European Central Bank also dramatically provided liquidity and reduced rates during Europe's Sovereign Debt Crisis. In early 2016, short-term interest rates were still below 0.50% in the U.S. and were actually negative in the Eurozone, while longer-term interest rates and derivatives contracts appear to reflect market expectations that rates will eventually increase. We show that these long-term interest rates reflect risk neutral probability distributions that were very positively skewed, and even bimodal, for the distribution for 3-month LIBOR over 3, 5 and 8-10 year horizons.

Despite the lack of rigorous justification, for many years market participants used the Black-Scholes formula to estimate values for options on interest rates using the assumption that the short-term interest rate itself was lognormally distributed. However, market prices are not constrained by Black-Scholes pricing and presumably reflect traders' true assessments of probabilities for future interest rates, as well as the supply and demand equilibrium risk adjustments for hedges of those rates. To uncover the implicit state prices and risk-neutral densities for interest rates, in this paper we use the prices of interest rate caps and floors, which are portfolios of relatively long-term call and put options on interest rates. We show that butterfly spreads of time spreads of cap and floor prices give sensible implied risk-neutral densities and state prices that reflect key policy moves made by the Federal Reserve and the European Central Bank. For example, in May and June, 2013, the Federal Reserve announced its intention to start "tapering" asset purchases as the economy strengthens. Interest rates increased significantly in response and risk neutral rate distibutions shifted to higher rates and somewhat more symmetric distributions. In September 2013, the Fed delayed tapering and markets were quite surprised and rates dropped sharply. In December 2013, the Fed actually

started the tapering operation in Ben Bernanke's last month as Fed Chair. In December 2015, Fed Chair Janet Yellen confirmed the continuation of tapering, surprising markets, and rates increased, while the stock market jumped. In March, 2015, President Mario Draghi's European Central Bank began massive "Quantitative Easing," which drove rates down sharply and made the insurance price distribution for rates much more positively skewed and bimodal. We will show how these changes in policy show up in interest rate cap and floor prices and in the implied risk neutral densities, or state prices for various interest rate levels.

State prices and risk-neutral densities computed are largely distribution-free and model-free results, building from the arbitrage-based computations based on options with different strikes presented in Breeden and Litzenberger (1978). The use of option prices to derive risk neutral densities has subsequently been used by many other authors, including researchers at U.S., European, Asian and Latin American central banks. Unlike many of these papers which interpolated options prices between prices for quoted strikes and extrapolated beyond the range of quoted strikes, we use only quoted strikes to estimate histograms of risk neutral probabilities.

Our work builds on prior work on options, using arbitrage insights, led by the Nobel Prize winning works of Black and Scholes (1973) and Merton (1973). Merton demonstrated that many results could be obtained just by prohibiting arbitrage or using stochastic dominance rather than making distributional assumptions (such as lognormality) or preference assumptions (such as power utility). In particular, Merton's arbitrage result that option prices are convex functions of exercise prices directly implies positive state price densities. Ross's (1976) elegant article showed how options with different strike prices could span the state space and complete markets. This led to Breeden and Litzenberger's (1978) derivation of the state price density as proportional to the second derivative of the option pricing function with respect to the exercise price. Once one has the state price density, prices of many types of derivative securities can be found from their functional payoff relationships with the underlying asset's price, integrated with the pricing density, so this result stimulated many applications. Like Merton's and Ross's results, their result is arbitrage based and does not depend upon preferences, beliefs or the underlying probability distribution.

In the ensuing years, the Breeden-Litzenberger model and a variety of estimation techniques have been used by many authors to estimate state prices or risk neutral probability densities from option prices. These risk neutral densities have been used to price more complex derivative securities. Articles estimating risk neutral densities and/or applying them to price more complex securities include academic works by Banz and Miller (1978), Shimko (1993), Rubinstein (1994), Longstaff (1995), Jackwerth and Rubinstein (1996), Ait-Sahalia and Lo (1998), Ait-Sahalia, Wang and Yared (2001), Longstaff, Santa-Clara and Schwartz (2001), Ait-Sahalia and Duarte (2003), Carr (1998, 2004, 2012), Bates (2000), Li and Zhao (2006), Figlewski (2008), Zitzewitz (2009), Birru and Figlewski (20010a,b), Kitsul and Wright (2012), Ross (2013), and Martin (2013), to name a few.

Central banks have also estimated "option implied (risk-neutral) probability distributions" using these techniques. Central bank applications are discussed in articles of Bahra (1996, 1997), Clews, Panigirtzoglous and Proudman (2000), and Smith (2012) of the Bank of England,; Maltz (1995,1997) of the Federal Reserve Board of New York; and Durham (2007), Kim (2008) and Kitsul and Wright (2012) of the Federal Reserve Bank of Washington. Kocherlakota's (2013) research group at the Federal Reserve Bank of Minneapolis uses Shimko's (1993) statistical method of applying the Breeden-Litzenberger formula to regularly estimate and publish risk neutral density functions and tail risks (e.g., risk neutral probabilities of moves of +/- 20% or more) for many assets, such as stocks, crude oil, wheat, real estate, and foreign exchange.

A typical Wall Street or Central Bank application is to estimate the volatility surface over time and across strike prices for an underlying asset, which then is used with the Black-Scholes formula to give the option pricing function. Volatility estimation is parameterized so as to give a thrice-continuous option pricing function, from which the second derivatives can be estimated and used as state prices and implied risk neutral densities. Many papers have been written about alternative ways to best estimate the volatility surface to replicate observed option prices.

Our approach is a bit different, as it is wholly arbitrage based and nonparametric, using market prices from well-traded markets. We use long maturity cap and floor prices and, by

calculating the price difference between a longer maturity and a shorter maturity cap, a "caplet," we create forward caps for the 3rd and 5th years, as well as years 8-10. We then use butterfly spreads of these forward caps to derive the arbitrage prices for triangular payoffs for various possible interest rate ranges. We show that if the risk-neutral density is approximated by a linear function over the rate range covered by the butterfly, then these prices will be the same as the prices of more intuitive Arrow "digital options" that pay $1 in a range, and zero elsewhere.

In Section II, we show how the triangular payoffs of the butterfly spreads aggregate in portfolios to trapezoidal payoffs, and how tail spreads of caps and floors can complete the portfolio and give riskless payoffs. From this demonstration, we are able to then find implied state prices and risk neutral probabilities for each rate range for future 3-month LIBOR. This approach is used in Section IV with LIBOR cap and floor prices, where we examine the computed risk-neutral densities for 3-month LIBOR, 3 and 5 years out, for the period from December 31, 2003 to the present, early 2016. We find risk-neutral densities that are relatively symmetric until the Great Recession of 2008/2009. With the drop in LIBOR to near zero, the shift to a very positively skewed risk-neutral density becomes increasingly prominent in the 2009 to 2013 period. Prior to that, Section III derives equilibrium relationships between risk neutral probabilities and true probabilities and shows how changing betas of nominal bonds affect this relationship as well.

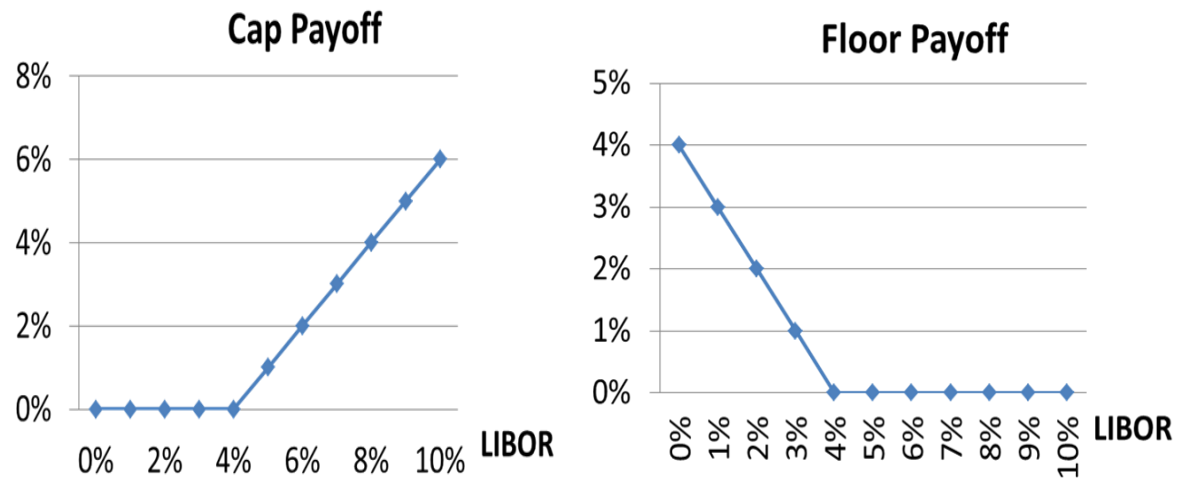
Section V shows before and after state prices/risk neutral densities for interest rates around major actions of the Federal Reserve Board in the 2008-2016 period, a time of historic levels of Fed intervention in markets and the economy. Sections VI and VII show similar analyses around policy actions by the European Central Bank, as it dealt with the sovereign debt crises in Europe in the 2010-2015 period. These analyses show not just the impact of the Fed's and ECB's actions on the level of the short rate or a long rate, but also on the entire state price distribution and risk neutral density for future 3-month dollar and Euro LIBOR possibilities. Section VIII shows the movements in risk neutral densities in the USA and Europe in 2013-2015 when the Federal Reserve "tapered" their historic levels of asset purchases and then finally lifted off rates in December 2015. Section IX presents some concluding remarks.

II. Triangles, Trapezoids and Tail Spreads:

Interest Rate Cap and Floor Payoffs, State Prices and Risk Neutral Densities

An interest rate cap with a strike rate of 4% pays off the larger of zero or what LIBOR is less the 4% strike rate, so caps win increasingly as rates increase and bond prices fall, much like portfolios of put options on bond prices. Floors pay off when rates drop, so floors are like portfolios of call options on bond prices. A 5-year cap or floor makes quarterly insurance payments over the 5-year period, based upon what LIBOR ends up being each quarter.

Figure 1



It is easy to verify that a portfolio that is long a 5-year cap and short a 4-year cap for the same strike rate and notional principal would receive payments only in year 5, as the net cash flows for years 1 to 4 would all be zero. This portfolio is called a "caplet" for year 5. A cap is a portfolio of quarterly caplets. "Floorlets" are defined correspondingly as time spreads of floors with the same strike rate. Interest rate cap and floor prices have moved dramatically in the past few years, as interest rates have plummeted. A 10-year, 4% cap traded at $13.30 on June 30, 2007, then dropped 75% to $3.28 on December 31, 2008 and then tripled back to $10.25 on December 31, 2009, and then dropped again by 70% to $3.50 on December 31, 2012. Note that many banks that hedge would say that the cost of hedging and "locking in" a 4% LIBOR rate for the next 10 years went from approximately 133 basis points ($13.30/10) per year in 2007 down to 33 bp/year at the end of 2008, then up to 102 bp at the end of 2009, and then down again to 35 bp/year at the end of 2012. Floor prices moved in opposite directions, but by similar amounts.

Breeden and Litzenberger (1978) demonstrated with discrete underlying asset prices that "butterfly spreads" (spreads of spreads) of options create unit payoffs in certain states of the world. If one allows the underlying state variable to have continuous values at the future payoff date, like in 5 years, then spreads of caplets have payoffs as in Figure 2A and the butterfly spread produces a triangular payoff in Figure 2B that starts at zero for all rates below the lower strike rate, then increases linearly and peaks at 1% at the middle strike rate and then declines linearly back to zero at the upper strike rate and remains for all higher rates.

Figure 2A

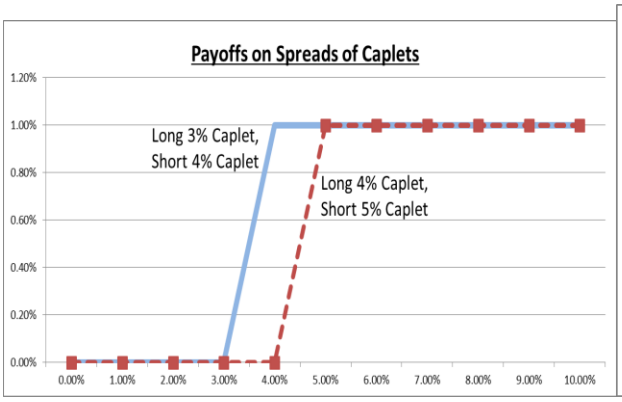
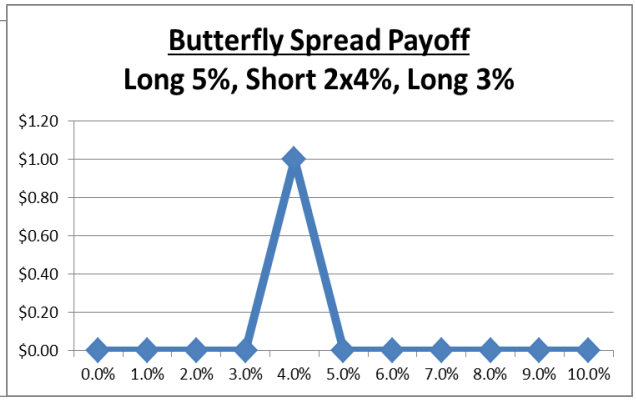


Figure 2B



Thus, the butterfly spread is a bet that pays off if rates are between the lower (3%) and upper (5%) rates of the spread, with a peak payoff of $1 in the middle (4%). In equilibrium, informally stated, B-L showed that the value of such a payoff should depend upon the probability of being in that rate range, multiplied by the conditional expectation of the marginal utility of $1 of consumption if that state range occurs, relative to the marginal utility of $1 of consumption today. We will call the cost of the 3%-4%-5% butterfly spread the "**state price** for a $1 payoff if the 3-month LIBOR rate is 4% in 5 years." By arbitrage, it would have to be that in equilibrium, as these payoffs can be constructed by this cap portfolio for that cost. Thus, state prices reflect both probability and risk adjustments, in that payoffs that occur in bad economies, when consumption is low, have higher values, due to higher marginal utilities then.

If one purchased a portfolio of 7 butterfly spreads with strike rates centered on 2% to 8%, respectively, the butterfly spreads would have overlapping payoffs, as illustrated in Figure 2C, which gives a total payoff pattern that is a trapezoid, as shown in Figure 2D:

Figure 2C

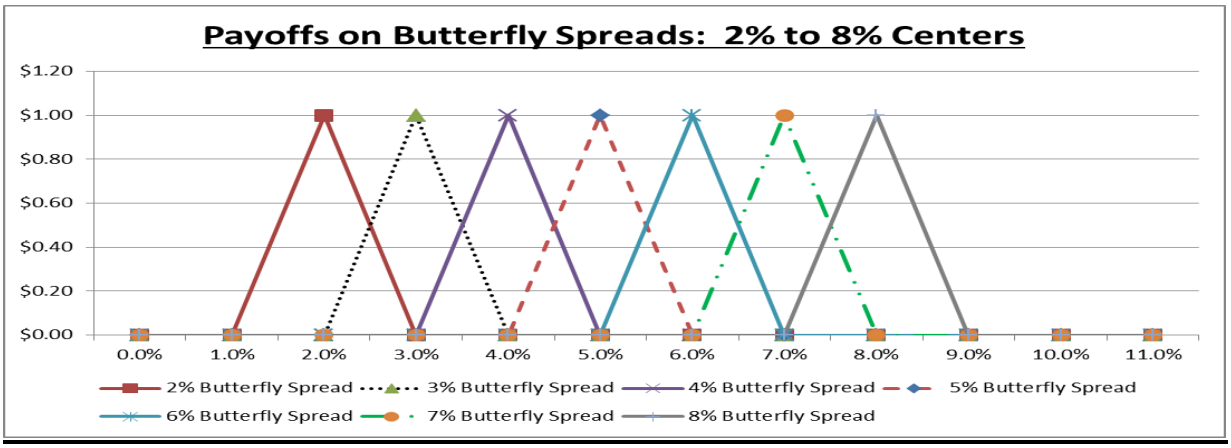
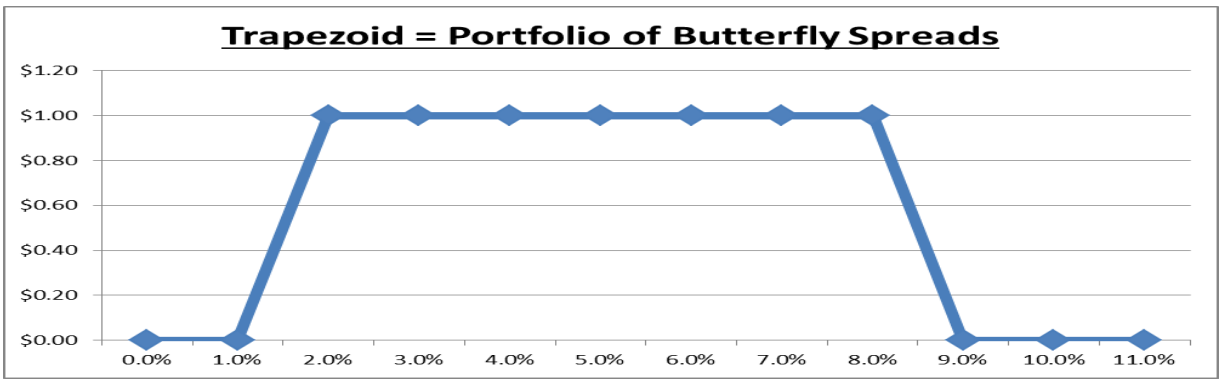


Figure 2D



A riskless $1.00 payoff is created starting with the trapezoid of Figure 2D (a portfolio of butterfly spreads) and then "completing it" by adding on complementary tail spreads for the left and right tails. As Breeden and Litzenberger (1978) showed, a spread that is long a 2% floorlet and short a 1% floorlet gives the complementary payoff pattern for the left tail, as indicated in Figure 2E. For the right tail, one simply uses a spread of caplets that is long an 8% and short a 9% to get the complementary tail payoff. Figure 2E shows the payoffs on the two portfolios of floors and caps that give the "tail spreads" for left and right tails. Figure 2F shows how these tail spreads combine with the trapezoid of butterfly spreads to give a riskless $1.00 payoff.

Figure 2E

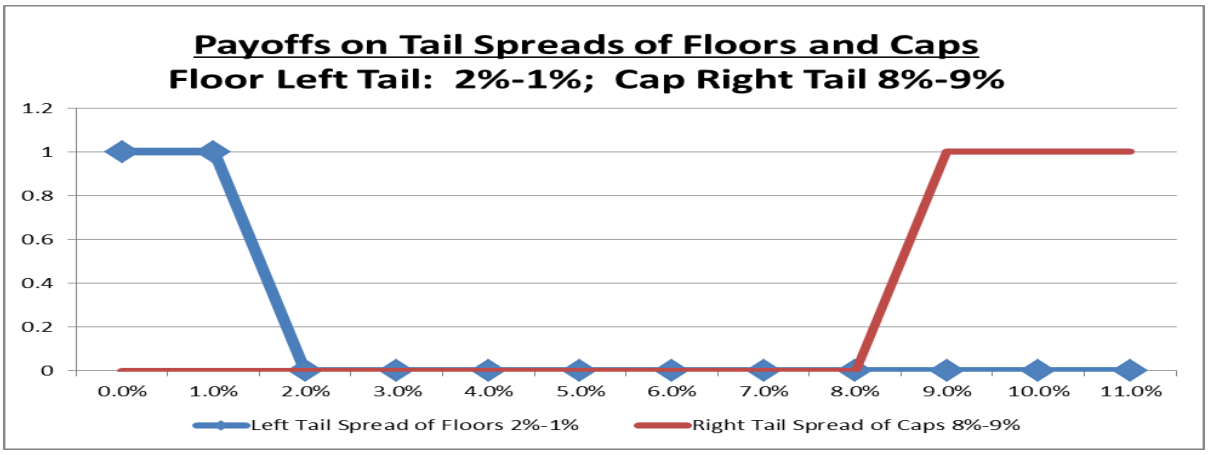
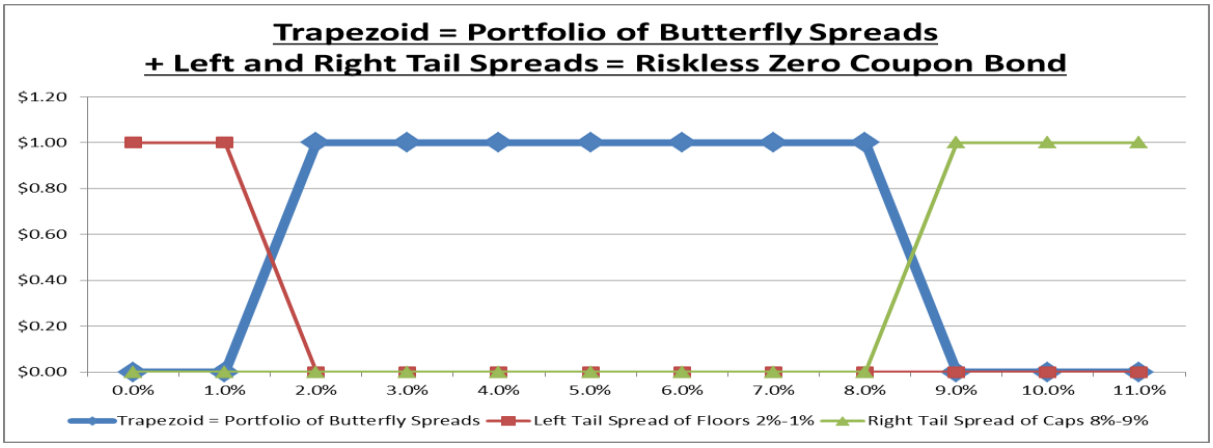


Figure 2F



To summarize, we have shown that the portfolio of 7 butterfly spreads (centered on strike rates from 2% to 8%), plus the left tail spread of floorlets (with strikes of 2% and 1%), plus the right tail spread of caplets (with strikes of 8% and 9%) gives a riskless payoff of $1.00. As a result, to prevent arbitrage, the cost of this "complete portfolio" must be the cost of a zero coupon bond for this maturity. We can then divide each component of the complete portfolio by the total and get fractions that are like a risk neutral density. Let us now explore whether prices for these triangular payoffs might be similar to the prices for digital option payoffs in the same ranges.

Within the range of rates for a butterfly spread, e.g., from 3% to 5%, the risk–neutral density will not be constant, as the true probability density likely changes throughout the range,

as do the conditional expected marginal utilities of $1.00. However, as a first approximation, let us assume that the risk neutral density changes **linearly** with the interest rate inside of the range. With that linear approximation, we can prove the following Proposition:

Proposition: The relationship between butterfly spread values and digital option values:
If the risk-neutral density (RND) is a linear function of the interest rate within the range of the butterfly strikes, then the value of a digital option that pays off $1.00 over the middle half of the range is equal to the value of the butterfly.

Proof: Let x be the interest rate, such that $x = c$ at the lower strike of the butterfly, $x = c + 1$ at the mid-point strike of the butterfly, and $x = c + 2$ at the high strike of the butterfly.

Assume that between c and c+2 the risk-neutral density = RND $= a + b(x - c)$

The forward value of a digital option that pays off $1.00 between $x = c + 0.5$ and $x = c + 1.5$ is:

$$\int_{c+0.5}^{c+1.5} [a + b(x - c)] \cdot 1 dx = a + b$$

The forward value of a butterfly is $\int_{c}^{c+1} \{[a + b(x - c)](x - c)\} dx + \int_{c+1}^{c+2} \{[a + b(x - c)](c + 2 - x)]\} dx$

$$= \frac{1}{3} bx^3 + \frac{1}{2}(a - 2bc)x^2 + (bc^2 - ac)x \Big|_{c}^{c+1} - \frac{1}{3} bx^3 + (bc + b - \frac{1}{2} a)x^2 + (2a - 2bc - bc^2 + ac)x \Big|_{c+1}^{c+2} = a + b$$

Of course, since forward values are equal at the same date, present values are also equal.

Q.E.D.

From the Proposition, under the assumption that the risk neutral density is linear in interest rates between 3% and 5%, the price of the 3%/4%/5% butterfly (with triangular payoffs) would be identical to the price of a claim that paid off $1.00 when interest rates were between 3.5% and 4.5% (digital payoffs). The 4%/5%/6% butterfly would have a value equal to that of a digital option that pays off between rates of 4.5% and 5.5%, and so on.[2]

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[2] Do note that there is a macro inconsistency in applying this approach with RNDs linear in rates where the {a,b} coefficients change from rate range to rate range, as would be realistic. With overlapping triangles, this would give an RND for the 4% to 5% range that is different for the 3/4/5 butterfly than for the 4/5/6 butterfly. Thus, this Proposition's result is just an approximation that is for useful intuition about butterflies and digital options.

Thus, if the price of each component of our earlier "complete portfolio" of caplets and floorlets is divided by the riskless bond price (the sum of the portfolio's component prices), these normalized prices give the integrals of the "**risk-neutral density**" over 1.0 percentage ranges centered on the mid points of the butterflies, and the histogram of risk-neutral probabilities will sum to 1.0. The risk-neutral density for date T is just given by the state price distribution for the butterfly spreads and tail spreads at T, normalized by dividing by the sum of those state prices for date T. We will use the phrases "risk-neutral density" and "state prices" and "interest rate insurance prices" synonymously, as they are proportional and have the identical shape as a distribution across rate levels.

To illustrate this, let us take the prices for 5-year and 4-year caps and floors on December 31, 2012 and compute the butterfly spreads of the time spreads between the 5-year and 4-year securities. We use floorlets for 1% to 3% butterflies, caplets for 7% and 8%, and an average of caplets and floorlet butterflies for 4% to 6% centers). We use the floorlet and caplet prices to compute the costs of the left and right tail spreads. (Note that here and henceforth we start with butterflies around the 1% strike, as we assume that a floorlet with a zero strike has zero value.)

Figure 3

	Spread Cost	"Risk-Neutral Probability"
"0%" = Left tail spread: Long 1%, Short 0% floorlet	$0.290	0.297
1% Butterfly spread (Long 0%, Short 2 1%, Long 2%)	$0.320	0.328
2% Butterfly spread (Long 1%, Short 2 2%, Long 3%)	$0.180	0.184
3% Butterfly spread	$0.080	0.082
4% Butterfly spread	$0.037	0.038
5% Butterfly spread	$0.028	0.028
6% Butterfly spread	$0.014	0.014
7% Butterfly spread	$0.007	0.007
8% Butterfly spread	$0.007	0.007
9%+ = Right tail spread: Long 8%, Short 9% caplet	$0.015	0.015
Totals	$0.977	1.000

III. True Probabilities vs. Risk Neutral Probabilities (State Prices Normalized).

While it is tempting to think of shifts in the state prices and risk neutral densities as predominantly due to shifts the true probabilities of future interest rates, that is not necessarily true. Marginal utilities affect state prices and the risk neutral density, in addition to true

probability shifts. In this section, we will explore that relationship and derive the theoretical relationship between true and risk-neutral probabilities, both in a relatively general state preference context and for the special case of constant relative risk aversion utility function, combined with lognormally distributed consumption.

The basic time-state preference model used is the same as used by Breeden-Litzenberger (1978). With a complete market, each individual, k, chooses consumption plans that maximize a time-additive utility function, subject to the usual budget constraint:

$$\max L = u_0^k(c_0^k) + \sum_t \sum_{s \in s_t} \pi_{ts}^k u^k(c_{ts}^k, t) + \lambda^k [W_0^k - c_0^k - \sum_t \sum_s \phi_{ts} c_{ts}^k] \tag{1}$$

First-order conditions for a maximum give:

$$\frac{\partial L}{\partial c_0^k} = u_0'^k - \lambda^k = 0 \Rightarrow \lambda^k = u_0'^k \qquad \textit{Define}: u_{ts}'^k \equiv \frac{\partial u^k(c_{ts}^k, t)}{\partial c_{ts}^k}, \text{ evaluated at } c_{ts}^k \tag{2}$$

$$\frac{\partial L}{\partial c_{ts}^k} = \pi_{ts}^k u_{ts}'^k - \lambda^k \phi_{ts} = 0 \tag{3}$$

$$\Rightarrow \phi_{ts} = \frac{\pi_{ts}^k u_{ts}'^k}{u_0'^k} = \text{price of \$1 in time-state ts.} \tag{4}$$

$$\Rightarrow \frac{u_{ts}'^k}{u_0'^k} = \frac{\phi_{ts}}{\pi_{ts}^k} \qquad \Rightarrow \frac{1}{u_0'^k}\begin{pmatrix} u_{t_1 s_1}'^k \\ \vdots \\ u_{TS}'^k \end{pmatrix} = \begin{pmatrix} \frac{\phi_{t_1 s_1}}{\pi_{t_1 s_1}^k} \\ \vdots \\ \frac{\phi_{TS}}{\pi_{TS}^k} \end{pmatrix}, \tag{5}$$

so if ordered from high to low, price/probability ratios at the optimum are positively and monotically related to marginal utilities in different states, and negatively related to consumption across states.

$$\text{Note: } B_t = \sum_{s \in s_t} \phi_{ts} = \text{zero coupon bond price} \quad \Rightarrow B_t = \sum_{s \in s_t} \frac{\pi_{ts}^k u_{ts}'^k}{u_0'^k} = \frac{E^k(\tilde{u}_t'^k)}{u_0'^k} \tag{6}$$

$$\Rightarrow \frac{\phi_{ts}}{B_t} \equiv \text{"risk neutral probability" (normed state price)} \equiv \phi_{ts}^* = \frac{\pi_{ts}^k u_{ts}'^k}{u_0'^k}\left[\frac{u_0'^k}{E^k(\tilde{u}_t'^k)}\right] = \pi_{ts}^k \frac{u_{ts}'^k}{E^k(\tilde{u}_t'^k)} \tag{7}$$

Thus, $\pi_{ts}^{k} = \phi_{ts}^{*}\left[\dfrac{E^{k}(\tilde{u}_{t}'^{k})}{u_{ts}'^{k}}\right]$ and $\phi_{ts}^{*} = \pi_{ts}^{k}\dfrac{u_{ts}'^{k}}{E^{k}(\tilde{u}_{t}'^{k})}$ (8)

So, risk-neutral probabilities = "true" probabilities x [(conditional marginal utility for state s, time t consumption)/unconditional expected marginal utility for time t consumption]. The risk-neutral probability for a state is higher, the higher the probability of the state and the higher the marginal utility in state (the lower the consumption in the state).

Our analysis so far has been for price/probability ratios for general states of the world. Can we say anything about prices of claims that pay off if 3-month LIBOR is, say, r_j, where {j=1,…N} represents different possible interest rate levels at time t? Note that, in general, the interest rate could end up at the same level in a variety of different states. To analyze this, we partition all states at time t into sets of states that all have the same chosen interest rate's level, r_j, where j goes from 1 to N. Every state is included in one and only one of the partitions at time t by interest rate level $\{r_j\}: s \in s_{r_j}$

Let ϕ_{tr_j} = Price of $1.00 received at t if the interest rate $r = r_j$, with zero received otherwise.

$$\phi_{tr_j} = \sum_{s \in s_{r_j}} \phi_{ts} = \sum_{s \in s_{r_j}} \pi_{ts}\frac{u_{ts}'}{u_0'} \tag{9}$$

The definition of conditional probabilities: [$P(A/B) = \dfrac{P(A\&B)}{P(B)} \le 1$] implies $\Rightarrow \pi_{ts} = \pi_{ts|r_j}\pi_{tr_j}$

$$\phi_{tr_j} = \sum_{s \in s_{r_j}} \pi_{ts|r_j}\pi_{tr_j}\frac{u_{ts}'}{u_0'} = \pi_{tr_j}\left[\sum_{s \in s_{r_j}} \pi_{ts|r_j}\frac{u_{ts}'}{u_0'}\right] = \pi_{tr_j}E\left[\frac{u_{ts}'}{u_0'}\middle| r_j\right] \tag{10}$$

$$\Rightarrow \frac{\phi_{tr_j}}{\pi_{tr_j}} = \frac{E\left[u_{ts}'\middle| r_j\right]}{u_0'} \tag{11}$$

Inserting eq. 6 for the zero coupon bond gives:

$$\frac{\phi_{tr_j}*}{\pi_{tr_j}} = \frac{E\left[u'_{ts}\middle|r_j\right]}{E[u't]} \qquad \text{(note: re-type this equation)} \tag{12}$$

Thus, we see that the risk-neutral probability to true probability ratio at the optimum for r_j is equal to the expected marginal utility of consumption, conditional upon the interest rate being at the specified level, divided by the unconditional expected marginal utility of consumption at time t. So if we are looking at butterfly spreads or digital options centered upon LIBOR = 2%, we need to compute the conditionally expected marginal utility of consumption, given that 2% rate.

If one wanted to gain further insight into the sizes of the potential fluctuations in risk neutral/true price to probability ratios, one could make two further assumptions, one on preferences and one on distributions: (A1) constant relative risk aversion (CRRA) utility and (A2) lognormally distributed consumption.

A1: If we assume power utility:

$$Let\ \ u_t^k(c_{ts}^k) = \frac{e^{-\rho t}(c_{ts}^k)^{1-\gamma}}{1-\gamma}; \quad \text{then } u_{ts}^{\prime k} = e^{-\rho t}\left(c_{ts}^k\right)^{-\gamma} \quad \text{and } RRA = \frac{-u''c}{u'} = \gamma \tag{13}$$

Suppressing the k superscript:

$$\frac{\phi_{ts}^*}{\pi_{ts}} = \frac{u'_{ts}}{E^k(\tilde{u}'_t\ )} = \frac{e^{-\rho t}c_{ts}^{-\gamma}}{E[e^{-\rho t}\tilde{c}_t^{-\gamma}]} = \frac{c_{ts}^{-\gamma}}{E[\tilde{c}_t^{-\gamma}]} \tag{14}$$

A2: If we assume consumption is lognormally distributed:

Note: Lognormal $Y \equiv e^x$ where $x \sim N(\mu,\sigma^2) \Rightarrow E(\tilde{Y}) = e^{\mu+\frac{1}{2}\sigma^2}$ (15)

Assume lognormal $c_{ts} = c_0 e^{g_{ts}t}$, where $\tilde{g}_t t \sim N(t\mu_t, \sigma_t^2 t) \Rightarrow -\gamma\tilde{g}_t t \sim N(-\gamma\mu_t t, \gamma^2\sigma_t^2 t)$ (16)

$$\frac{\phi_{ts}^*}{\pi_{ts}} = \frac{c_{ts}^{-\gamma}}{E[\tilde{c}_t^{-\gamma}]} = \frac{c_0^{-\gamma} e^{-\gamma g_{ts} t}}{E[c_0^{-\gamma} e^{-\gamma \hat{g} t}]} = \frac{e^{-\gamma g_{ts} t}}{E[e^{-\gamma \hat{g} t}]} = \frac{e^{-\gamma g_{ts} t}}{e^{-\gamma \mu_t t + \frac{1}{2}\gamma^2 \sigma_t^2 t}} \tag{17}$$

Taking logs of both sides, we get:

$$\log\left(\frac{\phi_{ts}^*}{\pi_{ts}}\right) = -\gamma g_{ts} t - \left(-\gamma \mu_t t + \frac{1}{2}\gamma^2 \sigma_t^2 t\right) \quad = \gamma[\mu_t - g_{ts}]t - \frac{1}{2}\gamma^2 \sigma_t^2 t \tag{18}$$

$$\log\left(\frac{\phi_{ts}^*}{\pi_{ts}}\right) = \gamma\left[\mu_t - g_{ts} - \frac{1}{2}\gamma \sigma_c^2\right]t \tag{19}$$

As expected, higher growth states for consumption have lower $\left(\frac{\phi_{ts}^*}{\pi_{ts}}\right)$ ratios. One could input different estimates of relative risk aversion and different states' growth rates and consumption volatility into eq. 19 and estimate the log of the risk neutral probability to the true probability.

Returning to the general case (not assumed lognormal or CRRA), Eq. 12 shows that the risk neutral probability for a certain rate level will exceed (be less than) the true probability if marginal utility, conditional upon that rate level, exceeds (is less than) the unconditional expected marginal utility for that date. This would typically be true when real consumption is less (more) in the state than it is expected to be on average for that date.

IV. Expected Marginal Utility, Conditional upon a Certain Nominal Interest Rate, Varies Over Time: Changing Real Betas for Nominal Bonds

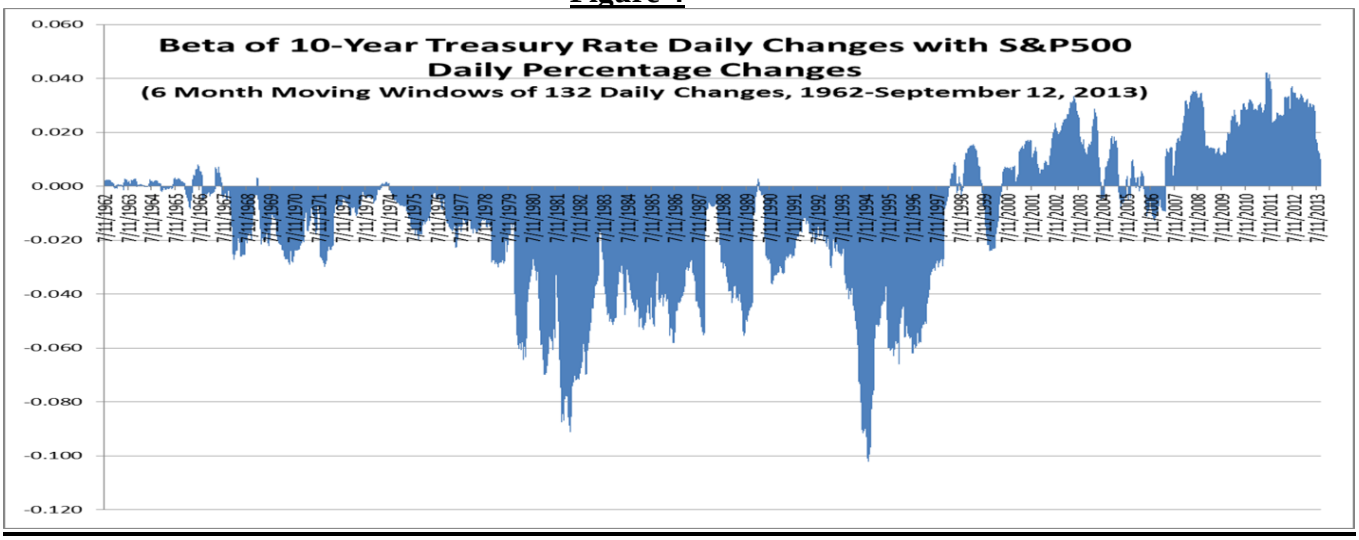
Next, let's consider issues related to the fact that our interest rate cap and floor prices are based on nominal interest rates, not real rates. Of course, marginal utilities are based upon real consumption, not nominal consumption. The relation between inflation and real economic growth is an unstable one over time. In the big recessions in 1974-1975 and 1981-1982, we had high inflation and high nominal interest rates (giving negative realized bond returns) at times of recession (with stocks and real consumption negative), which would indicate positive real consumption and stock market betas for holders of long-term nominal bonds. We would

characterize this as "supply-oriented inflation," as it was a situation of high inflation caused by constrained supplies of oil and grains.

In contrast, in recent years of the Great Recession and its weak recovery (2007-2013), market participants are well aware of the very positive correlation of daily interest rate changes with the stock market. The logic is that aggregate demand issues are the dominant inflation risks, and higher stock prices are viewed as leading better economic growth, which would give higher inflation and interest rates. Thus, we have in these recent years, negative real consumption and stock market betas for nominal long-term bonds, in contrast to the positive betas in the 1974-1975 and 1981-1982 recessions. This recent relationship is consistent with the "flight to quality" reactions that drive up nominally riskless bond prices on days when the stock market plummets.

To verify that the betas of nominal bonds have indeed changed signs over the years, as supply and demand inflation uncertainties alternate in volatility, we gathered daily data on nominal interest rates from the Federal Reserve's website for 1962 to September 12, 2013. We also gathered daily index prices for the Standard and Poor 500 from Yahoo! Finance's website, which gives them back to 1950 daily. Using windows of 3 or 6 months of daily data windows, the graphic results are very similar, so only the 6-month window data are shown here. Figure 4 shows the moving window regression betas of daily changes in nominal rates with daily percentage changes in the S&P 500 from 1962 to 2013.

Figure 4



Betas for bond returns will be opposite in sign from those for rates. The negative correlations and betas for rates in the 1970s and 1980s correspond to positive betas for nominal long-term bonds relative to the S&P 500 and relative to real consumption growth. The positive correlations of rates with stock returns in recent years demonstrates the negative betas for long term nominal bonds, which reflect their flight to quality appeal. From Figure 4, recent interest rate beta estimates averaged approximately 0.03, which would imply that a 10% increase in stock prices is associated with a 30 basis points move in the 10-year interest rate, which is not implausible.

The fact that the betas of interest rates and bond returns have changing betas over time lead us to conclude that the ratios of risk neutral probabilities to true probabilities should also change, depending upon which regime we are in. If we are in a situation like that of the 2007-2013 period, then low interest rates (1% or 2%) are associated with low real growth forecasts and high conditional marginal utilities, so the risk neutral probabilities for those low rate states should exceed true probabilities, and the reverse should be true for states with high interest rates (perhaps 6% to 9% rates).

If we return to a situation where high inflation is associated with very poor economic growth and poor stock returns (as in the 1970s and 1980s), then the pattern would reverse, and low rate states would have low ratios of risk neutral probabilities to true probabilities.

Figure 5 shows an illustration of how true probabilities should be related to risk neutral probabilities if one made the CRRA-Lognormal model assumptions, which yielded equation 19. Two alternative regimes are estimated – the 1998-2011 recent period, where nominal interest rates were positively correlated with real consumption growth, and the 1977 to 1997 period, where interest rates were negatively related to stock returns and real consumption growth. The higher the amount of relative risk aversion, the higher the premium paid for insurance against low consumption growth. In those situations of low consumption growth, the relatively high risk neutral probabilities (state prices) have to be scaled down to get proper estimates of true probabilities. In contrast, for good scenarios which have high real consumption growth, risk

neutral probabilities are low, due to low marginal utility, and have to be scaled up more to properly estimate true probabilities.

Figure 5

Illustration of True Probabilities Related to Risk Neutral Probabilities

True probability = K*Risk Neutral x exp(Gamma*(gts - mu))

Assumes: CRRA-Lognormal real growth model

Real Growth on Nominal Rate: 1998 to 2011 Data					
Intercept	-3.71	(t= -2.2)			
Slope	1.42	(t= 3.8)			
MuCgrowth	3				
		Relative Risk Aversion (Gamma)			
Nominal Rate	Real Growth	2	4	8	
		Ratio of True Probability to Risk Neutral*			
1	-2.29	0.90	0.81	0.65	
2	-0.87	0.93	0.86	0.73	
3	0.55	0.95	0.91	0.82	
4	1.97	0.98	0.96	0.92	
5	3.39	1.01	1.02	1.03	
6	4.81	1.04	1.08	1.16	
7	6.23	1.07	1.14	1.29	
8	7.65	1.10	1.20	1.45	
9	9.07	1.13	1.27	1.63	
10	10.49	1.16	1.35	1.82	

*=Up to a scalar multiple

Real Growth on Nominal Rate: 1977 to 1997 Data					
Intercept	4.11	(t= 3.2)			
Slope	-0.12	(t= -0.8)			
MuCgrowth	3				
		Relative Risk Aversion (Gamma)			
Nominal Rate	Real Growth	2	4	8	
		Ratio of True Probability to Risk Neutral*			
1	3.99	1.02	1.04	1.08	
2	3.87	1.02	1.04	1.07	
3	3.75	1.02	1.03	1.06	
4	3.63	1.01	1.03	1.05	
5	3.51	1.01	1.02	1.04	
6	3.39	1.01	1.02	1.03	
7	3.27	1.01	1.01	1.02	
8	3.15	1.00	1.01	1.01	
9	3.03	1.00	1.00	1.00	
10	2.91	1.00	1.00	0.99	
11	2.79	1.00	0.99	0.98	
12	2.67	0.99	0.99	0.97	
13	2.55	0.99	0.98	0.96	
14	2.43	0.99	0.98	0.96	
15	2.31	0.99	0.97	0.95	
16	2.19	0.98	0.97	0.94	

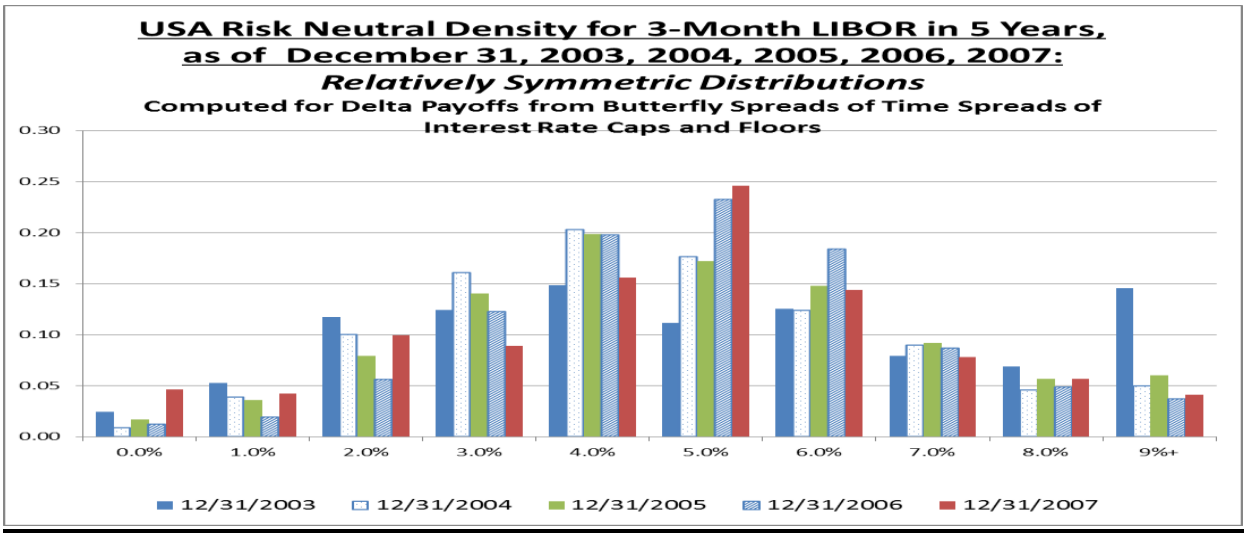
Given the slightly negative relation of nominal rates to stock returns and real growth in the 1977 to 1997 period (in good part due to the correlation of high inflation and interest rates with the big recession in 1981-1982), Figure 5's 2nd panel shows that the relation can reverse if that is the regime. Low interest rates were correlated with higher real growth and low marginal utilities and low risk neutral probabilities. These have to be scaled up to properly estimate true probabilities in this regime.

With these complexities of changing real risks and equilibrium risk premia of nominal bonds, the analyses of changes in the distributions for future interest rates that are in the following sections should most precisely be viewed in terms of changing state prices and risk neutral probabilities, which are what we uncover from prices of interest rate caps and floors.

V. State Prices and Risk Neutral Densities from Data on Cap and Floor Prices

In this section, we use the prices of interest rate caps and floors that were obtained from Bloomberg to estimate the market's implied distribution of "state prices" and "risk neutral densities" for various possible future rates for 3-month LIBOR. Plotting this gives us the shape of the risk-neutral density implied from market prices. Figure 7 shows the densities implied from prices of interest rate caps and floors. Floors are used for rates 0% to 3%, as they best trace out the density for lower interest rates. Averages of cap and floor implied densities are used for rates from 4% to 6%, and then the cap densities are used for 7%, 8% and 9% plus rates. Caps are most active for higher interest rates, as markets are more active for insurance that is out of the money, and therefore cheaper. We use yearend data for 2003 to 2007 for time spreads of 5-year and 4-year caps and floors, which means that these average distributions are for 3-month LIBOR from 4 to 5 years from the pricing date:

Figure 7



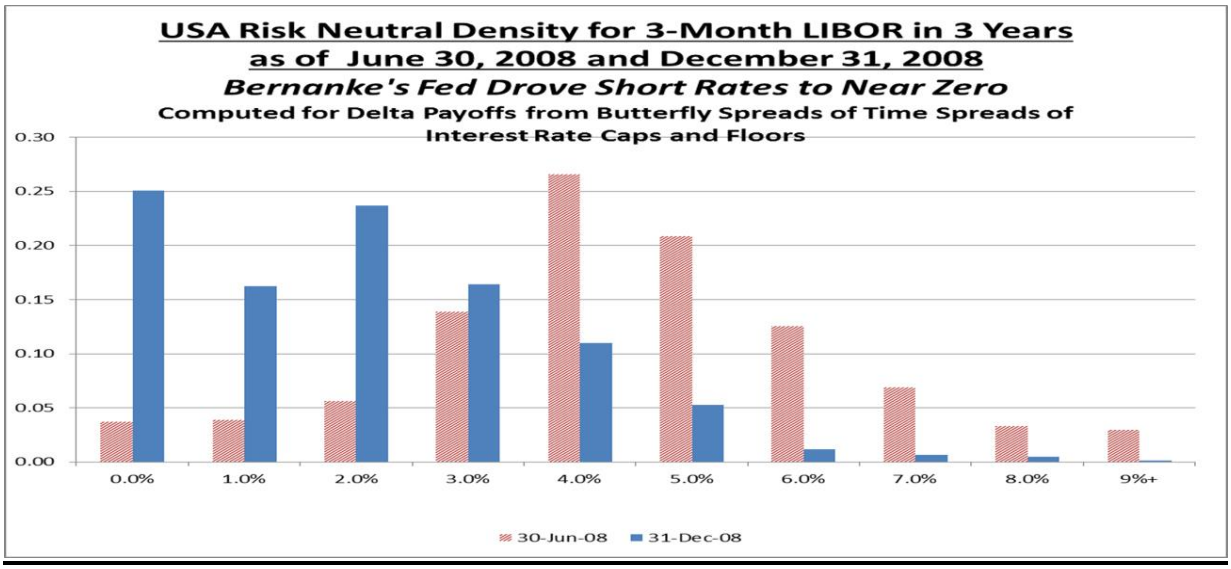
From Figure 6, we see that putting the information from caps and floors together gives one a relatively **symmetric** picture of risk-neutral probabilities for the 3-month LIBOR rate in 5 years, as seen from yearends **2003-2007.** The mode for long-term forecasts of 3-month LIBOR during these years centered around a projected 3-month LIBOR rate of about 4% to 5%, depending upon the year. In later sections, we will examine in more detail the dramatic moves in these insurance prices in response to Federal Reserve actions in the Financial Panic of 2008/2009 and in the European Sovereign Debt Crisis of 2010-2013, shifting to very postively skewed and sometimes bimodal distributions.

VI. Responsiveness of Risk Neutral Densities to Fed Policy Actions

Let us examine moves in the insurance price distribution in 4 major actions: (1) December 2008, when Bernanke's Fed took rates to zero, (2) during the U.S. budget crisis in Q3 2011, when the President and Congress were unable to reach agreement, (3) in June 2013, when the Fed made clear its intent to begin "tapering" their purchases of securities ("quantitative easing"), (4) December 2013, when tapering began, and (5) December 2015, when Chair Yellen's Fed actually began raising rates ("liftoff").

Major Policy Move #1: September 2008 was when Lehman Brothers filed for bankruptcy and many financial institutions were troubled (Wachovia, Merrill Lynch, Morgan Stanley) and did mergers or substantial capital raising. In response to the great fears and stock price drops around the globe, the U.S. Federal Reserve stepped in strongly in December 2008 to provide liquidity and reduce short-term rates to near-zero levels, and long-term rates dropped to lows for the prior 75 years (2.25%). Figure 8 shows the major shift in the risk neutral density to lower rates between June 30, 2008 and December 2008. In addition to this reflecting the true probabilities of lower rates in 5 years, these state prices likely also reflect changes in conditional consumption betas for these delta securities, with low rate ones having increasingly negative consumption betas, while high rate delta securities had more positive consumption betas.

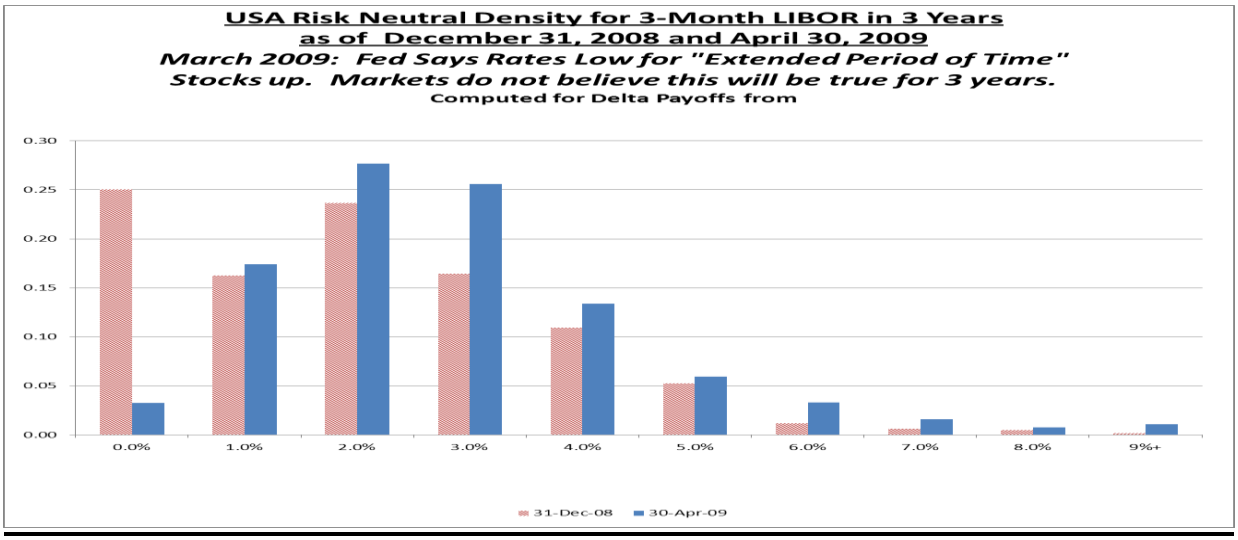
Figure 8



Major Policy Move #2

When the Fed announced in **March 2009** that they were going to keep rates low "for an extended period of time," Figures 9A and 9B, which compare risk-neutral distributions for LIBOR on December 31, 2008 and April 30, 2009, show that the markets apparently felt that the strengthening economy would not allow the Fed to maintain rates this low for 3 or 5 years, as the risk neutral distribution for rates then actually shifted to the right, towards higher rates. The stock market hit its low on March 9, 2009, and then began a very strong bull market that took stock prices higher by more than 50% in the remainder of 2009. Market participants apparently believed that an over heated economy might have higher inflation and actually require that rates be pushed up within 3 to 5 years.

Figure 9A

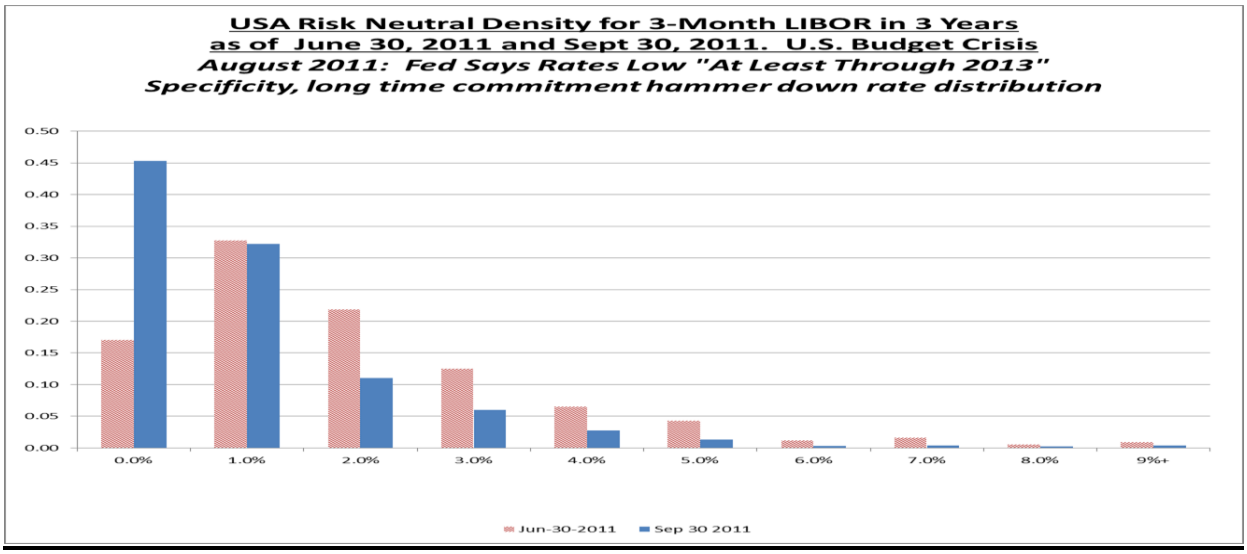


Major Policy Move #3

In early August, 2011, during the budget impasse between the President and Congress on raising the U.S. Federal Debt Ceiling, the stock market plummeted by 10% in one week and there was a huge flight to quality and drop in interest rates. In response, the Fed made a statement that surprised markets by specifically saying that they expected rates to main extremely low "at least through 2013," i.e., for more than 2 years. This specificity and long time commitment of the Fed's interventions dramatically affected the expectations for LIBOR rates even 5 years out. State prices for 1% delta securities doubled, while state prices for rates of 3%

to 6% LIBOR in 3 years dropped in half between June 30, 2011 and September 30, 2011, as Figure 10 shows:

Figure 10



Note that while the histograms for June 30, 2011 and September 30, 2011 cross in Figure 10, the cumulative frequency distributions (not shown) do not cross. If, for all levels of rates, the September 30, 2011 cumulative frequency distribution (starting from zero rates) is greater than or equal to the June 30, 2013 cdf, then we can say that the market's projected rate distribution is lower in the sense of "First Order Stochastic Dominance" (FOSD) of Rothchild-Stilgitz (1971).

The effect of the August 2011 specificity and long time commitment on the distribution for longer term, 5-year forecasts for 3-month LIBOR was even more dramatic. The gradually strengthening economy of 2009-2011 had shifted the risk neutral density for LIBOR 5 years out to a relatively symmetric distribution, with low prices for 0% and 1% rates, and a mode of 4%, with highest probability densities from 3% to 5%, then tapering off gradually. After the Fed's announcement, the mode for rates out 5-years dropped from 3% to 4% to less than 1%, and the risk-neutral density shifted to very positive skewness. Clearly, the Fed's actions and the stock market fall during the budget impasse transformed the 2011 state prices and risk neutral density.

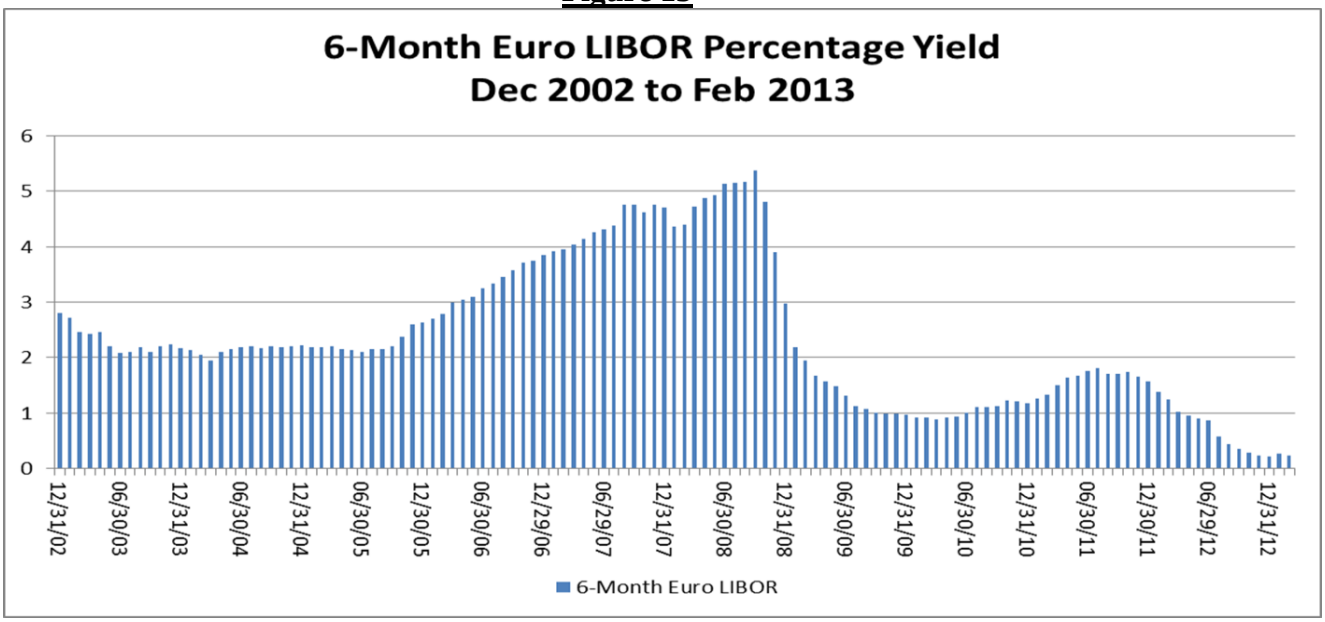
In the next section, we will similarly present what happened in Europe during this same time. From 2010 to 2013, Europe dealt with sovereign debt crises in Greece, Ireland, Portugal, Spain and Italy, so major policy actions were taken by the European Central Bank at that time.

VII. Euro Area State Prices and Risk Neutral Densities for Euro LIBOR.

In this section and the next, we do parallel analysis for state prices/risk neutral densities for the Euro Area. Prices for caps and floors for various maturities and strike rates for Euro LIBOR were obtained from Bloomberg. Principal cap and floor market prices used are for caps and floors with two or more years to maturity, based upon 6-month Euro LIBOR, paid semiannually.

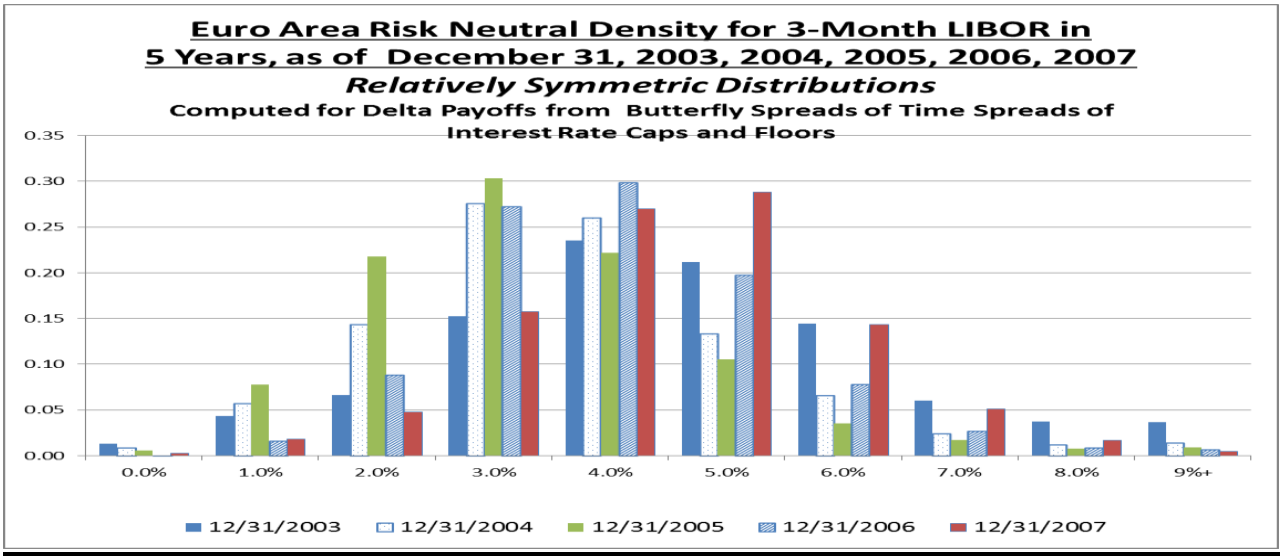
Stock prices in the Eurozone in 2003-2013 largely paralleled moves in USA stock prices, but with a larger increase up in 2003-2007 period, a larger fall in 2008-2009, and a smaller comeback from 2009-2013, due to the European Sovereign Debt Crisis from 2010-2013. Euro LIBOR started at a higher yield than in the USA, with 2%+ in 2003, then rose similarly to 5% in 2007-2008. Euro LIBOR fell a few months later and less than did USA LIBOR in 2008-2009, as the European Central Bank was slightly less stimulative than the Federal Reserve was at that time (as Europe appeared less affected by the Great Recession than the USA). Indeed, in 2010 and 2011, when the stock market recovery was under way, the ECB under President Jean-Claude Trichet actually raised interest rates, something the Fed did not do after 2008. These rate increases were reversed when Mario Draghi became ECB President on November 1, 2011, and made clear the ECB's commitment to provide massive liquidity in the face of the Euro Sovereign Debt Crisis. Figure 13 shows the movements in 6-month Euro LIBOR for the past 10 years:

Figure 13



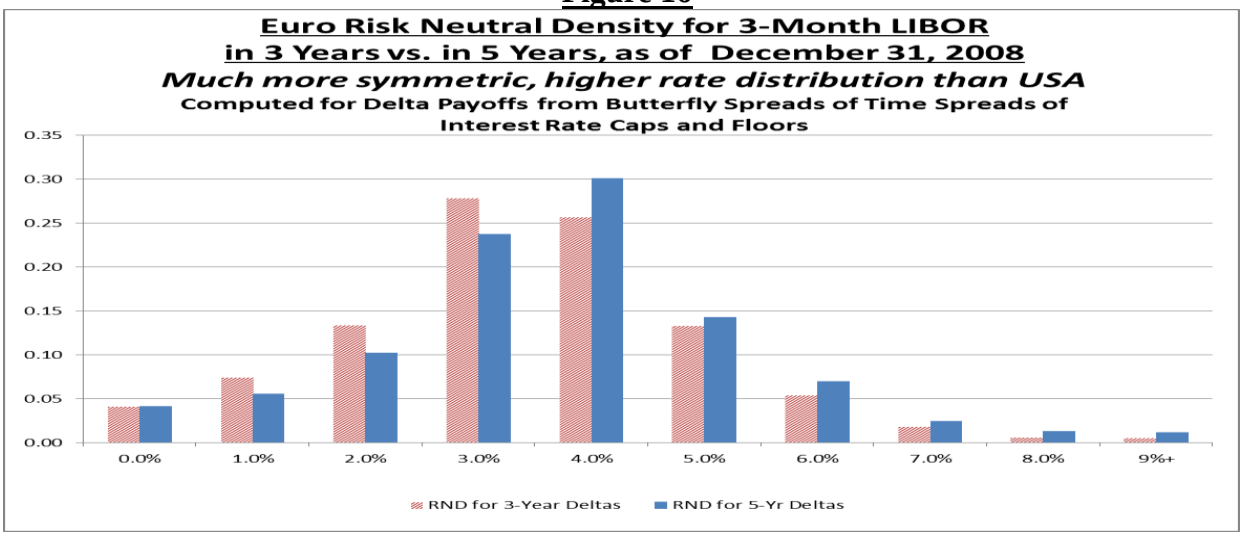
Butterfly spreads of time spreads of interest rate caps and floors are again used to compute prices of triangular payments (as described in Section II) to arrive at risk neutral densities for 6-month LIBOR 5 years ahead. Figure 14 gives the distributions at yearends for 2003-2007. The big picture looks quite similar to that of the USA in Figures 6. From 2003 to 2007, the market's state prices/risk neutral densities for 6-month Euro LIBOR, 5 years hence, were approximately symmetric, with modal rates between 3% and 5%, depending upon the year.

Figure 14



In Figures 14 and 15, we see similar distributional results for the Euro as for the dollar, with large uncertainties in 2003, evolving to tight distributions in 2006. However, note that 9%+ tail risk in 2003 was much smaller in Europe than it was in the USA. As of December 31, 2008, shortly after the fall in Lehman Brothers on September 15th, the dramatic fall of stocks globally in October, and the credit markets "seizing up" in November (*New York Times*, November 21 headline), European markets' implied state prices/risk neutral densities for 5 years out were quite different from those of the USA. Figure 16 shows the Euro Area 3 and 5-year insurance price distributions on December 31, 2008. We see a relatively symmetric distribution for Euro LIBOR 3 and 5 years hence, whereas Figure 8 showed that U.S. dollar LIBOR had a very long tail with positive skewness. Perhaps at this point Europe was not nearly as worried of a deep and lasting recession as the USA was.

Figure 16



However, we will see in the next section that the ensuing European Sovereign Debt Crisis caused the ECB to provide highly stimulative policy responses, and market beliefs in 2013 had risk neutral distributions that were quite similar in shape to those in the USA, with very positive skewness starting from very low interest rates.

VI. ECB Policy Impacts in the European Sovereign Debt Crisis

In this section, we examine key events in the European Sovereign Debt Crisis of 2010 to 2013 and policy actions by the European Central Bank in dealing with the crisis. Using timelines provided by the BBC (13 Jun 2012), Yahoo (23 Feb 2013) and a Reuters Special Report (2 Mar 2013), we examine three key dates in the Sovereign Debt Crisis and the ECB's monetary stimulus, led by President Mario Draghi: (1) Q2 2010, when the first bailout for Greece was announced, (2) Q3-Q4 of 2011, when Draghi took over and started cutting rates, (3) September 2012, when Draghi proclaimed that the ECB would do "whatever it takes" to save the Euro, and (4) Q1 2015, when the ECB began a massive "Quantitative Easing" program.

Dow Jones Euro Stoxx stock price indexes plummeted in 2010 around the first Greece bailout, as well as in 2011 around the second Greece bailout and serious concerns over Spain and Italy's debts, and then again in 2012 over intensified concerns again on Spain and Italy, as reflected in the yields on their debts. These moves are seen in Figure 17, which focuses on Euro

area stock price index moves at month ends from December 2009 to February 2013. The movements of 6-month Euro LIBOR over the same period are given in Figure 18. Note that the ECB actually increased short term rates in 2010 and the first half of 2011, before short rates were dramatically reduced in late 2011 and again in the second half of 2012.

Figure 17

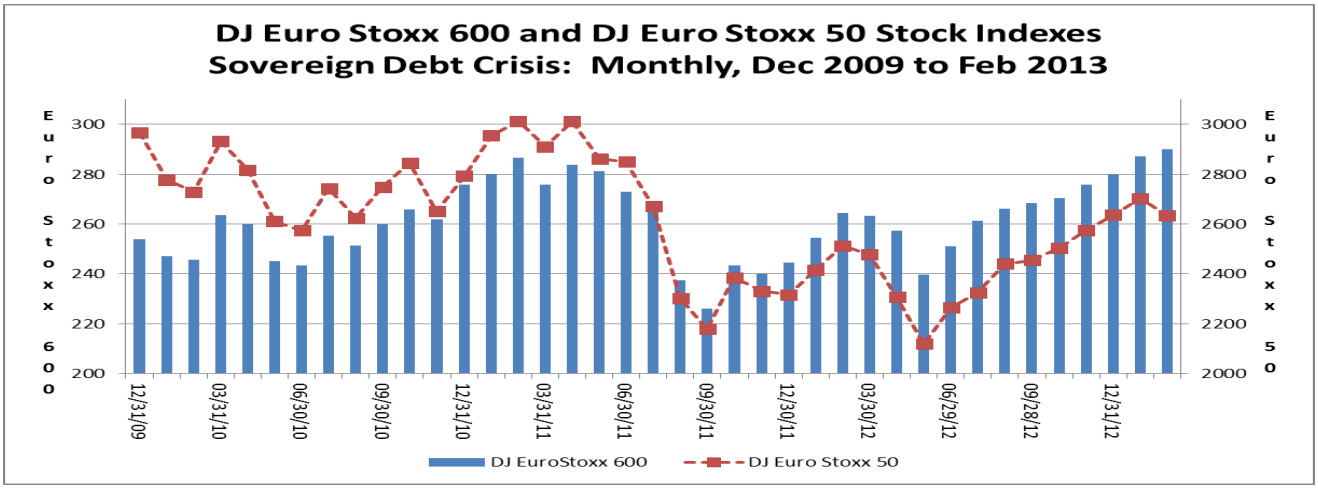
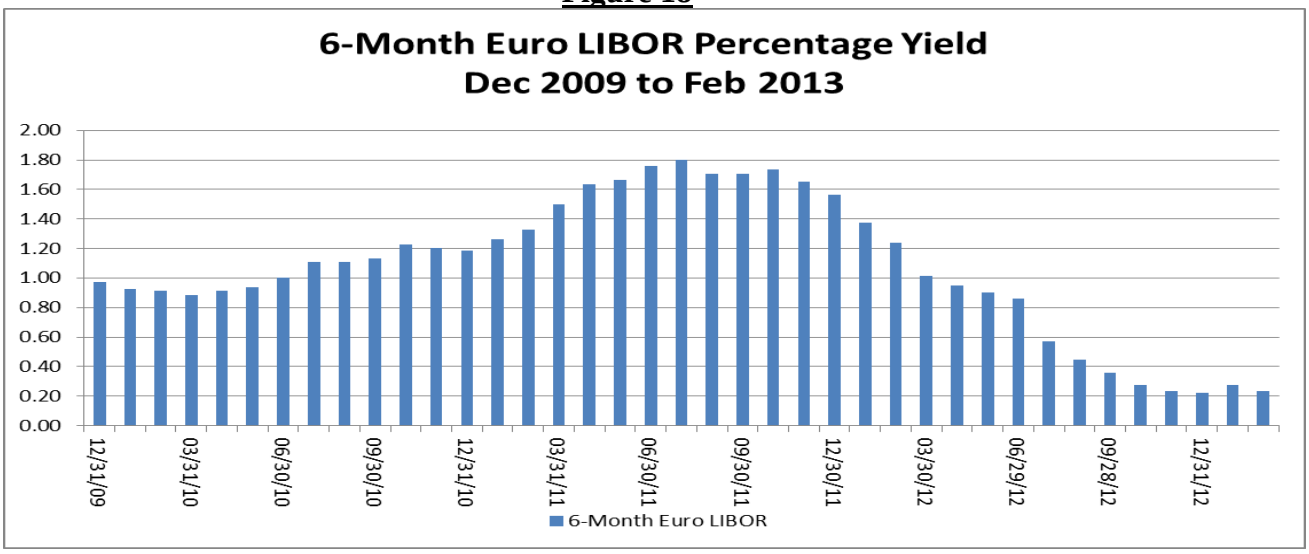


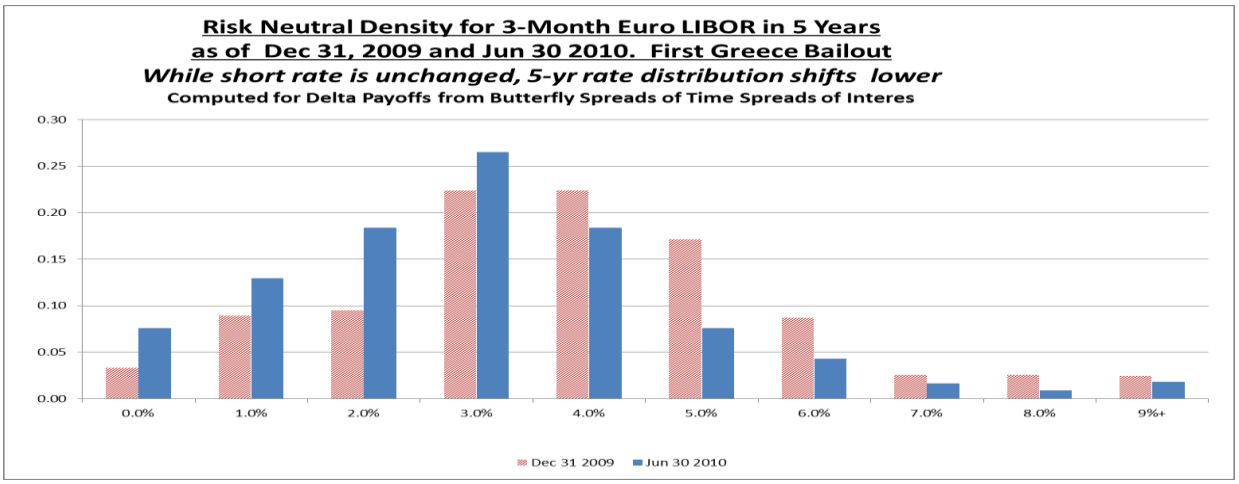
Figure 18



When Greece's budget deficit was restated from 3.7% to 12.7% in January 2010 and major irregularities found, their budget crisis ensued. The first Greece bailout was agreed in April 2010 and expanded in May 2010. While current Euro LIBOR did not decline then (see Figure 18), Figure 19 shows that the market's implied state prices and risk neutral distribution

for 6-month Euro LIBOR in 5 years did shift noticeably to lower yields. While the ECB did not move, markets did, anticipating that the ECB would have to change its stance and lower rates.

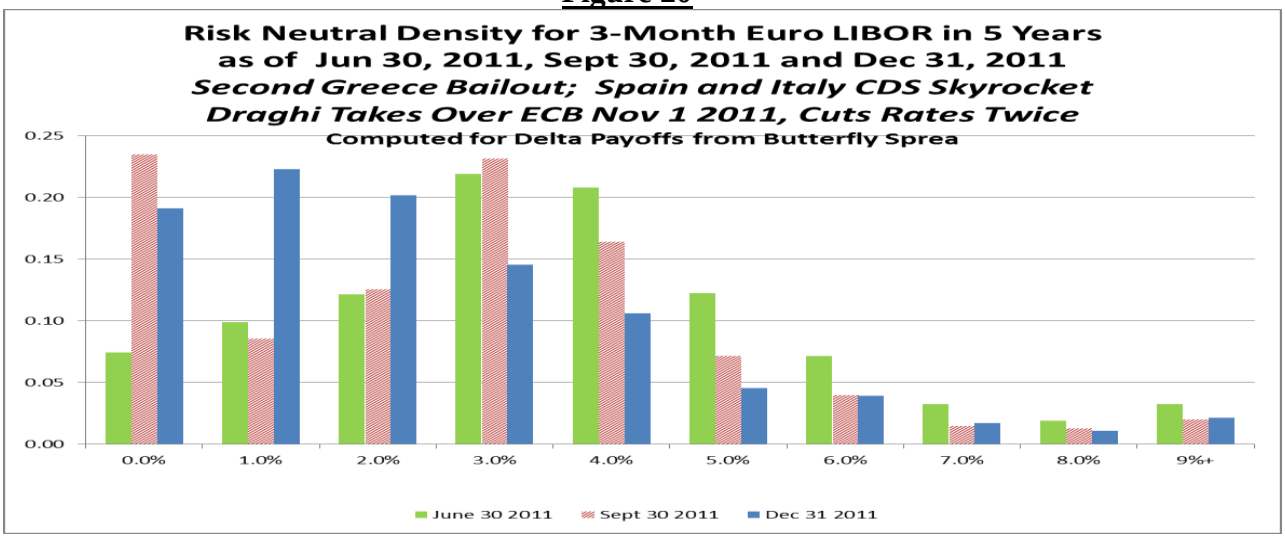
Figure 19



Following the first bailout agreements in April-May 2010, Eurozone stock prices rebounded sharply (15%+) from mid-2010 to the first half of 2011. And then, in July 2011, the worries began to intensify again and there was widespread discussion of a possible Greek exit from the Euro. In August 2011, credit default swap insurance costs jumped for Italy and Spain and their bonds' yields surged, as worries became acute that the debt crisis would spread to these larger countries, which had much greater amounts of debt. Eurozone stock prices plummeted through the summer by more than 25% from early 2011 peaks to the month-end lows on September 30, 2011. On November 1, 2011, Mario Draghi replaced Jean-Claude Trichet as President of the European Central Bank and quickly cut short-term policy rates twice by the end of 2011.

Figure 20 gives the implied state prices/risk neutral densities as they moved through this tumultuous period (which also had the USA's budget impasse in August 2011). Note that the prices of "left tail spreads" labeled as "0% LIBOR" skyrocketed from June 30th to September 30, 2011, illustrating the fear that was in the minds of investors. When Draghi cut rates twice in late 2011, the risk neutral distribution shifts left to lower projected rates in 5 years, as there were much higher values of 1% and 2% butterflies, and much lower values for 3% to 5% butterflies.

Figure 20

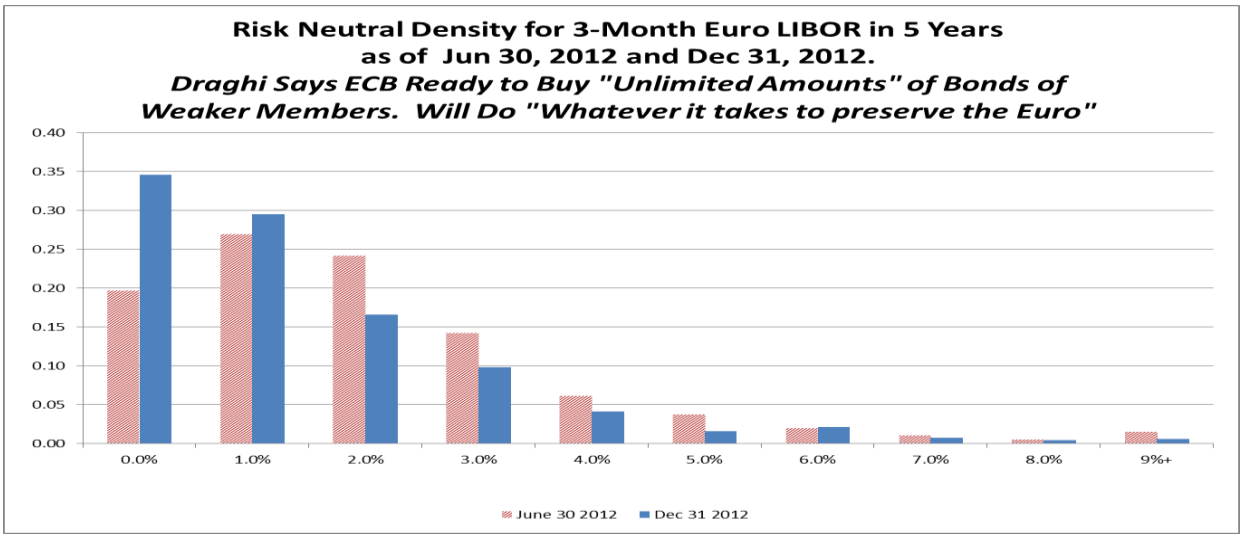


The tonic of lower rates worked initially in the Eurozone, much as it had in the USA, and stock prices rebounded by about 20% from September 30, 2011 to March 31, 2012. But then the fears re-surfaced that the larger economies of Italy and Spain would default on their debts and cause massive write downs for European and global banks. In just two months, from March 31 to May 31, 2012, Eurozone stock prices dropped a sharp 10%.

In July 2012, the ECB again cut rates sharply. And then on September 6, 2012, President Draghi said that the ECB stood ready to buy "unlimited amounts" of bonds of its weaker members. And finally, on September 25, 2012, as Reuters said, Draghi at a London business conference "dropped a bombshell" and surprised almost everyone by stating that: "Within our mandate, the ECB is ready to do whatever it takes to preserve the Euro." Then he went on to say "And believe me, it will be enough."

Figure 21 shows the significant shift in the 5-year Euro LIBOR risk neutral distribution that occurred with these strong moves. The state prices for butterfly spreads paying off for interest rates from 0% to 2% increased substantially, whereas those for rates 3% and higher dropped significantly. Right tail risks for rates, already low, became much lower yet. Stock prices rather steadily marched upward from mid-2012 to early 2013 for the Eurozone. Once again, we can verify that the market's distributions for December 31, 2012 rates were lower than June 30, 2012 in the sense of first order stochastic dominance.

Figure 21



VII. Impact of 2013-2014 Fed Consideration of "Tapering" Asset Purchases

On May 22, 2013, Chairman of the Federal Reserve, Ben Bernanke, testified before Congress that the Fed might "taper" its purchases of Treasury bonds and mortgages, which were running at $85 billion per month, an historic amount. In saying this, Bernanke noted that the labor market had shown steady improvement and the unemployment rate, while still high, had dropped considerably since the program of "quantitative easing" had started. Over the next 6 days, the 10-year Treasury rate increased by 21 basis points, from 1.94% to 2.15%. In testimony a month later, June 19th, the Fed Chairman made even stronger statements, and markets around the world reacted sharply. Bernanke said that the tapering would likely begin "later this year" if growth picks up as the Fed projects and unemployment continues to decline and inflation moves closer to the Fed's 2% target. Bernanke said "In particular, the housing sector, which has been a drag on growth since the crisis, is now obviously a support to growth." Rising home prices are increasing household wealth and strengthening consumer confidence and spending, he noted. (Wall Street Journal, June 20, 2013). Both 3-year and 5-year risk neutral densities for US LIBOR moved sharply, as the 10-year Treasury rate moved 32 basis points from June 18 to June 21, 2013. Figure 23 shows the moves in the 3-year risk neutral distributions, while Figure 23C shows that Europe's risk neutral distribution also moved sharply to higher rates.

Figure 23A

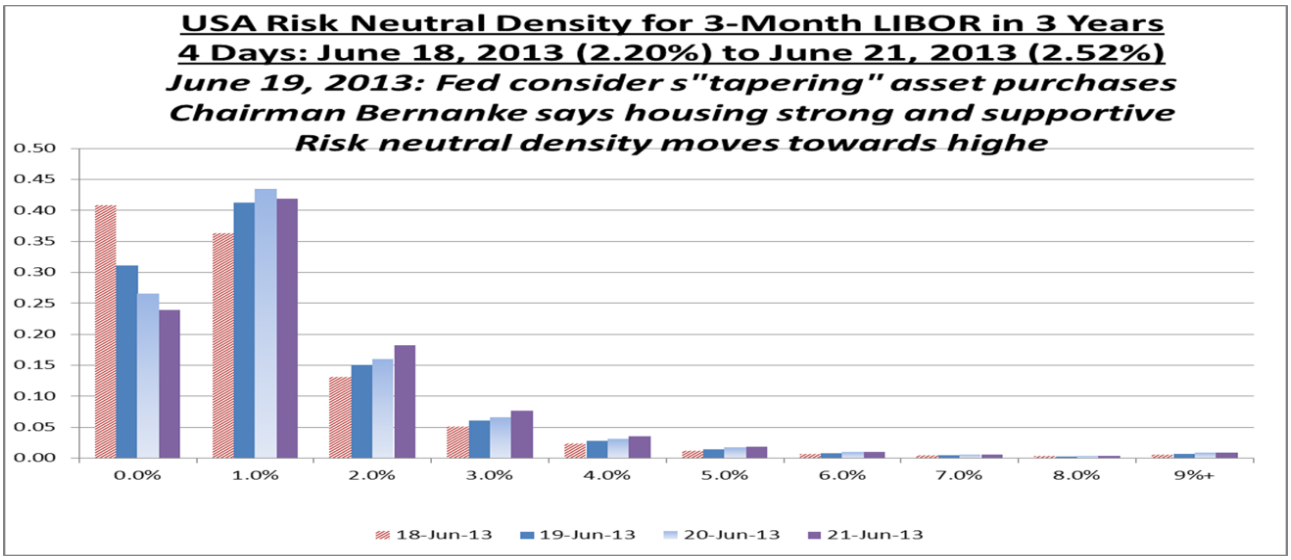
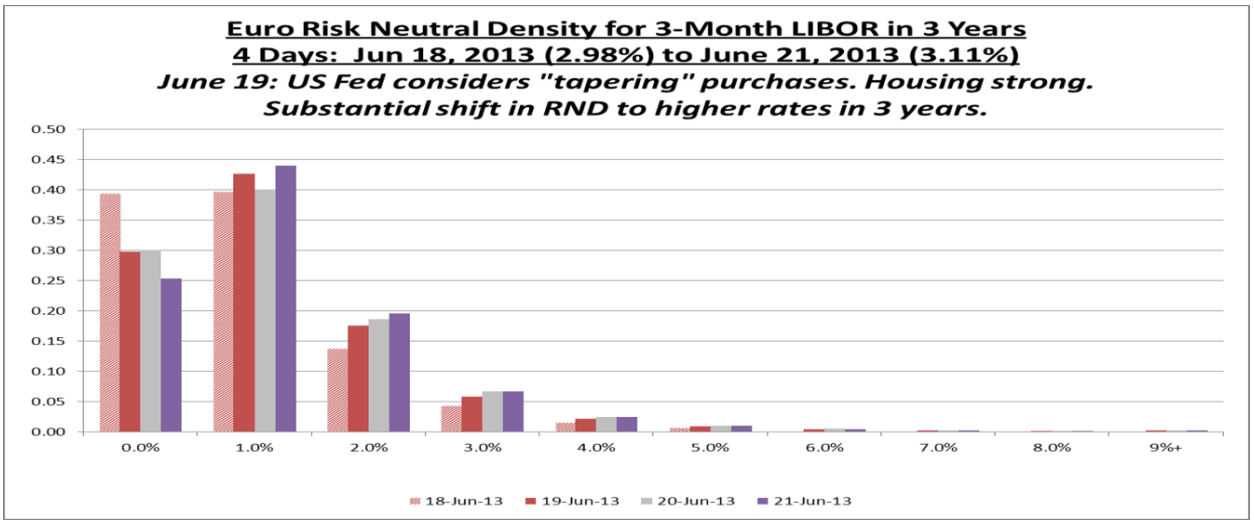


Figure 23



During the summer of 2013, the stock market, after falling sharply following Bernanke's June comments, recovered and moved back towards historic highs (1700+) on the Standard and Poor 500 index. Market participants appeared to believe that tapering would begin with at the Fed's meeting on September 18th, perhaps with a monthly reduction of $10 billion of asset purchases. Interest rates drifted upward and touched 3.00% on the 10-year for the first time since the financial crisis, while the stock market regained its strength and the unemployment rate drifted lower. To see the cumulative impact of the Central Bank communications and market

assessments of the economy, Figure 24 shows the 5-year risk neutral distributions for May 21st (before Bernanke's first mention of tapering) and for September 16th, prior to the Fed's September meeting. Both 3-year and 5-year distributions shifted quite dramatically over the summer to higher rates and to much more spread out and symmetric distributions, yet ones that now possessed visible right tail risk (9%+ LIBOR) for the 5-year forward scenario.

Figure 24

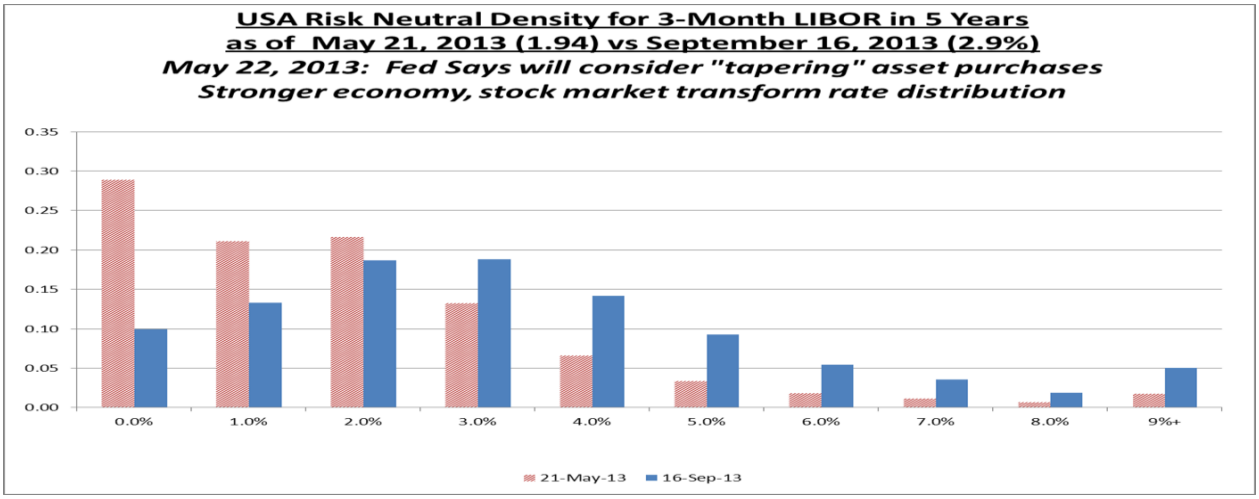
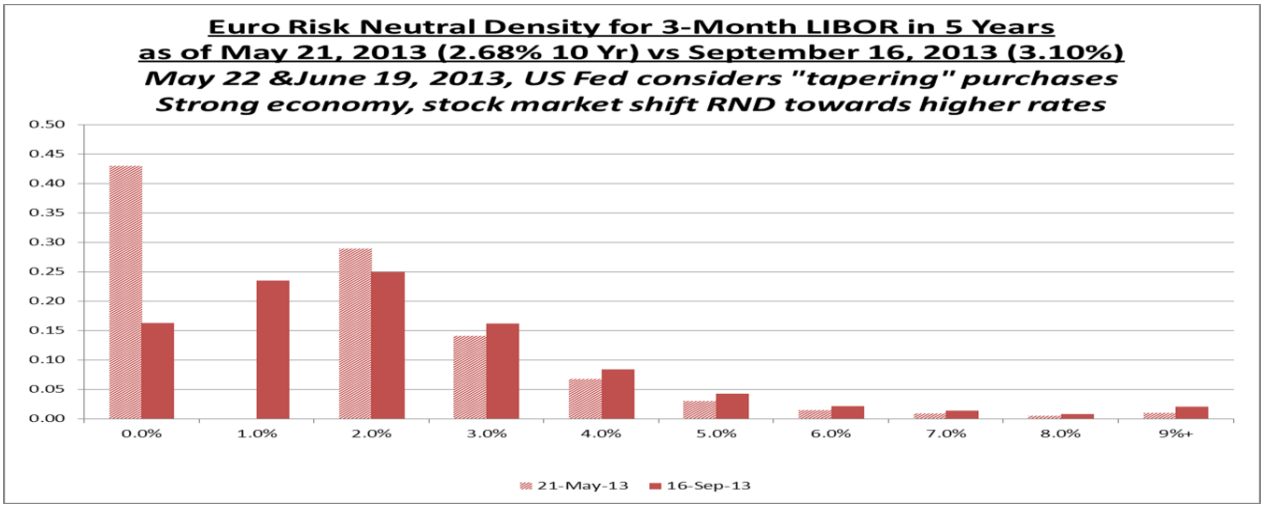


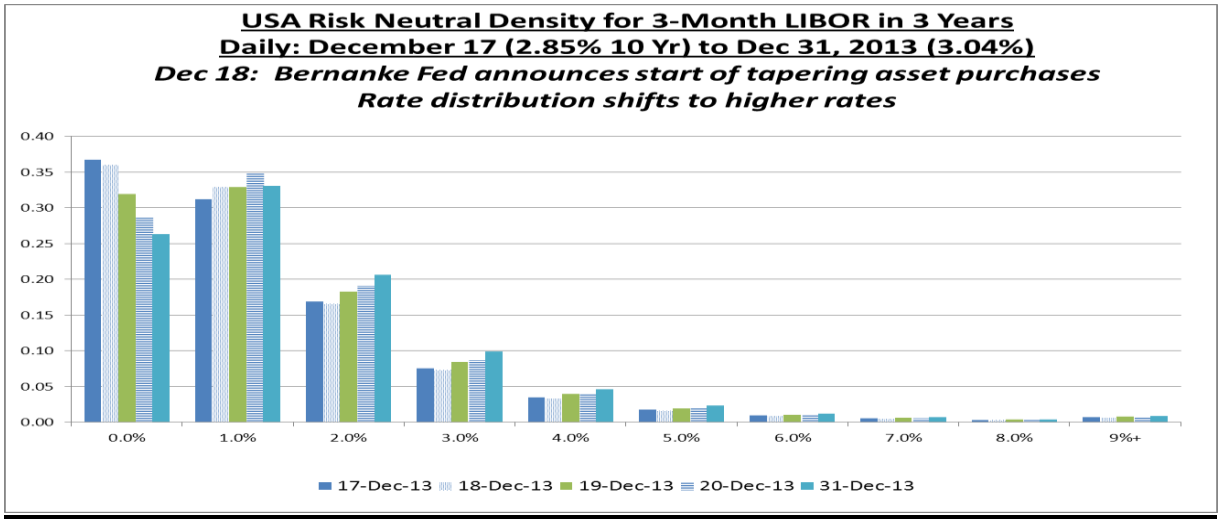
Figure 25 gives the corresponding graphs for the Euro, showing that rate distributions there shifted similarly, although a bit less than in the U.S. From these graphs, right tail risk in Europe was smaller than for the USA on September 16th, perhaps reflecting the market's view that Europe is recovering more slowly than the U.S. at present.

Figure 25

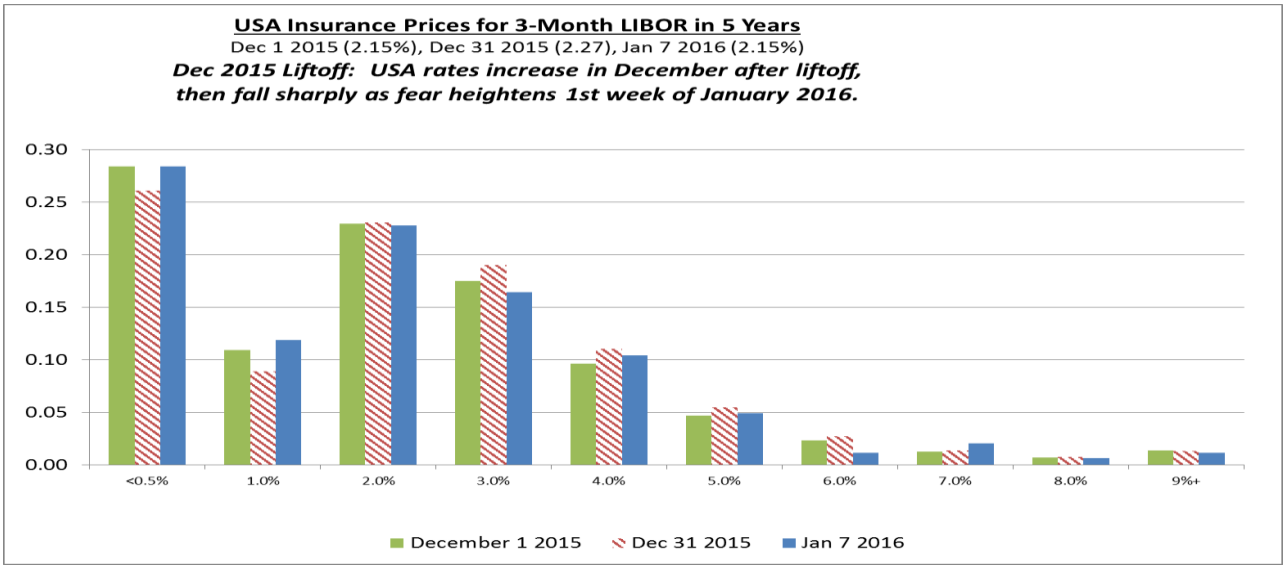


On December 18, 2013, Chairman Bernanke surprised many market participants when he said that the Fed would begin tapering by $10 billion per month in January, citing "improvement in economic activity and labour market conditions…" The market impact was significant, as shown by Figure 27. Rates moved higher, especially noticeably for the 3-year distribution, and more tail risk appears in the 5-year distribution.

Figure 27A



(Add a Yellen December 2015 example of liftoff here.)



VIII. **Conclusions**

While many markets do not have options traded for longer term maturities, like 5 or 10 years, interest rate caps and floors have been traded in size for such maturities for many years. We have shown how butterfly spreads of forward-starting caps and floors can be used to estimate state prices and risk neutral densities for LIBOR projected for the 3rd and 5th years. These risk-neutral densities have shifted from relatively symmetric distributions to highly skewed distributions in the past few years, as interest rates have approached zero, making application of the Black-Scholes option pricing model to short term interest rates particularly problematic.

We have shown that policy actions taken by the Federal Reserve and the European Central Bank can and do affect the entire probability distribution for future interest rates, not just means and variances. The tools we use are wholly arbitrage based and nonparametric, so they do not rely upon assumptions about statistical estimates for the volatility process and surface, nor on the pricing function for actual market prices for options. These tools should be helpful to policy makers and to market participants in measuring policy impacts. The state price distributions that we show are core pricing densities that underly the pricing of interest rate dependent securities and derivatives. Their movements can substantially affect the relative valuation of many such securities.

References

Ait-Sahalia,Yacine and Andrew W. Lo, 1998, Nonparametric estimation of state-price densities implicit in financial asset prices, *Journal of Finance*, April, pp. 499-547.

Ait-Sahalia, Yacine, Yubo Wang and Francis Yared, 2001, Do option markets correctly price the probabilities of movement of the underlying asset? *Journal of Econometrics* 102, 67-110.

Ait-Sahalia,Yacine and Jefferson Duarte, 2003, Nonparametric option pricing under shape restrictions, *Journal of Econometrics* 116, 9-47.

Bahra, Bhupinder, 1996, Probability distributions of future asset prices implied by option prices, *Bank of England Quarterly Bulletin*, August, 299-311.

Bahra, Bhupinder, 1997, Implied risk-neutral probability density functions from option prices: theory and application, Bank of England mimeo.

Banz, Rolf W. and Merton H. Miller, 1978, Prices for state-contingent claims: some estimates and applications. *Journal of Business*, October 51:4: pp. 653-672.

Bates, David S., 2000, Post-'87 crash fears in the S&P 500 futures option market, *Journal of Econometrics* 94, 181-238.

Beber, Alessandro and Michael Brandt, 2006, The effect of macroeconomic news on beliefs and preferences: Evidence from the options market, *Journal of Monetary Economics*, 53, 1997-2039.

Birru, Justin and Stephen Figlewski, 2010, Anatomy of a meltdown: The risk neutral density for the S&P 500 in the fall of 2008, NYU mimeo February.

Birru, Justin and Stephen Figlewski, 2010, The Impact of the Federal Reserve's Interest Rate Target Announcement on Stock Prices: A Closer Look at How the Maket Impounds New Information, NYU mimeo, October.

Black, Fischer and Myron Scholes, 1973, The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, pp. 637-654.

Breeden, Douglas T. and Robert H. Litzenberger, 1978, Prices of State Contingent Claims Implicit in Option Prices, *Journal of Business*, October, 51:4: pp. 621-651.

Breeden, Douglas T., 1979, An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities, *Journal of Financial Economics* 7, September, pp. 265-296.

Breeden, Douglas T., 1986, Consumption, Production, Inflation, and Interest Rates: A Synthesis, *Journal of Financial Economics* 16, May, pp. 3-39.

Brown, Roger and Stephen M. Schaefer, 1993, Interest rate volatility and the term structure of interest rates, *Philosophical Transactions of the Royal Society, Physical Sciences and Engineering* 347, 563-576.

Carr, Peter, 2004, "On the Market Completing Role of Standard and Barrier Options," New York University mimeo.

Carr, Peter and Dilip Madan, 1998, Towards a Theory of Volatility Trading, in R. Jarrow, ed., *Volatility: New Estimation Techniques for Pricing Derivatives*, London: Risk Books, 417-427.

Carr, Peter and Jiming Yu, 2012, Risk, Return and Ross Recovery, *Journal of Derivatives* 20, pp. 38-59.

Clews, Roger,Nikolaos Panigirtzoglou and James Proudman, 2000, Recent developments in extracting information from options markets, *Bank of England Quarterly Bulletin*, February.

Cox, John C. and Stephen A. Ross, 1976, The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics* 3, pp. 145-166.

Cox, John C., Stephen A. Ross and Mark E. Rubinstein, 1979, Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, pp. 229-263.

Duffie, Darrell and Chi-Fu Huang, 1986, Multiperiod security markets with differential information, *Journal of Mathematical Economics* 15, 283-303.

Duffie, Darrell, Jun Pan, and Kenneth Singleton, 2000, Transform analysis and asset pricing for affine jump-diffusions, *Econometrica* 68, No. 6, 1343-1376.

Durham, J. Benson, 2007, Implied interest rate skew, term premiums, and the "Conundrum," Federal Reserve Board Discussion Series paper no. 55, Washington, D.C.

Figlewski, Stephen, 2008, Estimating the implied risk neutral density for the U.S. market portfolio, in Volatility and Time Series Econometrics: Essays in Honor of Robert F. Engle (Tim Bollerslev, Jeffrey R. Russell and Mark Watston, eds.). Oxford University Press, 2008.

Grauer, Frederick L. A. and Robert H. Litzenberger, 1979, The Pricing of Nominal Bonds and Commodity Futures Contracts Under Uncertainty, *Journal of Finance*, March, pp. 69-84.

Harvey, Campbell R., 1991, The Term Structure and World Economic Growth, *Journal of Fixed Income*, June.

Ilinski, Kirill, 2001, Finding the Basket, *Wilmott Magazine*, August.

Jackwerth, Jens C. and Mark Rubinstein, 1996, Recovering probability distributions from contemporary security prices, *Journal of Finance* 51, pp. 1611-1631.

Kitsul,Yuriy and Jonathan Wright, 2012, The Economics of Options-Implied Probability Density Functions, NBER Working Paper 18195, June.

Kocherlakota, Narayana, 2013, Federal Reserve Bank of Minneapolis website on Risk Neutral Density functions. Biweekly report and Methodology tabs.

Li, Haitao and Feng Zhao, 2006, Nonparametric estimation of state-price densities using interest rate options, University of Michigan mimeo.

Longstaff, Francis, 1995, Option pricing and the martingale restriction, *Review of Financial Studies* 8, pp. 1091-1124.

Longstaff, Frances A., Pedro Santa-Clara and Eduardo S. Schwartz, 2001, The relative valuation of interest rate caps and swaptions: Theory and empirical evidence, *Journal of Finance* 56:6, 2067-2110, December.

Malz, Allan M., 1995, Using option prices to estimate realignment probabilities in the European Monetary System, Federal Reserve Bank of New York Staff Reports No. 5, September.

Malz, Allan M., 1997, Option implied probability distributions and currency excess returns, Federal Reserve Bank of New York Staff Reports No. 32, November.

Martin, Ian, 2013, Simple Variance Swaps, Stanford University GSB mimeo, January.

Merton, Robert C., 1973, Theory of Rational Option Pricing, *The Bell Journal of Economics and Management Science* 4, pp: 141-183.

Ross, Stephen A., 1976, Options and Efficiency, *Quarterly Journal of Economics* 90, pp. 75-89.

Ross, Stephen A., 2012, The Recovery Theorem, forthcoming *Journal of Finance*.

Rubinstein, Mark, 1981, A discrete-time synthesis of financial theory, *Research in Finance*, Volume 3.

Rubinstein, Mark, 1994, Implied binomial trees, *Journal of Finance* 49, 771-818.

Shimko, David, 1993, Bounds of probability, *Risk*, Vol. 6, No. 4, 33-37.

Smith, Tom, 2012 Q3, Option-implied Probability Distributions for Future Inflation, *Bank of England Quarterly Bulletin.*

Trolle, Anders B. and Eduardo S. Schwartz, 2012, The swaption cube, Swiss Finance Institute, UCLA and NBER mimeo, July.

Zitzewitz, Eric, 2009, Quantifying the Nightmare Scenarios, *Freakonomics* blog, March 2.