No-Arbitrage Taylor Rules with Switching Regimes

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ABSTRACT

We develop a continuous-time regime-switching model for the term structure of interest rates, in which the spot rate follows the Taylor rule, and government bonds at different maturities are priced by no-arbitrage. We allow the coefficients of the Taylor rule and the dynamics of inflation and output gap to be regime-dependent. We estimate the model using government bond yields and find that the Fed is proactive in controlling inflation in one regime but is accommodative for growth in another. Our model significantly improves the explanatory power of macroeconomic variables for government bond yields. Without the regimes, inflation and output can explain less than 50% of the variations of *contemporaneous* bond yields. With the regimes, the two variables can explain more than 80% of the variations of *contemporaneous* bond yields. Proactive monetary policies are associated with more stable inflation and output gap and therefore could have contributed to the "Great Moderation."

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1 Introduction

Dynamic term structure models (hereafter DTSMs), such as the affine term structure models (ATSMs) of Duffie and Kan (1996) and Dai and Singleton (2000) and the quadratic term structure models (QTSMs) of Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2003), are among the most popular term structure models in the academic literature.[1] DTSMs assume that a finite number of latent state variables drive the evolution of the entire yield curve. DTSMs are analytically tractable and at the same time can capture many stylized features of bond yield dynamics. One fundamental issue with DTSMs, however, is that they do not clearly specify the economic nature of the latent state variables. Although the latent variables have been interpreted as the level, slope, or curvature of the yield curve, they have not been explicitly tied down to fundamental macroeconomic variables.

In recent years, a fast growing literature that explicitly relates term structure dynamics to fundamental macroeconomic variables has been developed.[2] In one of the pioneering papers in this literature, Ang and Piazzesi (2003) introduce the Taylor rule into traditional arbitrage-free affine models. Taylor (1993) proposes that US monetary policy in 1980s and early 1990s can be described by an interest-rate feedback rule of the form

$$r_t = 0.04 + 1.5\,(\pi_t - 0.02) + 0.5\,g_t, \tag{1}$$

where r_t denotes the Fed's operating target for the federal funds rate, π_t is the inflation rate, and g_t is the output gap. By relating the spot rate to macro variables through the Taylor rule and pricing bonds at different maturities via no-arbitrage, this approach conveniently introduces the macro variables into term structure models.[3]

While the no-arbitrage Taylor rule approach provides a convenient way to relate the term structure dynamics explicitly to fundamental macro variables, there is still a large fraction of the variations of bond yields that cannot be explained by inflation and output gap. For example, simple regressions of government bond yields on *contemporaneous* inflation and output gap show that the two macro variables explain less than 50% of the variations of bond yields, and the explanatory power declines monotonically with maturity. Latent variables, interpreted as monetary policy shocks, are still needed to capture fully the variations of bond yields.

While most studies in this literature assume a fixed Taylor rule, there is ample evidence that monetary policies in the US have experienced dramatic changes during the last half century. For example, Clarida, Gali, and Gertler (2000) find substantial differences in the estimated Taylor

[1]See Piazzesi (2006) and Dai and Singleton (2002) for comprehensive surveys on the huge literature on DTSMs.

[2]See Diebold, Piazzesi, and Rudebush (2005) and Rudebush (2009) for surveys on this literature.

[3]An alternative approach is to analyze term structure dynamics in a New-Keynesian general equilibrium model as in Bekaert, Cho, and Moreno (2010).

rule for the postwar US economy before and after Volcker's appointment as Fed Chairman in 1979. They find that the Taylor rule in the Volcker-Greenspan period is more responsive to changes in inflation than in the pre-Volcker period and argue that "the Volcker-Greenspan rule is stabilizing." Bernanke (2004) also attributes the "Great Moderation," the substantial decline in macroeconomic volatility in the US since early 1980s, at least partly to the improved monetary policies during this time.

Time-varying monetary policies could significantly affect both the short and long end of the yield curve because (i) the spot rate is determined by inflation and output gap through the Taylor rule, and (ii) long bond yields are determined by expectations of the future spot rate adjusted for risk premium. Moreover, since there are likely to be more changes in monetary policies over a longer horizon, the effects could be more severe for long bond yields. As evidenced by the "Great Moderation," different monetary policies could also lead to different dynamics of the macro variables, which in turn could affect the dynamics of bond yields. Therefore, a model that ignores the changing monetary policies are likely to be misspecified and could lead to misleading conclusions about the relation between bond yields and macro variables.

In this paper, we develop a new macro term structure model that explicitly captures the changing nature of monetary policies in the US. Instead of assuming a structural break in monetary policies before and after Volcker, we model monetary policy changes using a regime-switching model, in which the coefficients of the Taylor rule and the dynamics of the macro variables behave differently in different economic regimes. Our regime-switching model parsimoniously captures the Fed's dual objectives in controlling inflation and stimulating growth. It allows the possibility that the Fed might react differently to changes in inflation or output gap depending on the urgency of achieving either objective at a given time. This could lead to different coefficients in the Taylor rule and different dynamics of the macro variables. Whereas the structural-break view suggests that changing monetary policies are more important in understanding historical rather than future bond yields, the regime-switching approach suggests that monetary policies will keep changing in the future as the Fed constantly re-balances its emphasis on inflation or growth. Therefore, changing monetary policies will remain important in modeling the term structure in the future under our approach.

Specifically, we develop a continuous-time regime-switching model for the term structure of interest rates with the regime-dependent Taylor rule and macro variables. We develop a closed-form approximation of bond price under the regime-switching model, which dramatically simplifies the empirical implementations of the model. We estimate our model using Bayesian Markov Chain Monte Carlo (MCMC) methods based on quarterly observations of US Treasury yields, inflation and output gap between 1952 and 2007. In our empirical analysis, we consider three models with different levels of complexity. In the first model, M1, we consider a single-regime model that serves as a benchmark model for the two-regime models we consider. In the second model M2, we consider a two-regime model in which the coefficients of the Taylor rule are regime-dependent while the dynamics of inflation and output gap remain the same in both regimes. This model illustrates the contribution of changing monetary policies on yield dynamics. In the third model M3, we allow both the monetary policies and the dynamics of inflation and output gap to be regime-dependent. The last model incorporates the potential effects of monetary policies on the dynamics of the macro variables.

Our empirical results on M1 show that in a single-regime model, inflation and output gap can explain only less than 50% of the variations of *contemporaneous* bond yields. In addition, the explanatory power declines monotonically over maturity. The estimate of the coefficient of inflation in the Taylor rule is about 0.9. This value violates the so-called "Taylor principle," which requires the coefficient of inflation in the Taylor rule to be above 1.0 to ensure the stability of the economy.

Our empirical results on M2 show that regimes in monetary policies significantly improve the explanatory power of inflation and output gap for bond yields: the R^2s of regressions of the observed yields on model yields increase from less than 50% under M1 to above 80% across all maturities under M2, and there is no obvious dependence of the R^2s under M2 on maturities. Our parameter estimates identify two distinct monetary policy regimes. In one regime, the Fed is proactive in controlling inflation with the estimated coefficient of inflation in the Taylor rule close to 1.5, which satisfies the Taylor principle. In another regime, the Fed is more accommodative for growth with the estimated inflation coefficient close to 0.85.

The excellent performance of M2, especially the high R^2s, is an important result that is significantly different from that of most existing regime-switching models on the term structure. In typical regime-switching models, the dynamics of the state variables (e.g., inflation and output gap in our model) are regime-dependent. Since most financial time series exhibit fat-tailed distribution and volatility clustering, one can almost always improve model performance by introducing regimes in state variables. However, the superior performance of M2 is achieved even we restrict inflation and output gap to follow the same process under both regimes. This shows that the key to the model improvement comes from regime-dependent monetary policies.

Our estimates of M3 lead to similar monetary policies as those in M2. However, inflation and output gap behave very differently under the two monetary policy regimes. Under the proactive regime, inflation has low mean and volatility, strong mean reversion, weak feedback from output gap to inflation, and output gap has strong mean reversion and low volatility. Under the accommodative regime, inflation has high mean and volatility, weak mean reversion, strong feedback from output gap to inflation, and output gap has weak mean reversion and high volatility. These results suggest that proactive monetary policies are associated with a more stable economy with steady inflation and economic growth. On the other hand, the R^2s of regression of the observed yields on model yields remain close to those under M2. Again, this suggests that the main explanatory power for bond yields comes from regime-dependent monetary policies rather than from regime-dependent macro variables.

In summary, our regime-switching model significantly improves the explanatory power of macro variables for government bond yields. Without the regimes, inflation and output can explain about 50% of the variations of *contemporaneous* bond yields. With the regimes, the two variables can explain more than 80% of the variations of *contemporaneous* bond yields. Our model shows that a majority of the variations in bond yields can be traced back to fundamental economic variables under a realistic model of the monetary policies.

Since the pioneering work of Hamilton (1989), a huge literature has been developed in economics and finance to model the interest rates using regime-switching models. Many studies, such as Gray (1996), Garcia and Perron (1996), Bekaert, Hodrick and Marshall (2001), Ang and Bekaert (2002a,b), Hong, Li, and Zhao (2005), and Li and Xu (2010) among others, focus on modeling the spot rates using regime-switching models. Regime-switching term structure models, such as that of Naik and Lee (1994), Bansal and Zhou (2002), Bansal, Tauchen and Zhou (2004), and Dai, Singleton, and Yang (2006) among others, are still under the framework of DTSMs. While Ang, Bekaert, and Wei (2008) incorporate macro variables in their regime-switching term structure model, they mainly focus on the term structure of real rates and expected inflation. Our paper extends the literature by explicitly modeling regime-dependent monetary policies and macro variables and relating nominal bond yields to economic fundamentals. Although Sims and Zha (2006) argue against regime switches in the US monetary policy, their empirical analysis uses only the short rate. Our analysis shows that term structure data contains rich information on potential switches in monetary policy; therefore, the results based on the short rate might not be robust.

Our paper complements that of Ang, Boivin, Dong, and Loo-Kung (2009) (hereafter ABDL), who demonstrate the importance of time varying monetary policies for term structure modeling. While ABDL model the coefficients of the Taylor rule as continuous-time stochastic processes, we allow the coefficients to follow only two distinctive regimes. To the extent that there are costs involved in and delayed reactions to the changing monetary policies, a regime-switching model could be a reasonable parsimonious way of capturing the Fed's policy decisions. Our paper further strengthens the case of time-varying monetary policies by showing that even with only two regimes in the Taylor rule, inflation and output gap can explain more than 80% of the variations of *contemporaneous* bond yields.

In another closely related paper, Bikbov and Chernov (2008) also study monetary policy regimes and the term structure of interest rates. Our paper differs from Bikbov and Chernov

(2008) in terms of research objective, modelling technique, and empirical conclusion. While the main objective of Bikbov and Chernov (2008) is to infer monetary policies from term structure data, our goal is to develop a term structure model with a realistic monetary policy to link macro variables to bond yields explicitly. Given their focus on monetary policy, Bikbov and Chernov (2008) rely heavily on the structural VAR approach of the rational expectations literature. In contrast, we develop a continuous-time regime-switching model, which is a simple extension of the existing models and is very convenient for bond pricing. Finally, while Bikbov and Chernov (2008) document different regimes in monetary policy, they do not explicitly study the importance of policy regimes for bond pricing. On the other hand, we show that monetary policy regimes significantly improve the explanatory power of macro variables for bond yields. Moreover, monetary policy regimes identified from our model have clear economic interpretations and match the history of the US economy in the past few decades very well. Our model also has interesting implications for the US monetary policy before the current global financial crisis.

The rest of this paper is organized as follows. In section 2, we develop a continuous-time regime-switching model for the term structure of interest rates with no-arbitrage Taylor rule. Section 3 develops MCMC methods for estimating and comparing the regime-switching term structure models. Section 4 discusses the data and empirical results. Section 5 concludes, and the appendix provides the technical details.

2 Regime-Switching Term Structure Model with Taylor Rules

In this section, we introduce our continuous-time regime-switching model on the term structure of interest rates with no-arbitrage Taylor rule. We first briefly review the traditional single-regime model, which serves as a benchmark for the regime-switching model. We then introduce our regime-switching model and develop a closed-form approximation to zero-coupon bond price under the model. Finally, we discuss the explicit two-regime model used in our empirical analysis.

2.1 No-Arbitrage Taylor Rule without Regimes

In the following single-regime model, the spot rate is governed by the Taylor rule:

$$r(t) = \delta_0 + \delta^\top z(t), \tag{2}$$

where $r(t)$ is the spot rate, $\delta = (\delta_\pi, \delta_g)^\top$, $z(t) = (\pi(t), g(t))^\top$, $\pi(t)$ represents inflation, and $g(t)$ represents output gap.

In the absence of arbitrage, there exists a risk-neutral measure $\mathbb{Q}$ such that the time-t price of a zero coupon bond that matures at T equals

$$P(t, T, z(t)) = E_t^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)\, ds\right)\right].$$

Therefore, by pricing government bonds at different maturities via no-arbitrage, this model relates inflation and output gap to the zero-coupon bond yields.

To obtain closed-form solutions to the zero-coupon bond prices, we assume $z(t)$ follows affine diffusions under the risk-neutral measure $\mathbb{Q}$:

$$\begin{aligned} dz(t) &= \kappa\left(\theta - z\left(t\right)\right)dt + \Sigma\, dW^{\mathbb{Q}}\left(t\right) \\ &= \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \begin{pmatrix} \theta_1 - \pi(t) \\ \theta_2 - g(t) \end{pmatrix} dt + \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix} dW^{\mathbb{Q}}\left(t\right), \end{aligned} \tag{3}$$

where $\theta = (\theta_1, \theta_2)^\top$ represents the long-run mean of $(\pi\left(t\right), g\left(t\right))$, κ represents the speed of mean reversion matrix, Σ represents the instantaneous covariance matrix of $(\pi\left(t\right), g\left(t\right))$, and $W^{\mathbb{Q}}(t)$ is a two-dimensional Brownian motion under $\mathbb{Q}$-measure. We choose Σ to be a lower-triangular matrix since we do not have enough information to identify each term of the matrix fully. We could equivalently choose Σ to be a lower triangular. We also adopt the following affine specification of the market price of risk to guarantee that $z\left(t\right)$ is affine under the physical measure $\mathbb{P}$ as well:

$$\lambda(t) = \Sigma^{-1}\left\{ \begin{pmatrix} \lambda_{10} \\ \lambda_{20} \end{pmatrix} + \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} \begin{pmatrix} \pi(t) \\ g(t) \end{pmatrix} \right\} = \Sigma^{-1}(\lambda_0 + \lambda z(t)). \tag{4}$$

Both the spot rate and the state variables follow affine specifications, and thus the zero-coupon bond price is given by

$$P\left(t, \tau, z(t)\right) = \exp\left(A(\tau) + B^\top(\tau) z\left(t\right)\right),$$

where $\tau = T - t$, $A\left(\tau\right)$ and $B\left(\tau\right)$ solve a system of ordinary differential equations (ODEs),

$$\begin{aligned} &-A' + \theta^\top \kappa^\top B + \frac{1}{2}\textbf{Trace}\left(\Sigma^\top B B^\top \Sigma\right) - \delta_0 = 0, \\ &-B' - \kappa^\top B - \delta = 0. \end{aligned}$$

Under certain model specifications, the ODEs have a closed-form solution. Otherwise, they can be easily solved numerically. For the rest of the paper, we refer to the model with a Taylor rule in (2), a state variable in (3), and a market price of risk specification in (4) as M1. This is the first model considered in our empirical analysis and serves as a benchmark for the two new regime-switching models.

2.2 No-Arbitrage Taylor Rule with Switching Regimes

In this section, we introduce a regime-switching model for the term structure of interest rates with no-arbitrage Taylor rule. In the most general version of the model, we allow both monetary policies (i.e., the coefficients of the Taylor rule) and the dynamics of the macro variables to be regime-dependent. We obtain a closed-form approximation of the zero-coupon bond prices under the regime-switching model, which greatly simplifies its empirical implementation.

Let $s_t \in \{1, \cdots, N\}$ be a discrete state variable that represents the regime that the economy is in. We assume that the regime variable s_t follows a continuous-time Markov chain with a transition matrix under the $\mathbb{Q}$ measure,

$$Q(t) = \{q_{ij}(t)\}_{i,\, j=1,\cdots,N},$$

where $q_{ij}(t) > 0$ for all $j \neq i$, $q_{ii}(t) < 0$ and $\sum_{j=1}^{N} q_{ij}(t) = 0$ for all i. Suppose $s(t) = i$, then, intuitively, over a small time interval Δ, the probability that the economy remains in regime i equals $1 - q_{ii}(t)\,\Delta$, and the probability that the economy switches from regime i to regime j equals $q_{ij}(t)\,\Delta$.

In our model, we allow both the Taylor rule and the macro variables to be regime-dependent. Suppose $s_t = i$, then the Taylor rule is given as

$$r_i(t) = \delta_{i0} + \delta_i^{\top} z_i(t),$$

and the dynamics of the macro variables are given as

$$dz_i(t) = \mu_i(t, z_i(t))\, dt + \Sigma_i(t, z_i(t))\, dW^{\mathbb{Q}}(t),$$

where $W^{\mathbb{Q}}(t)$ is a two-dimensional Brownian motion under $\mathbb{Q}$, μ_i is a two-dimensional vector, and Σ_i is a 2×2 matrix for all $i = 1, 2, \cdots, N$.[4]

Given the above assumptions, zero-coupon bond prices should depend on the regime variable s_t and the macro variables in all the regimes, i.e., $P_{s_t}(z_1(t), z_2(t), ..., z_N(t))$. Consequently, the evolution of bond price should depend on the evolution of both the regime variable and the macro variables. Suppose that $s_{t_-} = i$ at time t_-, then under $\mathbb{Q}$,

$$dP_{s_t}(t, T) = dP_{s_{t-}}(t, T) + P_{s_t}(t, T) - P_{s_{t-}}(t, T),$$

where intuitively $dP_{s_{t-}}(t, T)$ represents the continuous evolution of bond price if the economy remains in regime i, and $P_{s_t}(t, T) - P_{s_{t-}}(t, T)$ represents the discrete jump in the bond price when the economy switches to a new regime.

An application of Itô's Lemma leads to the following expression for the continuous evolution of bond price

$$dP_i(t,T) = \left(\frac{\partial P_i(t,T)}{\partial t} + \mathcal{A} P_i(t,T) \right) dt + \sum_{i=1}^{N} \left(\frac{\partial P_i(t,T)}{\partial z_i} \right)^{\top} \Sigma_i(t, z_i)\, dW^{\mathbb{Q}}(t),$$

therefore,

$$E_t^{\mathbb{Q}}\left[dP_i(t,T)\right] = \left(\frac{\partial P_i(t,T)}{\partial t} + \mathcal{A} P_i(t,T) \right) dt, \tag{5}$$

[4]We restrict $z(t)$ to be two-dimensional because the standard Taylor rule only includes inflation and output gap. We can easily incorporate other macro variables that can be important for bond pricing into $z(t)$ in the most general setup of our model.

where $\mathcal{A}$ is the infinitesimal generator given by

$$\mathcal{A} = \sum_{i=1}^{N} \mu_i^\top(t, z_i)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \textbf{Trace}\left(\Sigma_i(t, z_i)\Sigma_j^\top(t, z_j)\frac{\partial^2}{\partial z_i \partial z_j^\top}\right).$$

By the properties of the Markov chain and the fact $q_{ii}(t) = -\sum_{j\neq i} q_{ij}(t)$, the expected change in bond price when the economy switches from regime i to a new regime j equals:

$$\begin{aligned} E_t^{\mathbb{Q}}\left[P_{s_t}(t,T) - P_{s_{t-}}(t,T)\right] &= \sum_{j\neq i}^{N} q_{ij}(t)[P_j(t,T) - P_i(t,T)]\,dt \\ &= \sum_{j=1}^{N} q_{ij}(t)P_j(t,T)\,dt. \end{aligned} \tag{6}$$

As $E_t^{\mathbb{Q}}\left[\frac{dP_{s_t}(t,T)}{P_{s_{t-}}(t,T)}\right] = r_{s_{t-}}(t)\,dt$, combining equations (5) and (6), we obtain the following PDE that $P_i(t,T)$ must satisfy

$$\frac{\partial P_i(t,T)}{\partial t} + \mathcal{A}P_i(t,T) + \sum_{j=1}^{N} q_{ij}(t)P_j(t,T) - r_i(t)\,P_i(t,T) = 0. \tag{7}$$

The above PDE differs from the standard pricing PDE by the term $\sum_{j=1}^{N} q_{ij}(t)P_j(t,T)$, which represents the discrete jump in bond prices due to the changes in the regime variable s_t. Under the special case where $q_{ij}(t) = 0$ for all j and t (meaning that the economy will never leave the current regime i), the above PDE simplifies to the standard pricing PDE.

Let $P(t,T)$ be an N-dimensional vector whose ith element, $P_i(t,T)$, is the zero coupon bond price when $s_t = i$ and $R(t)$ a diagonal matrix whose ith element, $r_i(t)$, is the spot rate when $s_t = i$. Then we obtain the following system of PDEs that bond price in each regime must satisfy

$$\frac{\partial P(t,T)}{\partial t} + \mathcal{A}P(t,T) + [Q(t) - R(t)]P(t,T) = 0. \tag{8}$$

Therefore, we have generalized the fundamental pricing PDE for single-regime models to multi-regime models.

One challenge we face is that obtaining a closed-form solution for the zero-coupon bond price is difficult even under the affine specifications due to the extra term $\sum_{j=1}^{N} q_{ij}(t)P_j(t,T)$. We develop a closed-form approximation to zero-coupon price under some simplifying assumptions for the regime-switching model. We will show in the appendix that the approximation is very accurate in our empirical applications.

While in the most general case, both the transition matrix $Q(t)$ and the zero-coupon bond price $P_i(t,T)$ can depend on the macro variables in all regimes, we make the simplifying assumption that $Q(t)$ is a constant matrix and $P_i(t,T)$ is exponentially affine in $z_i(t)$

$$P_i(t,T,z_i(t)) \equiv e^{A_i(\tau)+B_i^\top(\tau)z_i(t)}.$$

We also assume that the macro variables $z_i\,(t)$ follow affine diffusions with the following drift and volatility terms

$$\mu_i(t, z_i) = \kappa_i(\theta_i - z_i), \text{and } \Sigma_i = \text{constant matrix.}$$

These simplified assumptions allow us to obtain a closed-form approximation to the pricing PDE for the regime-switching model. Specifically, substituting $\frac{\partial P_i(t,T)}{\partial t}$, $\frac{\partial P_i(t,T)}{\partial z_i}$, and $\frac{\partial^2 P_i(t,T)}{\partial z_i \partial z_i^\top}$ into the PDE, we have

$$-P_i[A_i' + (B_i')^\top z_i] + P_i(\theta_i - z_i)^\top \kappa_i^\top B_i + \frac{1}{2} P_i \mathbf{Trace}\left(\Sigma_i^\top B_i B_i^\top \Sigma_i\right) + \sum_{j=1}^{N} q_{ij} P_j - P_i(\delta_{i0} + \delta_i^\top z_i) = 0.$$

After further simplification, we have

$$\left(-A_i' + \theta_i^\top \kappa_i^\top B_i + \frac{1}{2}\mathbf{Trace}\left(\Sigma_i^\top B_i B_i^\top \Sigma_i\right) + q_{ii} - \delta_{i0}\right) + \sum_{j \neq i} q_{ij} e^{A_j - A_i + B_j^\top z_j - B_i^\top z_i} + z_i^\top \left(-B_i' - \kappa_i^\top B_i - \delta_i\right) = 0.$$

Taking a first-order approximation and letting $z_j = z_i = z$, we have[5]

$$e^{(B_j - B_i)^\top z} \approx 1 + z^\top (B_j - B_i).$$

Thus, the PDE becomes

$$\left(-A_i' + \theta_i^\top \kappa_i^\top B_i + \frac{1}{2}\mathbf{Trace}\left(\Sigma_i^\top B_i B_i^\top \Sigma_i\right) + q_{ii} - \delta_{i0}\right) + \sum_{j \neq i} q_{ij} e^{A_j - A_i} \left\{1 + z^\top (B_j - B_i)\right\} + z^\top \left(-B_i' - \kappa_i^\top B_i - \delta_i\right) = 0.$$

Since the PDE has to hold for all values of z, it can be reduced to a system of ODEs:

$$-A_i' + \theta_i^\top \kappa_i^\top B_i + \frac{1}{2}\mathbf{Trace}\left(\Sigma_i^\top B_i B_i^\top \Sigma_i\right) + q_{ii} - \delta_{i0} + \sum_{j \neq i} q_{ij} e^{A_j - A_i} = 0$$

$$-B_i' - \kappa_i^\top B_i + \sum_{j \neq i} q_{ij} e^{A_j - A_i}[B_j - B_i] - \delta_i = 0,$$

where A_i is a scalar, B_i is a two-dimensional vector for $i = 1, ..., N$. Thus, this is a system of $(2+1) \times N$ ODEs with initial conditions $A_i(0) = 0$ and $B_i(0) = 0$. We obtain the zero-coupon bond prices by solving the above system of ODEs numerically.

[5] The linear approximation approach has been used in other studies, such as Bansal and Zhou (2002). The main difference is that we consider a continuous-time model that explicitly models macro variables.

2.3 An Explicit Two-Regime Model

While the above section develops a general N-regime model, in this section, we consider an explicit two-regime model that will be later considered in our empirical analysis. We also consider models with more than two regimes but find that they do not significantly outperform the two-regime model.

We first assume a regime-dependent Taylor rule. That is, in regime $i = \{1, 2\}$,

$$r_i(t) = \delta_{i0} + \delta_{i\pi}\pi(t) + \delta_{ig}g(t). \tag{9}$$

The dynamics of the state variables are regime-dependent as well. In regime i, $z_i(t)$ follows

$$\begin{aligned} dz_i(t) &= \kappa_i\left(\theta_i - z_i(t)\right)dt + \Sigma_i\, dW^{\mathbb{Q}}(t) \\ &= \begin{bmatrix} \kappa_{11}^i & \kappa_{12}^i \\ \kappa_{21}^i & \kappa_{22}^i \end{bmatrix} \begin{pmatrix} \theta_{i1} - \pi(t) \\ \theta_{i2} - g(t) \end{pmatrix} dt + \begin{bmatrix} \sigma_{i1} & 0 \\ \rho_i\sigma_{i2} & \sqrt{1-\rho_i^2}\sigma_{i2} \end{bmatrix} dW^{\mathbb{Q}}(t). \end{aligned} \tag{10}$$

Therefore, in our most general setup, κ, θ, and Σ are all regime-dependent. This setup is a convenient reduced-form way of identifying the potential effects of monetary policies on the dynamics of macro variables, although it cannot establish any causal relation between the two. To obtain the dynamics of $z_i(t)$ under the $\mathbb{P}$ measure, we assume the following standard affine market prices of risk for $z_i(t)$ at regime i:

$$\lambda_i(t) = \Sigma_i^{-1}\left\{ \begin{pmatrix} \lambda_{10}^i \\ \lambda_{20}^i \end{pmatrix} + \begin{bmatrix} \lambda_{11}^i & \lambda_{12}^i \\ \lambda_{21}^i & \lambda_{22}^i \end{bmatrix} \begin{pmatrix} \pi(t) \\ g(t) \end{pmatrix} \right\} = \Sigma_i^{-1}(\lambda_{i0} + \lambda_i z_i(t)). \tag{11}$$

We allow all the prices of risk parameters to be regime-dependent. Therefore, the dynamics of $z_i(t)$ at regime i under the $\mathbb{P}$ measure are given by

$$dz_i(t) = \kappa_i^{\mathbb{P}}\left(\theta_i^{\mathbb{P}} - z_i(t)\right)dt + \Sigma_i\, dW^{\mathbb{P}}(t),$$

where $\kappa_i^{\mathbb{P}} = \kappa_i - \lambda_i$ and $\theta_i^{\mathbb{P}} = (\kappa_i - \lambda_i)^{-1}(\kappa_i\theta_i + \lambda_{i0})$.

In the two-regime model, investors face not only the diffusion risk, but also the risk of regime switch. This risk is reflected in the difference in the transition matrix of s_t under $\mathbb{Q}$ and $\mathbb{P}$ measures. For the two-regime model, due to the restrictions on the transition matrix, we have under the $\mathbb{Q}$ measure

$$Q = \begin{bmatrix} q_{11} & -q_{11} \\ -q_{22} & q_{22} \end{bmatrix}, \tag{12}$$

and under the $\mathbb{P}$ measure

$$P = \begin{bmatrix} p_{11} & -p_{11} \\ -p_{22} & p_{22} \end{bmatrix}.$$

Dai, Singleton, and Yang (2006) show that Λ_{ij}, the ratio between $Q(i,j)$ and $P(i,j)$ (the ith and jth elements of the matrix Q and P, respectively), represents the excess return of a security that pays one dollar if the regime variable s_t changes from i to j and zero otherwise.

To satisfy the restrictions on the transition matrices under $\mathbb{Q}$ and $\mathbb{P}$ measures, we specify the market prices of risk for the regime switch as

$$\Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}.$$

Therefore, the transition matrix under the $\mathbb{P}$ measure is

$$P = \Lambda Q = \begin{bmatrix} q_{11}\Lambda_{11} & -q_{11}\Lambda_{11} \\ -q_{22}\Lambda_{22} & q_{22}\Lambda_{22} \end{bmatrix}. \tag{13}$$

In our empirical analysis, in addition to the single-regime benchmark model M1, we consider two versions of the two-regime model. For the rest of the paper, we refer to the model with a Taylor rule in (9), a state variable in (3), and a market price of risk specification in (4) as M2(δ), and the model with a Taylor rule in (9), a state variable in (10), and a market price of risk specification in (11) as M3(δ,π,g). Therefore, M2 (δ) only allows the regime-dependent Taylor rule, whereas the most sophisticated model M3(π,g,δ) allows both regime-dependent monetary policies and macro variables. The appendix provides the details on the solutions of the two regime-switching models.

3 Estimating Regime-Switching Models Using MCMC

In this section, we discuss the MCMC methods for estimating and comparing the two-regime term structure model using the yields of zero-coupon government bonds. MCMC methods have been used by Ang, Dong, and Piazzesi (2006) and others in estimating macro term structure models with the Taylor rule. They are especially appropriate for the regime-switching model given the latent regime variable.[6]

Suppose we observe a time series of yields of M zero-coupon bonds with different maturities, $Y(t) = \{y_t(\tau_1), y_t(\tau_2), ..., y_t(\tau_M)\}$ for $t = 1, 2, ..., T$. We assume that the actual yields are observed with independent pricing errors

$$\begin{aligned} y_t(\tau_m) &= \hat{y}_{s_t}(\tau_m) + \varepsilon_m(t),\ \varepsilon_m(t) \sim N\left(0, \sigma_m^2\right), \\ \hat{y}_{s_t}(\tau_m) &= -\frac{A_{s_t}(\tau_m) + B_{s_t,\pi}(\tau_m)\,\pi(t) + B_{s_t,g}(\tau_m)\,g(t)}{\tau_m}, \end{aligned}$$

[6]MCMC methods have been widely used in the literature to estimate continuous-time finance models with latent variables. For example, Eraker, Johannes, and Polson (2002) and Li, Wells, and Yu (2009, 2010) estimate stochastic volatility models with jumps using index return and option prices.

where $\hat{y}_{s_t}(\tau_m)$ represents the model yield given the current regime s_t, and $\varepsilon_m(t)$ represents the pricing errors for the yield with maturity τ_m.

The main objective of our analysis is to make inferences about model parameters Θ and the regime variables $\mathbf{S} = \{s_t\}_{t=1}^{T}$ based on the observed yields $\mathbf{Y} = \{y_t(\tau_m)\}_{t=1,\ldots,T}^{m=1,\ldots,M}$ and macro variables $\mathbf{Z} = \{\pi(t), g(t)\}_{t=1}^{T}$. That is, we need to estimate $p(\Theta, \mathbf{S}|\mathbf{Y}, \mathbf{Z})$, the posterior distribution of $(\Theta, \mathbf{S})$ given $(\mathbf{Y}, \mathbf{Z})$. For $s = 1$ and 2, the parameters that we need to estimate are

$$\Theta = \left\{\kappa_s^{\mathbb{P}}, \theta_s^{\mathbb{P}}, \Sigma_s, \delta_s, \lambda_{s0}, \lambda_s, \Lambda_{jj} \text{ and } q_{jj} \text{ for } j = 1, 2, \sigma_m^2 \text{ for } m = 1, 2, ..., M\right\}.$$

We also need to filter the latent regime variable $\mathbf{S} = \{s_t\}_{t=1}^{T}$.

Our MCMC methods simulate posterior samples of $(\Theta, \mathbf{S})$ from complicated posterior distributions $p(\Theta, \mathbf{S}|\mathbf{Y}, \mathbf{Z})$. We estimate the means and standard deviations of $(\Theta, \mathbf{S})$ using the means and standard deviations of the simulated posterior samples. By Bayes rule,

$$p(\Theta, \mathbf{S}|\mathbf{Y}, \mathbf{Z}) \propto p(\mathbf{Y}, \mathbf{Z}, \mathbf{S}, \Theta) = p(\mathbf{Y}|\mathbf{Z}, \mathbf{S}, \Theta)p(\mathbf{Z}, \mathbf{S}|\Theta)p(\Theta),$$

where the likelihood function equals

$$p(\mathbf{Y}|\mathbf{Z}, \mathbf{S}, \Theta) = \prod_{m=1}^{M}\prod_{t=1}^{T}\frac{1}{\sigma_m}\exp\left\{-\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\},$$

$$\begin{aligned} p(\mathbf{Z}, \mathbf{S}|\Theta) &= \prod_{t=1}^{T-1} p(z_{t+1}, s_{t+1}|z_t, s_t, \Theta) = \prod_{t=1}^{T-1} p(z_{t+1}|z_t, s_{t+1})\, p(s_{t+1}|s_t) \\ &= \prod_{t=1}^{T-1} \frac{1}{\left|\Phi_{s_{t+1}}\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\varepsilon_{t+1}^{s_{t+1}}\right)^{\top}\Phi_{s_{t+1}}^{-1}\varepsilon_{t+1}^{s_{t+1}}\right\}\exp(\Lambda Q\Delta)_{s_t, s_{t+1}}, \end{aligned}$$

$\varepsilon_{t+1}^{s_{t+1}} = z_{t+1} - z_t - \kappa_{s_{t+1}}^{\mathbb{P}}\theta_{s_{t+1}}^{\mathbb{P}}\Delta + \kappa_{s_{t+1}}^{\mathbb{P}} z_t\Delta$, and $\Phi_{s_{t+1}} = \Delta\Sigma_{s_{t+1}}\Sigma_{s_{t+1}}^{\top}$. We note that the likelihood function $p(\mathbf{Z}, \mathbf{S}|\Theta)$ is based on a discretized version of the continuous-time model of the macro variables over the small time interval Δ. More details on how to obtain the likelihood function are provided in the Appendix.

The joint posterior distribution $p(\Theta, \mathbf{S}|\mathbf{Y}, \mathbf{Z})$ is very complicated. However, the Clifford-Hammersley theorem indicates that the joint posterior is equivalent to its complete conditionals, i.e.,

$$p(\Theta, \mathbf{S}|\mathbf{Y}, \mathbf{Z}) \Longleftrightarrow p(\Theta|\mathbf{S}, \mathbf{Z}, \mathbf{Y}), p(\mathbf{S}|\Theta, \mathbf{Z}, \mathbf{Y}).$$

Therefore, we simulate the posterior samples of each parameter and the regime variable (s_t for each t) from the complete conditionals as follows. Given initial values $\left(\Theta^{(0)}, \mathbf{S}^{(0)}\right)$,

- draw $\Theta^{(j+1)} \sim p\left(\Theta|\mathbf{S}^{(j)}, \mathbf{Z}, \mathbf{Y}\right)$,

- draw $\mathbf{S}^{(j+1)} \sim p\left(\mathbf{S}|\Theta^{(j+1)}, \mathbf{Z}, \mathbf{Y}\right)$.

We choose priors that are as uninformative as possible and discard the first 20,000 simulations as burn in. We use the mean of the last 30,000 simulations as the parameter and state estimates. More detailed discussions on how to choose the priors and update the model parameters are given in the appendix.

There are different ways to evaluate the performance of the regime-switching models. One intuitive approach is to consider the following in-sample regression of observed yields on model yields

$$y_t(\tau_m) = \gamma_0 + \gamma_1 \hat{y}_{s_t}(t, \tau_m) + e_m(t). \tag{14}$$

This shows how much of the variations of contemporaneous bond yields can be explained by the macro variables.

Formally, we compare the relative model performance using the Bayes factor, a well-adopted procedure in the Bayesian statistics literature. A detailed discussion of the Bayes factor can be found in Kass and Raftery (1995). Denote the data by $\mathbf{D} = (\mathbf{Y}, \mathbf{Z})$. Consider two models, $\mathbf{M}_1$ and $\mathbf{M}_2$, with associated parameters Θ_1 and Θ_2. Given prior odds $\frac{p(\mathbf{M}_1)}{p(\mathbf{M}_2)}$, the Bayesian method compares models using the posterior odds

$$\frac{p(\mathbf{M}_1|\mathbf{D})}{p(\mathbf{M}_2|\mathbf{D})} = \underbrace{\frac{p(\mathbf{D}|\mathbf{M}_1)}{p(\mathbf{D}|\mathbf{M}_2)}}_{\text{Bayes Factor}} \cdot \frac{p(\mathbf{M}_1)}{p(\mathbf{M}_2)}.$$

If $p(\mathbf{M}_1) = p(\mathbf{M}_2) = 0.5$, we compare the models using the Bayes factor. Therefore, the Bayes factor is a summary of the evidence provided by the data in favor of one model as opposed to another. Unlike the likelihood ratio type of tests, which evaluates the likelihood at the maximized values of the model parameters, the Bayes factor averages all the possible parameter values. The literature has relied on the following interpretations of the Bayes factor in evaluating models: 1-3.2, not worth more than a bare mention; 3.2-10 substantial; 10-100, strong; and greater than 100, decisive. More detailed discussions on how to compute the Bayes factor for the regime-switching models are provided in the appendix.

4 Empirical Results

4.1 Data

We use the dataset similar to the one used in Ang and Piazzesi (2003) in our empirical analysis. The data contain yields of the zero coupon the US Treasury bonds with maturities of 3 months, 1, 2, 3, 4, and 5 years observed at a quarterly frequency from Q.2 of 1952 to Q.3 of 2007. We have deflator inflation rate and demeaned output gap observed at quarterly frequency from Q.2 of 1952 to Q.3 of 2007.

Figure 1 provides the time series plots of the three-month bond yields, inflation, and output gap and reveal several interesting facts about the data. First, the yields are highly correlated with the two macro variables. Second, we do observe the "Great Moderation" in the macro activities: the volatilities of inflation and output gap have declined dramatically since the early 1980s. Table 1 provides the results of the regressions of bond yields at different maturities on *contemporaneous* inflation and output gap during the whole sample. It is interesting to see that the R^2s decline monotonically from about 48% at one-quarter maturity to 38% at the five-year maturity. Since bond yields are linear functions of the two macro variables with cross section restrictions on the coefficients under standard affine models, the R^2s reflect the explanatory power of the macro variables for contemporaneous bond yields.

4.2 Estimates of the Single-Regime Model

We first discuss the parameter estimates of the single-regime model, which are given in Table 2. For ease of discussion, we report the parameter estimates for the Taylor Rule in the following equation:

$$r(t) = \underset{(0.0026)}{0.0187} + \underset{(0.0523)}{0.9055}\,\pi(t) + \underset{(0.1097)}{0.1863}\,g(t).$$

Interestingly, most Taylor coefficients are estimated with high precision. The most striking result is that the coefficient of inflation, δ_π, is roughly 0.9 and is less than one. This estimate violates the so-called "Taylor principle," which requires the policy rate to respond to inflation by more than one-for-one to ensure the stability of the economy. Our estimate suggests that for 1% increase in inflation, the Fed increases the nominal rate by less than 1% and effectively lowers the real rate. As pointed out by Bernanke (2004), "if policy makers do not react sufficiently aggressively to increases in inflation, spontaneously arising expectations of increased inflation can ultimately be self-confirming and even self-reinforcing."

Next, we report the parameters through the following equation of the dynamics of the macro variables under the $\mathbb{P}$ measure:

$$d\begin{pmatrix} \pi(t) \\ g(t) \end{pmatrix} = \begin{bmatrix} \underset{(0.0356)}{0.0536} & \underset{(0.0527)}{-0.4602} \\ \underset{(0.0787)}{0.1341} & \underset{(0.1079)}{0.7308} \end{bmatrix} \begin{pmatrix} \underset{(0.0335)}{0.0303} - \pi(t) \\ \underset{(0.0169)}{-0.0003} - g(t) \end{pmatrix} dt$$
$$+ \begin{bmatrix} \underset{(0.0003)}{0.0071} & 0 \\ \underset{(0.0682)}{0.0204} \cdot \underset{(0.0008)}{0.0167} & \sqrt{1 - \underset{(0.0682)}{0.0204}^2} \cdot \underset{(0.0008)}{0.0167} \end{bmatrix} dW^{\mathbb{P}}(t)\,.$$

These estimates are generally consistent with the actual data observed in the market. For example, the long-run mean for inflation is about 3%, with a speed of mean reversion of 0.054 and volatility of 0.7%. The long-run mean of output gap is close to 0, with a much higher speed of mean reversion of 0.73 and volatility of 1.7%. There is negative feedback from output gap to inflation with a coefficient of -0.46. This suggests that high output gap tends to lead to a high level of inflation. On the other hand, there is positive feedback from inflation to output gap with a coefficient of 0.13. This suggests that high inflation tends to lead to a low output gap in the future. The instantaneous correlation between inflation and output gap, ρ, is not significantly different from zero. The estimates of the market price of risk parameters suggest that under the $\mathbb{Q}$ measure, both macro variables exhibit stronger mean-reversion, and there is stronger feedback from inflation to output gap but a weaker one from output gap to inflation.

Next, we examine the explanatory power of the two macro variables in capturing contemporaneous bond yields under M1. The results for regressing the observed bond yields $y_t(\tau_m)$ on model yields $\hat{y}_t(\tau_m)$ for each maturity τ_m over the entire sample period under the single-regime model are reported in Table 3. The estimates of the intercept at different maturities, $\hat{\gamma}_0$, are close to zero, while the estimates of the linear coefficient, $\hat{\gamma}_1$, are close to one, suggesting that the model can capture the average level of the yields reasonably well. The standard errors of the residuals in the regression are about 2%, which are very similar to the estimated standard errors of the pricing errors ($\hat{\sigma}_m$, for $m = 1, 2, ..., 6$) in Table 2. This suggests that the average pricing error in terms of yield is about two basis points under this model. The most important result is that the R^2s across all maturities are less than 50% and decline monotonically with maturity. While the R^2 at three months is about 48%, it declines to about 38% at five-year maturity. These results are consistent with those based on the simple regression of bond yields directly on contemporaneous inflation and output gap.

Figure 2 provides the time series plots of the observed and model-implied yields at six different maturities under the single-regime model. It is clear that the model cannot satisfactorily capture the observed yields. In particular, at short maturities, the observed yields are much higher than the model yields. At the five-year maturity, the model yields are higher than the observed yields before the 1980s and lower after the 1980s. Overall, our results show that the single-regime model fails to capture some important features of the data.

4.3 Estimates of the Two-Regime Models

In this section, we discuss the estimates of the two-regime models, M2(δ) and M3(π, g, δ). In M2(δ) , we allow monetary policies, i.e., the coefficients of the Taylor rule, to be regime-dependent. In M3(π, g, δ) , we allow both monetary policies and the dynamics of the macroeconomic variables to be regime-dependent. These two models allow us to examine the incremental contributions of regime-dependent monetary policies and macro dynamics in explaining bond yields.

4.3.1 Estimates of M2(δ)

Table 4 presents the parameter estimates for the two-regime model M2(δ). For ease of discussion, we report the parameter estimates for the Taylor rules under two different regimes in the following equations:

$$\begin{aligned}
\text{Under regime 1} \quad &: \quad r_1(t) = \underset{(0.0026)}{0.0246} + \underset{(0.0726)}{1.5165}\,\pi(t) + \underset{(0.0962)}{0.1250}\,g(t), \\
\text{Under regime 2} \quad &: \quad r_2(t) = \underset{(0.0019)}{0.0106} + \underset{(0.0403)}{0.8568}\,\pi(t) + \underset{(0.0550)}{0.3629}\,g(t).
\end{aligned}$$

It is interesting to see that most Taylor coefficients are quite different under the two regimes. The coefficient of inflation is about 1.5 (0.85) under regime 1 (2) and satisfies (violates) the "Taylor principle." Therefore, the Fed is more (less) active in controlling inflation in the first (second) regime. The coefficient of output gap is about 0.13 (0.36) under regime 1 (2), suggesting that the Fed is more accommodative for growth under the second regime than the first one.

Next, we report the parameters through the following equation of the dynamics of the macro variables under the physical measure:

$$\begin{aligned}
d\begin{pmatrix} \pi(t) \\ g(t) \end{pmatrix} &= \begin{bmatrix} \underset{(0.0530)}{0.0566} & \underset{(0.0641)}{-0.4691} \\ \underset{(0.0772)}{0.0854} & \underset{(0.1263)}{0.6956} \end{bmatrix} \begin{pmatrix} \underset{(0.0253)}{0.0461} - \pi(t) \\ \underset{(0.0027)}{-0.0008} - g(t) \end{pmatrix} dt \\
&+ \begin{bmatrix} \underset{(0.0003)}{0.0070} & 0 \\ \underset{(0.0675)}{0.0376} \cdot \underset{(0.0008)}{0.0165} & \sqrt{1 - \underset{(0.0675)}{0.0376^2}} \cdot \underset{(0.0008)}{0.0165} \end{bmatrix} dW^{\mathbb{P}}(t)
\end{aligned}$$

The parameters under M2(δ) are reasonably close to those under M1. For example, the long-run mean for inflation is about 4.6%, with a speed of mean reversion of 0.057 and volatility of 0.7%. The long-run mean of output gap is close to 0, with a much higher speed of mean reversion of about 0.70 and volatility of 1.65%. There is negative feedback from output gap to inflation with a coefficient of -0.47. This suggests that high output gap tends to lead to a high level of inflation. On the other hand, there is positive feedback from inflation to output gap with a coefficient of 0.085. This suggests that high inflation tends to lead to a low output gap in the future. The instantaneous correlation between inflation and output gap, ρ, is again not significantly different from zero.

Next, we examine the explanatory power of the two macro variables in capturing contemporaneous bond yields under M2(δ). Table 5 reports the results of regressing the observed bond yields on model yields under M2(δ) at all maturities. Similar to the single-regime model, the intercepts are close to zero, while the linear coefficients are close to one, suggesting that the model can capture the average level of the yields reasonably well. The standard errors of the residuals under M2(δ) are close to 1%, which are about half of that under M1. The most striking result here is that by allowing δs to depend on the regimes, the R^2s increase from below 50% to above 80%. Moreover, while under the single-regime model the R^2s decline monotonically with maturity, there is no significant difference in R^2s under M2(δ) across maturities. The R^2s range from the lowest at 81.8% at five-year maturity to 84.4% at two-year maturity. The results on R^2s are consistent with our regime-switching model because changing regimes are more important in pricing long-term bonds, whose maturities tend to span different economic cycles.

Figure 3 provides the time series plots of the observed and model-implied yields at all maturities. Unlike the results under M1, the differences between the observed and model yields become much smaller under M2(δ). In particular, while the single-regime model has difficulties in capturing the high spikes in bond yields in the early 1980s, M2(δ) can fit bond yields during the same time period much better.

Figure 4 provides the time series plots of the filtered regime variable. It presents the posterior probabilities of the regime that the economy is in. It is interesting to see that the economy switches to the first regime since late 1970s and early 1980s, which is the beginning of the "Great Moderation," and remains in the first regime until the early 1990s. Boivin (2006) estimates Taylor rules with drifting coefficients and find important changes in the rule coefficients around early 1980s. Except for a brief period between 1992 and 1993, the economy is in the second regime, the economy stays in the first regime until early 2000s. After the burst of the "internet bubble," the Fed becomes more aggressive in stimulating economic growth. Hamilton, Pruitt, and Borger (2010) estimate market-perceived monetary policy rule using macroeconomic news. They also identify a change in Fed's policy rule after 2000.

One important point to emphasize here is that the regime does not change dramatically at a high frequency. In fact, the economy remains in one regime for a long time, sometimes decades. This important fact shows that the increased explanatory power of our model is not due to a fast changing regime variable that plays the role of a latent variable. In the following, we report the estimates of the transition matrix over a quarter ($\Delta = 0.25$ year) of the regime variable s_t under both the $\mathbb{Q}$ and $\mathbb{P}$ measures:

$$\begin{aligned}
\exp\{Q\Delta\} &= \begin{bmatrix} 0.9862 & 0.0138 \\ 0.0077 & 0.9923 \end{bmatrix}, \\
\exp\{\Lambda Q\Delta\} &= \begin{bmatrix} 0.9496 & 0.0504 \\ 0.0243 & 0.9757 \end{bmatrix}.
\end{aligned}$$

These estimates suggest that under $\mathbb{Q}$ and $\mathbb{P}$ measures, both regimes are very persistent. That is, the probability for the economy to remain in either regime 1 or 2 is very high. The differences in the transition matrix between the two measures are quite small, suggesting that investors are not overly concerned about the switching risk.

The empirical results of M2(δ) are very interesting and quite different from those in the existing literature. The huge literature on empirical financial time series shows that regime switching models tend to improve the in-sample fit for financial time series with a fat-tailed distribution and time varying volatility. Mainly, the reason is that regime-switching models generally incorporate a mixture of two normal innovations, which tend to generate fat-tailed distributions. By switching between two regimes and allowing models with different levels of volatilities in the two regimes, such models can also generalize time-varying volatility. However, M2(δ) allows regime switching only in the Taylor coefficients but not in the macroeconomic variables. Therefore, the increased explanatory power of bond yields comes mainly from the regime-dependent monetary policies rather than the normal channel of standard regime-switching models. M2(δ) demonstrates the potential for macroeconomic variables to explain bond yields: by allowing slow-moving monetary policies, inflation and output gap can explain more than 80% of the variations of bond yields with R^2s roughly constant across maturities.

4.3.2 Estimates of M3(π, g, δ)

Table 6 reports the parameter estimates of M3(π, g, δ). For ease of discussion, we report the parameter estimates for the Taylor rules under two different regimes in the following equations:

$$\begin{aligned}
\text{Under regime 1} \quad &: \quad r_1(t) = \underset{(0.0040)}{0.0187} + \underset{(0.0938)}{1.5060}\,\pi(t) + \underset{(0.1459)}{0.2489}\,g(t), \\
\text{Under regime 2} \quad &: \quad r_2(t) = \underset{(0.0029)}{0.0135} + \underset{(0.0820)}{0.7286}\,\pi(t) + \underset{(0.1080)}{0.4445}\,g(t).
\end{aligned}$$

It is interesting to see that most Taylor coefficients are quite different under the two regimes. The coefficient of inflation is about 1.5 (0.73) under regime 1 (2) and satisfies (violates) the "Taylor principle." Therefore, the Fed fights inflation more (less) aggressively in the first (second) regime. The coefficient of output gap is about 0.25 (0.44) under regime 1 (2), suggesting that the Fed is more aggressive in stimulating growth under the second regime than the first one. Hence, the Taylor rules estimated under M2(δ) are similar to those estimated under M3(π, g, δ).

Next, we report the parameters through the following equations of the dynamics of the macro variables under the physical measure:

$$\begin{aligned}
\text{Under regime 1} \quad : \quad d\begin{pmatrix} \pi(t) \\ g(t) \end{pmatrix} &= \begin{bmatrix} \underset{(0.0592)}{0.3558} & \underset{(0.0763)}{-0.1808} \\ \underset{(0.1689)}{0.5676} & \underset{(0.1433)}{1.0773} \end{bmatrix} \begin{pmatrix} \underset{(0.0032)}{0.0216} - \pi(t) \\ \underset{(0.0026)}{0.0040} - g(t) \end{pmatrix} dt \\
&+ \begin{bmatrix} \underset{(0.0004)}{0.0046} & 0 \\ \underset{(0.1300)}{0.1678} \cdot \underset{(0.0013)}{0.0126} & \sqrt{1 - \underset{(0.1300)}{0.1678^2}} \cdot \underset{(0.0013)}{0.0126} \end{bmatrix} dW^{\mathbb{P}}(t).
\end{aligned}$$

$$\text{Under regime 2} \quad : \quad d\begin{pmatrix} \pi(t) \\ g(t) \end{pmatrix} = \begin{bmatrix} \underset{(0.0465)}{0.0166} & \underset{(0.0795)}{-0.5245} \\ \underset{(0.0649)}{0.0703} & \underset{(0.1866)}{0.6457} \end{bmatrix} \begin{pmatrix} \underset{(0.0272)}{0.0658} - \pi(t) \\ \underset{(0.0033)}{-0.0028} - g(t) \end{pmatrix} dt$$

$$+ \begin{bmatrix} \underset{(0.0005)}{0.0076} & 0 \\ \underset{(0.0802)}{-0.0407} \cdot \underset{(0.0010)}{0.0183} & \sqrt{1 - \underset{(0.0802)}{0.0407^2}} \cdot \underset{(0.0010)}{0.0183} \end{bmatrix} dW^{\mathbb{P}}(t).$$

The above two equations show that inflation and output gap behave very differently under the two different regimes. For example, under regime 1, the long-run mean for inflation is about 2.2%, with a speed of mean reversion of 0.36 and volatility of 0.46%; under regime 2, the long-run mean for inflation is about 6.6%, with a speed of mean reversion of 0.017 and volatility of 0.76%. Under regime 1, the long-run mean of output gap is 0.4% (not statistically significant), with a speed of mean reversion of about 1.08 and volatility of 1.26%. Under regime 2, the long-run mean of output gap is -0.28% (not statistically significant), with a speed of mean reversion of about 0.65 and volatility of 1.83%. Under regime 1, there is negative feedback from output gap to inflation with a coefficient of -0.18. Under regime 2, the negative feedback coefficient from output gap to inflation is about -0.52. On the other hand, there is positive feedback from inflation to output gap with a coefficient of 0.57 under regime 1 and 0.07 under regime 2. The instantaneous correlation between inflation and output gap, ρ, is again not significantly different from zero under both regimes.

Overall, the estimated parameters of M3(π, g, δ) suggest that inflation and output gap are much more stable under the first than under the second regime. In the first regime, (i) inflation has lower mean and volatility, stronger mean reversion, weaker feedback from output gap to inflation; and (ii) output gap has stronger mean reversion and lower volatility. In the second regime, (i) inflation has higher mean and volatility, weaker mean reversion, stronger feedback from output gap to inflation; and (ii) output gap has weaker mean reversion and higher volatility.

Table 7 reports the R^2s of the regressions of the observed yields to model-implied yields in (14) at all maturities. Similar to those in M2(δ), the intercepts are close to zero, while the linear coefficients are close to one, suggesting that the model can capture the average level of the yields reasonably well. The standard errors of the residuals are close to 1%, which are about half of that of the single-regime model. One interesting result here is that by allowing the dynamics of inflation and output gap to depend on regimes, the R^2s remain close to those under M2(δ): there is no significant difference in R^2s across maturities, and the R^2s range from the lowest at 81.7% at five-year maturity to 84.3% at two-year maturity. The results on R^2s suggest that regime-dependent monetary policies are the most important factor in pricing bond yields, and regime-dependent macroeconomic variables do not provide much incremental contributions to the pricing of bond yields beyond regime-dependent monetary policies.

Figure 5 provides the time series plots of the actual and model-implied yields at all maturities. Similar to the results under M2(δ), the differences between model and observed yields become much smaller under M3(π, g, δ). In particular, while the single-regime model has difficulties in capturing the high spikes in bond yields in the early 1980s, both M2(δ) and M3(π, g, δ) can fit bond yields during the same time period much better.

Figure 6 provides the time series plots of the filtered regime variable. It presents the posterior probabilities of the regime that the economy is in. The results are very similar to those under M2(δ). The economy switches to the first regime in the early 1980s, the beginning of the "Great Moderation," and remains in the first regime until early 1990s. Except for a brief period between 1992 and 1993, in which the economy is the second regime, the economy remains in the first regime until the early 2000s. After the burst of the "internet bubble," the Fed becomes more aggressive in stimulating economic growth. In the following, we report the estimates of the transition matrix over a quarter of the regime variable s_t under both the $\mathbb{Q}$ and $\mathbb{P}$ measures:

$$\begin{aligned}
\exp\{Q\Delta\} &= \begin{bmatrix} 0.8341 & 0.1659 \\ 0.4353 & 0.5647 \end{bmatrix}, \\
\exp\{\Lambda Q\Delta\} &= \begin{bmatrix} 0.9380 & 0.0620 \\ 0.0371 & 0.9629 \end{bmatrix}.
\end{aligned}$$

These estimates suggest that both regimes are more persistent under the $\mathbb{P}$ than the $\mathbb{Q}$ measure. For example, the probability for the economy to remain in regime 1 is 0.9380 (0.8341) under the $\mathbb{P}$ ($\mathbb{Q}$) measure, while the probability for the economy to remain in regime 2 is 0.9629 (0.5647) under the $\mathbb{P}$ ($\mathbb{Q}$) measure. Therefore, the market prices the bonds as if the probability of regime switching is higher under the $\mathbb{Q}$ measure. The differences in the estimated transition matrix under M3(δ, π, g) and M2(δ) are probably due to the fact that the dynamics of the state variables are misspecified under M2(δ).

Table 8 provides the results on the model comparison based on the Bayes factor. It basically compares the posterior odds ratio between the two models. The rule of thumb is that a Bayes factor over 15 is overwhelming evidence of superior model performance. Clearly, M2(δ) is much better than M1, whereas M3(π, g, δ) strongly dominates M2(δ). Since the two regime-switching models perform equally well in capturing bond yields, the advantages of M3(π, g, δ) over M2(δ) mainly come from its ability in capturing the dynamics of inflation and output gap.

In summary, our results on M3(π, g, δ) and M2(δ) show that the monetary policies in the US exhibit distinctive features in different economic environments. In one regime, the Fed fights inflation more aggressively, while in another, it is more concerned about stimulating economic growth. We find that regime-dependent monetary policies are essential in capturing government bond yields. Without regimes, inflation and output gap can explain only less than 50% of variations of contemporaneous bond yields. With the regimes, the explanatory power is increased to more than 80%. In addition, inflation and output gap become much more stable and less volatile under the more aggressive regime, suggesting that monetary policies could have contributed partially to the "Great Moderation."

4.4 Monetary Policies since 2000

It has been argued that one of the contributors to the current global financial crisis is the loose monetary policy in the US after 2000 (see Bernanke 2010). The Fed has been blamed to have kept the Fed fund rate "too low for too long." In this section, we discuss the monetary policies in the US since 2000 through the lens of our regime-switching model M3(π, g, δ).

Figure 7 plots the observed and model-implied three-month yields, and the spot rate under the proactive and the accommodative Taylor rules under M3(π, g, δ). Between 2000 and 2001, the observed and model three-month yields, and the spot rate under the proactive Taylor rule are very close to each other. This suggests that right before the burst of the internet bubble, the Fed has adopted an aggressive monetary policy to slow down the economy. However, during the first half of 2001, both the observed and model yields decline dramatically. They deviate away from the spot rate under the proactive Taylor rule and converge to that of the accommodative one. This suggests that after the burst of the internet bubble, the Fed has shifted its monetary policy to be more responsive to growth. However, while the model-implied yields track the spot rate under the accommodative Taylor rule closely during the rest of the sample period, the observed yields keep declining until they reach 1% in early 2003 and remain there for about a year. In other words, had the Fed followed the accommodative policy identified by our model during this time, it should have stopped easing when the spot rate reached about 2% in the middle of 2002. Therefore, although we cannot conclude definitively from these results that the Fed has kept the interest rate "too low for too long," it appears that the Fed has deviated from its long-term policy stance and gone the extra length to stimulate the economy, possibly due to the concerns of deflation risk during this time.

5 Conclusion

We have developed a continuous-time regime-switching model for the term structure of interest rates with regime-dependent monetary policies and macro variables. Our estimates of the model using government bond yields show that the US monetary policies exhibit two distinct regimes. The Fed is more aggressive in fighting inflation and less aggressive in stimulating growth in one regime than in the other. We show that regime-dependent monetary policies significantly improve the explanatory power of macro variables for government bond yields. Without the regimes, inflation and output can explain about 40% of the variations of contemporaneous bond yields. With the regimes, the two variables can explain more than 80% of the variations of bond yields. Regime-dependent monetary policies also lead to more stable behaviors of inflation and output gap and therefore could have contributed to the "Great Moderation." Our model suggests that majority of the variations in bond yields can be traced back to fundamental economic variables.

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Appendix

In this appendix, we provide details on the implementations of the MCMC methods. We also present evidence on the accuracy of our approximation formula for zero-coupon bond prices under the regime-switching model.

A.1 MCMC Algorithms for Two-Regime Models

A.1.1 Model Solution, Discretization, and Likelihood

In this section, we provide the details on how to solve a two-regime model explicitly and how to discretize a two-regime model to obtain the likelihood function used in later MCMC analysis.

For the two-regime model characterized by (9), (10), (11), (12), and (13), given the regime variable s_t, the yield of a zero-coupon bond with time to maturity τ_m at t equals

$$\hat{y}_{s_t}(\tau_m) = -\frac{A_{s_t}(\tau_m) + B_{s_t\pi}(\tau_m)\pi_t + B_{s_tg}(\tau_m)g_t}{\tau_m},$$

where we obtain $A_{s_t}(\tau), B_{s_t\pi}(\tau)$, and $B_{s_tg}(\tau)$ for $s_t = 1$ and 2 by solving the following system of ODEs numerically:

$$\begin{aligned} 0 &= A_1'(\tau) - \left(\kappa_{11}^1\theta_{11} + \kappa_{12}^1\theta_{12}\right)B_{1\pi}(\tau) - \left(\kappa_{21}^1\theta_{11} + \kappa_{11}^1\theta_{12}\right)B_{1g}(\tau) \\ &\quad -\frac{1}{2}\sigma_{11}^2B_{1\pi}^2 - \rho_1\sigma_{11}\sigma_{12}B_{1\pi}B_{1g} - \frac{1}{2}\sigma_{12}^2B_{1g}^2 - q_{11} + \delta_{10} + q_{11}e^{A_2-A_1}, \end{aligned}$$

$$\begin{aligned} 0 &= A_2'(\tau) - \left(\kappa_{11}^2\theta_{21} + \kappa_{12}^2\theta_{22}\right)B_{2\pi}(\tau) - \left(\kappa_{21}^2\theta_{21} + \kappa_{22}^2\theta_{22}\right)B_{2g}(\tau) \\ &\quad -\frac{1}{2}\sigma_{21}^2B_{2\pi}^2 - \rho_2\sigma_{21}\sigma_{22}B_{2\pi}B_{2g} - \frac{1}{2}\sigma_{22}^2B_{2g}^2 - q_{22} + \delta_{20} + q_{22}e^{A_1-A_2}, \end{aligned}$$

$$\begin{aligned} 0 &= B_{1\pi}^1(\tau) + \kappa_{11}^1B_{1\pi}(\tau) + \kappa_{21}^1B_{1g}(\pi) + q_{11}e^{A_2-A_1}(B_{2\pi} - B_{1\pi}) + \delta_{1\pi}, \\ 0 &= B_{1g}^1(\tau) + \kappa_{12}^1B_{1\pi}(\tau) + \kappa_{22}^1B_{1g}(\pi) + q_{11}e^{A_2-A_1}(B_{2g} - B_{1g}) + \delta_{1g}, \end{aligned}$$

$$\begin{aligned} 0 &= B_{2\pi}^1(\tau) + \kappa_{11}^2B_{2\pi}(\tau) + \kappa_{21}^2B_{2g}(\pi) + q_{22}e^{A_1-A_2}(B_{1\pi} - B_{2\pi}) + \delta_{2\pi}, \\ 0 &= B_{2g}^1(\tau) + \kappa_{12}^2B_{2\pi}(\tau) + \kappa_{22}^2B_{2g}(\pi) + q_{22}e^{A_1-A_2}(B_{1g} - B_{2g}) + \delta_{2g}. \end{aligned}$$

In our estimation of the two-regime model, we assume the observed yields of M zero-coupon bonds to be equal to the model yields plus some observation errors,

$$y_t(\tau_m) = \hat{y}_{s_t}(\tau_m) + \varepsilon_m(t),$$

where $\varepsilon_m(t) \sim$i.i.d.$N\left(0, \sigma_m^2\right)$ for $m = 1, 2, ..., M$. Therefore, in our empirical analysis, the observables are $\mathbf{Y} = \{y_t(\tau_m)\}_{m=1,...,M}^{t=1,...,T}$ and $\mathbf{Z} = \{(\pi_t, g_t)\}_{t=1}^T$. For $s = 1$ and 2, the parameters that we need to estimate are

$$\Theta = \left\{\kappa_s^{\mathbb{P}}, \theta_s^{\mathbb{P}}, \Sigma_s, \delta_s, \lambda_{s0}, \lambda_s, \Lambda_{jj}, q_{jj} \text{ for } j = 1, 2, \sigma_m^2 \text{ for } m = 1, 2, ..., M\right\}.$$

We also need to filter the latent regime variable $\mathbf{S} = \{s_t\}_{t=1}^{T}$.

A key to our MCMC analysis is to write down explicitly the joint likelihood function $p(\mathbf{Y}, \mathbf{Z}, \mathbf{S}, \Theta)$. For this purpose, we consider a discretized version of the regime-switching model over a fixed time interval $\Delta = 0.25$ year. We make the explicit assumption that the regime variable first changes from s_t to s_{t+1}. The evolution of the macro variables between t and $t+1$, $z_{t+1} - z_t$, is then determined by s_{t+1}:

$$z_{t+1} - z_t = \kappa_{s_{t+1}}^{\mathbb{P}} \left(\theta_{s_{t+1}}^{\mathbb{P}} - z_t \right) \Delta + \Sigma_{s_{t+1}} u_{t+1} \sqrt{\Delta},$$

where $u_{t+1} \sim$i.i.d. $N(0,1)$ over all t.

Given the above discretized model of z_t, the joint likelihood of $(\mathbf{Y}, \mathbf{Z}, \mathbf{S}, \Theta)$ equals

$$\begin{aligned}
p(\mathbf{Y}, \mathbf{Z}, \mathbf{S}, \Theta) &= p(\mathbf{Y}|\mathbf{Z}, \mathbf{S})\, p(\mathbf{Z}, \mathbf{S}|\Theta)\, p(\Theta) \\
&= \prod_{m=1}^{M} \prod_{t=1}^{T} \frac{1}{\sigma_m} \exp\left\{ -\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)} \right]^2}{2\sigma_m^2} \right\} \prod_{t=1}^{T-1} p(s_{t+1}, z_{t+1}|s_t, z_t)\, p(\Theta) \\
&= \prod_{m=1}^{M} \prod_{t=1}^{T} \frac{1}{\sigma_m} \exp\left\{ -\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)} \right]^2}{2\sigma_m^2} \right\} \prod_{t=1}^{T-1} p(s_{t+1}|s_t)\, p_{s_{t+1}}(z_{t+1}|z_t)\, p(\Theta) \\
&= \prod_{m=1}^{M} \prod_{t=1}^{T} \frac{1}{\sigma_m} \exp\left\{ -\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)} \right]^2}{2\sigma_m^2} \right\}. \\
&\quad \prod_{t=1}^{T-1} \left\{ \exp(\Lambda Q \Delta) \right\}_{s_t, s_{t+1}} \frac{1}{\left| \Phi_{s_{t+1}} \right|^{\frac{1}{2}}} \exp\left\{ -\frac{1}{2} \left(\varepsilon_t^{s_{t+1}} \right)^{\top} \Phi_{s_{t+1}}^{-1} \varepsilon_t^{s_{t+1}} \right\} p(\Theta),
\end{aligned}$$

where $\Phi_{s_{t+1}} = \Delta \left(\Sigma_{s_{t+1}} \Sigma_{s_{t+1}}^{\top} \right)$, $\hat{y}_{s_t}^{(m)} = -\frac{A_{s_t}(\tau_m) + B_{s_t \pi}(\tau_m)\pi_t + B_{s_t g}(\tau_m) g_t}{\tau}$,

$$\varepsilon_t^{s_{t+1}} = z_{t+1} - z_t - \kappa_{s_{t+1}}^{\mathbb{P}} \left(\theta_{s_{t+1}}^{\mathbb{P}} - z_t \right) \Delta,$$

$\Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}$, and $Q = \begin{bmatrix} q_{11} & -q_{11} \\ -q_{22} & q_{22} \end{bmatrix}$.

A.1.2 Priors and Restrictions on Model Parameters

The following table lists the priors and restrictions we impose on the model parameters in our MCMC analysis. We choose the priors to be as uninformative as possible. We impose restrictions on the model parameters to ensure that the models are well-behaved.

Parameters	Range	Priors
$\delta_s = [\delta_{s0}, \delta_{s\pi}, \delta_{sg}]$	$\delta_{s\pi}, \delta_{sg} > 0$	$\delta_{s\pi},\, \delta_{sg} \sim N(0,100)$
$\kappa_s^{\mathbb{P}} = \begin{bmatrix} \kappa_{11}^{\mathbb{P}s} & \kappa_{12}^{\mathbb{P}s} \\ \kappa_{21}^{\mathbb{P}s} & \kappa_{22}^{\mathbb{P}s} \end{bmatrix}$	$\mathrm{eig}(\kappa_s^{\mathbb{P}}) > 0$	$\kappa_{ij}^{\mathbb{P}s} \sim N(0,100), i,j = 1,2$
$\theta_s^{\mathbb{P}} = \left(\theta_{s1}^{\mathbb{P}}, \theta_{s2}^{\mathbb{P}}\right)^\top$	$\theta_{s1}^{\mathbb{P}} > 0$	$\theta_{sj}^{\mathbb{P}} \sim N(0,100), j = 1,2$
$\Sigma_s = \begin{bmatrix} \sigma_{s1} & 0 \\ \rho_s\sigma_{s2} & \sqrt{1-\rho_s^2}\sigma_{s2} \end{bmatrix}$	$\sigma_{s1}, \sigma_{s2} > 0, \rho_s \in [-1,1]$	$\Sigma_s\Sigma_s^\top \sim IW\left(\left(T_0\hat{\Sigma}_0, T_0 - 1\right), T_0 = T/5\right)$
$\lambda_s = \begin{bmatrix} \lambda_{11}^s & \lambda_{12}^s \\ \lambda_{21}^s & \lambda_{22}^s \end{bmatrix}$	*None*	$\lambda_{ij}^s \sim N(0,100), i,j = 1,2$
σ_m	$\sigma_m > 0$	$\sigma_m^2 \sim IG(2,100), m = 1,2,...,M$
$Q = \begin{bmatrix} q_{11} & -q_{11} \\ -q_{22} & q_{22} \end{bmatrix}$	$q_{jj} < 0$	$q_{ij} \sim N(0,100), j = 1,2$
$\Lambda = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix}$	$\Lambda_{jj} > 0$	$\Lambda_{jj} \sim N(0,100), j = 1,2$

A.1.3 Posteriors for Model Parameters and Regime Variables

In this section, we provide details on updating the procedures and posteriors for all model parameters and regime variables. Except for a few parameters with closed-form posterior distributions, we have to use the Metropolis-Hastings (MH) algorithm to update most parameters and the regime variables.

- Posteriors for σ_m (closed form) with $\sigma_m^2 \sim IG(a,b)$

$$\begin{aligned} p(\sigma_m|\cdot) &\propto \frac{1}{(\sigma_m)^T}\exp\left\{-\frac{1}{2\sigma_m^2}\left[\sum_{t=1}^T\left(y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right)^2\right]\right\}\frac{b^a}{\Gamma(a)}\left(\sigma_m^2\right)^{-a-1}\exp\left\{-\frac{b}{\sigma_m^2}\right\} \\ &\propto \left(\sigma_m^2\right)^{-\frac{T}{2}-a-1}\exp\left\{-\frac{1}{\sigma_m^2}\left[\frac{1}{2}\sum_{t=1}^T\left(y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right)^2 + b\right]\right\} \Longrightarrow \\ \sigma_m^2 &\sim IG\left(\frac{T}{2} + a, \frac{1}{2}\sum_{t=1}^T\left(y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right)^2 + b\right). \end{aligned}$$

  Therefore, we simulate $\nu \sim \Gamma\left(\frac{T}{2} + a, \left[\frac{1}{2}\sum_{t=1}^T\left(y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right)^2 + b\right]^{-1}\right)$ and $\sigma_m = \sqrt{\frac{1}{\nu}}$.

- Posterior of $\theta^{\mathbb{P}}$ (MH) with $\theta_j^{\mathbb{P}s} \sim N(0, \sigma_h^s)$, for $s = 1$ (we have similar results for $s = 2$)

$$p\left(\theta_j^{\mathbb{P}(1)}\right) \propto \exp\left\{-\sum_{m=1}^M\sum_{t=1}^T\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\left\{-\frac{1}{2}\sum_{t=1,\text{ s.t. } s_{t+1}=1}^{T-1}\left(\varepsilon_t^{(1)}\right)^\top\Phi_1^{-1}\varepsilon_t^{(1)}\right\}p\left(\theta^{\mathbb{P}}\right)$$

where $\Phi_1 = \Delta\Sigma_1\Sigma_1^\top$ and $\varepsilon_t^{(1)} = z_{t+1} - z_t - \kappa_1^{\mathbb{P}}\left(\theta^{\mathbb{P}(1)} - z_t\right)\Delta$. We have the following updating procedures:

1. Simulate $\left[\theta_j^{\mathbb{P}(1)}\right]^{(g+1)} = \left[\theta_j^{\mathbb{P}(1)}\right]^{(g)} + \gamma N(0,1)$, for $j = 1, 2$.
2. Generate $U \sim$Uniform$(0,1)$.
3. Accept $\left[\theta_j^{\mathbb{P}(1)}\right]^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(\left[\theta_j^{\mathbb{P}(1)}\right]^{(g+1)} |\cdot\right) / p\left(\left[\theta_j^{\mathbb{P}(1)}\right]^{(g)} |\cdot\right)$.

- Posteriors for $\kappa_s^{\mathbb{P}}$ (MH) with $\kappa_{ij}^{\mathbb{P}s} \sim N(0, 100)$, for $s = 1$ (we have similar results for $s = 2$)

$$p\left(\kappa_{ij}^{\mathbb{P}(1)}\right) \propto \exp\left\{-\sum_{m=1}^{M}\sum_{t=1}^{T}\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\left\{-\frac{1}{2}\sum_{t=1,\text{ s.t. } s_{t+1}=1}^{T-1}\left(\varepsilon_t^{(1)}\right)^\top \Phi_1^{-1}\varepsilon_t^{(1)}\right\} p\left(\kappa^{\mathbb{P}}\right)$$

where $\Phi_1 = \Delta\Sigma_1\Sigma_1^\top$ and $\varepsilon_t^{(1)} = z_{t+1} - z_t - \kappa_1^{\mathbb{P}}\left(\theta_1^{\mathbb{P}} - z_t\right)\Delta$. We have the following updating procedures:

1. Simulate $\left[\kappa_{ij}^{\mathbb{P}(1)}\right]^{(g+1)} = \left[\kappa_{ij}^{\mathbb{P}(1)}\right]^{(g)} + \gamma N(0,1)$, for $j = 1, 2$.
2. Generate $U \sim$Uniform$(0,1)$.
3. Accept $\left[\kappa_{ij}^{\mathbb{P}(1)}\right]^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(\left[\kappa_{ij}^{\mathbb{P}(1)}\right]^{(g+1)} |\cdot\right) / p\left(\left[\kappa_{ij}^{\mathbb{P}(1)}\right]^{(g)} |\cdot\right)$.

- Posterior of Σ_s (MH) with $\Phi^s = \Delta\Sigma_s\Sigma_s^\top \sim IW(\psi, df)$, for $s = 1$ (we have similar results for $s = 2$)

$$\begin{aligned} p(\Sigma_1) \quad \propto \quad & \prod_{t=1,\text{s.t. } s_{t+1}=1}^{T-1}\frac{1}{|\Phi_1|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\sum_{n=1}^{N}\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\times \\ & \exp\left\{-\frac{1}{2}\sum_{t=1,\text{ s.t. } s_{t+1}=1}^{T-1}\left(\varepsilon_t^{(1)}\right)^\top \Phi_1^{-1}\varepsilon_t^{(1)}\right\} p(\Sigma_1) \end{aligned}$$

where $\Phi_1 = \Delta\Sigma_1\Sigma_1^T$ and $\varepsilon_t = z_{t+1} - z_t - \kappa_1^{\mathbb{P}}\left(\theta^{\mathbb{P}(1)} - z_t\right)\Delta$. We have the following updating procedures:

1. Simulate $\sigma_{sj}^{(g+1)} = \sigma_{sj}^{(g)} + \gamma N(0,1)$, $\rho_s^{(g+1)} = \rho_s^{(g)} + \gamma N(0,1)$, for $j = 1, 2$.
2. Generate $U \sim$Uniform$(0,1)$.
3. Accept $\sigma_{sj}^{(g+1)}$ or $\rho_s^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(\sigma_{sj}^{(g+1)}|\cdot\right)/p\left(\sigma_{sj}^{(g)}|\cdot\right)$ or $p\left(\rho_s^{(g+1)}|\cdot\right)/p\left(\rho_s^{(g)}|\cdot\right)$, respectively.

- Posterior for δ_s (MH) with $\delta_{sj} \sim N(0, 100)$, for $s = 1$ (we have similar results for $s = 2$)

$$p(\delta_{sj}|\cdot) \propto \exp\left\{-\sum_{m=1}^{M}\sum_{t=1}^{T}\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\left\{-\frac{1}{2}\frac{(\delta_{sj})^2}{\sigma_m^2}\right\}$$

We have the following updating procedures:

1. Simulate $\delta_{sj}^{(g+1)} = \delta_{sj}^{(g)} + \gamma N(0, 1)$, for $j = 0, \pi, g$.
2. Generate U ~Uniform$(0, 1)$.
3. Accept $\delta_{sj}^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(\delta_{sj}^{(g+1)}|\cdot\right)/p\left(\delta_{sj}^{(g)}|\cdot\right)$.

- Posterior of λ_0^s (MH) with $\lambda_{j0}^s \sim N(0, 100)$, for $s = 1$ (we have similar results for $s = 2$)

$$p\left(\lambda_{j0}^{(1)}\right) \propto \exp\left\{-\frac{1}{2}\sum_{t=1,\text{ s.t. } s_{t+1}=1}^{T-1}\left(\varepsilon_t^{(1)}\right)^{\top}\Phi_1^{-1}\varepsilon_t^{(1)}\right\}\exp\left\{-\frac{1}{2}\frac{\lambda_{j0}^2}{\sigma_m^2}\right\},\ j = 1, 2$$

where $\Phi_1 = \Delta\Sigma_1\Sigma_1^{\top}$ and $\varepsilon_t^{(1)} = z_{t+1} - z_t - \kappa_1^{\mathbb{P}}\left(\theta_1^{\mathbb{P}} - z_t\right)\Delta$. We have the following updating procedures:

1. Simulate $\left[\lambda_{j0}^{(1)}\right]^{(g+1)} = \left[\lambda_{j0}^{(1)}\right]^{(g)} + \gamma N(0, 1)$, for $j = 1, 2$.
2. Generate U ~Uniform$(0, 1)$.
3. Accept $\left[\lambda_{j0}^{(1)}\right]^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(\left[\lambda_{j0}^{(1)}\right]^{(g+1)}|\cdot\right)/p\left(\left[\lambda_{j0}^{(1)}\right]^{(g)}|\cdot\right)$.

- Posterior of λ_{ij}^s (MH) with $\lambda_{ij}^s \sim N(0, 100)$, the procedure is the same as that of λ_0^s.
- Posterior of Λ_{jj} (MH) with $\Lambda_{jj} \sim N(0, 100)$,

$$p(\Lambda_{jj}|\cdot) \propto \prod_{t=1}^{T-1}\exp\{\Lambda Q\Delta\}_{s_t, s_{t+1}}\exp\left\{-\frac{1}{2}\frac{\Lambda_{jj}^2}{\sigma_m^2}\right\}, j = 1, 2$$

We have the following updating procedures:

1. Simulate $\Lambda_{jj}^{(g+1)} = \Lambda_{jj}^{(g)} + \gamma N(0, 1)$, for $j = 1, 2$.
2. Generate U ~Uniform$(0, 1)$.
3. Accept $\Lambda_{jj}^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(\Lambda_{jj}^{(g+1)}|\cdot\right)/p\left(\Lambda_{jj}^{(g)}|\cdot\right)$.

- Posterior of q_{jj} (MH) with $q_{jj} \sim N(0,100)$,

$$p(q_{jj}|\cdot) \propto \exp\left\{-\sum_{m=1}^{M}\sum_{t=1}^{T}\frac{\left[y_t^{(m)}-\hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\prod_{t=1}^{T-1}\exp\{\Lambda Q\Delta\}_{s_t,s_{t+1}}\exp\left\{-\frac{1}{2}\frac{q_{jj}^2}{\sigma_m^2}\right\}, j=1,2$$

We have the following updating procedures:

1. Simulate $q_{jj}^{(g+1)} = q_{jj}^{(g)} + \gamma N(0,1)$, for $j = 1, 2$.
2. Generate $U \sim$Uniform$(0, 1)$.
3. Accept $q_{jj}^{(g+1)}$ if $u < \alpha$ where $\alpha = p\left(q_{jj}^{(g+1)}|\cdot\right)/p\left(q_{jj}^{(g)}|\cdot\right)$.

- Posterior for s_t (MH)

$$\begin{aligned} p(s_t|\cdot) &\propto \exp\left\{-\sum_{m=1}^{M}\frac{\left[y_t^{(m)}-\hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\{\Lambda Q\Delta\}_{s_{t-1},s_t}\exp\{\Lambda Q\Delta\}_{s_t,s_{t+1}} \\ &\quad \frac{1}{|\Phi_{s_t}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\varepsilon_{t-1}^{s_t}\right)^\top\Phi_{s_t}^{-1}\varepsilon_{t-1}^{s_t}\right\}. \end{aligned}$$

Thus, for $1 < t < T$,

$$\begin{aligned} p(s_t=1|\cdot) &\propto \exp\left\{-\sum_{m=1}^{M}\frac{\left[y_t^{(m)}-\hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\{\Lambda Q\Delta\}_{s_{t-1},1}\exp\{\Lambda Q\Delta\}_{1,s_{t+1}} \\ &\quad \frac{1}{|\Phi_1|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\varepsilon_{t-1}^{(1)}\right)^\top\Phi_1^{-1}\varepsilon_{t-1}^{(1)}\right\} \equiv \alpha_1. \end{aligned}$$

$$\begin{aligned} p(s_t=2|\cdot) &\propto \exp\left\{-\sum_{m=1}^{M}\frac{\left[y_t^{(m)}-\hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\{\Lambda Q\Delta\}_{s_{t-1},2}\exp\{\Lambda Q\Delta\}_{2,s_{t+1}} \\ &\quad \frac{1}{|\Phi_2|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\varepsilon_{t-1}^{(2)}\right)^\top\Phi_2^{-1}\varepsilon_{t-1}^{(2)}\right\} \equiv \alpha_2. \end{aligned}$$

We first simulate $z \sim$Bernoulli$\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)$, then $s_t = 1\ (2)$, if $z = 1\ (0)$.
For $t = 1$,

$$p(s_1|\cdot) \propto \exp\left\{-\sum_{m=1}^{M}\frac{\left[y_1^{(m)}-\hat{y}_{s_1}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\{\Lambda Q\Delta\}_{s_1,s_2}.$$

For $t = T$:

$$p(s_T|\cdot) \propto \exp\left\{-\sum_{m=1}^{M}\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}\exp\{\Lambda Q\Delta\}_{s_{T-1},s_T}$$
$$\frac{1}{|\Phi_{S_T}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\varepsilon_{t-1}^{S_T}\right)^{\top}\Phi_{s_T}^{-1}\varepsilon_{T-1}^{S_T}\right\}.$$

A.1.4 Computing the Bayes Factor

In this section, we provide the details on how to compute the Bayes factor for model comparison. One key challenge is to obtain the marginal likelihood of the data by integrating out all model parameters and latent regime variables. We discuss our implementation details in three steps: (i) how to obtain $p(\mathbf{D}|\Theta)$ assuming we know how to simulate s_t from $p\left(s_t|\mathcal{F}_{t-1},\Theta^{(g)}\right)$, (ii) how to simulate s_t from $p\left(s_t|\mathcal{F}_t,\Theta^{(g)}\right)$, and (iii) how to obtain $p(\mathbf{D})$.

How to obtain $p(\mathbf{D}|\Theta)$ Assuming we know how to generate $s_t^{(k)}$ $(k = 1,2,...,K)$ from $p\left(s_t|\mathcal{F}_{t-1},\Theta^{(g)}\right)$ for $t = 1,2,...,T$, we can use the following procedures to compute $p(\mathbf{D}|\Theta)$:

1. For each t and k, simulate $s_t^{(k)} \sim p\left(s_t|s_{t-1}^{(k)},\Theta^{(g)}\right) \propto \exp\{\Lambda Q\Delta\}_{2\times 2}$.

2. Integrate out the latent regime variables

$$\begin{aligned} p\left(z_t|\mathcal{F}_{t-1},\Theta^{(g)}\right) &= \int p\left(z_t|s_t,z_{t-1},\Theta^{(g)}\right)p\left(s_t|\mathcal{F}_{t-1},\Theta^{(g)}\right)ds_t \\ &= \frac{1}{K}\sum_{k=1}^{K}p\left(z_t|s_t^{(k)},z_{t-1},\Theta^{(g)}\right), \end{aligned}$$

   where

$$\sum_{k=1}^{K}p\left(z_t|s_t^{(k)},z_{t-1},\Theta^{(g)}\right) = \prod_{m=1}^{M}\frac{1}{\sqrt{2\pi}\sigma_m}\exp\left\{-\frac{\left[y_t^{(m)} - \hat{y}_{s_t}^{(m)}\right]^2}{2\sigma_m^2}\right\}.$$
$$\frac{1}{2\pi|\Phi_{s_t}|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(\varepsilon_t^{s_{t+1}}\right)^{\top}\Phi_{s_{t+1}}^{-1}\varepsilon_t^{s_{t+1}}\right\}.$$

3. Apply filtering to obtain $s_t^{(1)}, s_t^{(2)}, ..., s_t^{(K)}$ from $p\left(s_t|\mathcal{F}_{t-1},\Theta^{(g)}\right)$; if $t < T-1$, $t = t+1$, and go back to Step 1.

4. If $t = T$, then $p\left(\mathbf{D}|\Theta^{(g)}\right) = \prod_{t=2}^{T}p\left(z_t|\mathcal{F}_{t-1},\Theta^{(g)}\right)$.

Simulate s_t from $p\left(s_t|\mathcal{F}_t,\Theta^{(g)}\right)$ Given $s_t^{(1)},s_t^{(2)},...,s_t^{(K)}$ from $p\left(s_{t-1}|\mathcal{F}_{t-1},\Theta^{(g)}\right)$, we have

$$\begin{aligned} p\left(s_t|\mathcal{F}_t,\Theta^{(g)}\right) &\propto p\left(s_t|z_t,\mathcal{F}_{t-1},\Theta^{(g)}\right) \propto p\left(s_t,z_t|\mathcal{F}_{t-1},\Theta^{(g)}\right) \\ &= p\left(z_t|s_t,z_{t-1},\Theta^{(g)}\right) p\left(s_t|\mathcal{F}_{t-1},\Theta^{(g)}\right), \end{aligned}$$

where

$$\begin{aligned} p\left(s_t|\mathcal{F}_{t-1},\Theta^{(g)}\right) &= \int p\left(s_t|s_{t-1},\Theta^{(g)}\right) p\left(s_{t-1}|\mathcal{F}_{t-1},\Theta^{(g)}\right) ds_{t-1} \\ &= \frac{1}{K}\sum_{k=1}^{K} p\left(s_t|s_{t-1}^{(k)},\Theta^{(g)}\right). \end{aligned}$$

Therefore, we have

$$p\left(s_t|\mathcal{F}_t,\Theta^{(g)}\right) \propto p\left(z_t|s_t,z_{t-1},\Theta^{(g)}\right)\frac{1}{K}\sum_{k=1}^{K} p\left(s_t|s_{t-1}^{(k)},\Theta^{(g)}\right),$$

based on which we can simulate $s_t^{(1)},s_t^{(2)},...,s_t^{(K)}$ from $p\left(s_t|\mathcal{F}_t,\Theta^{(g)}\right)$ based on the following procedures for each $k=1,2,...,K$:

1. Define

$$\begin{aligned} \alpha_1 &= p\left(z_t|1,z_{t-1},\Theta^{(g)}\right)\frac{1}{K}\sum_{k=1}^{K}\exp\left\{\Lambda^{(g)}Q^{(g)}\Delta\right\}_{s_{t-1},1}, \\ \alpha_2 &= p\left(z_t|2,z_{t-1},\Theta^{(g)}\right)\frac{1}{K}\sum_{k=1}^{K}\exp\left\{\Lambda^{(g)}Q^{(g)}\Delta\right\}_{s_{t-1},2}. \end{aligned}$$

2. Draw $u\sim$ Bernoulli $\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)$, if $u=1$, then $s_t^{(k)}=1$; if $u=0$,then $s_t^{(k)}=2$.

Computing $p(\mathbf{D})$ Once we know how to estimate $p(\mathbf{D}|\Theta)$, we can compute $p(\mathbf{D})$ by integrating out the parameters Θ

$$p(\mathbf{D}) = \int p(\mathbf{D}|\Theta)\,\pi(\Theta)\,d\Theta,$$

where $\pi(\Theta)$ is the prior of Θ. In practice, we estimate $p(\mathbf{D})$ using an importance function $\pi^*(\Theta)$

$$p(\mathbf{D}) = \frac{\sum_{g=1}^{G} w_g p\left(\mathbf{D}|\Theta^{(g)}\right)}{\sum_{g=1}^{G} w_g},$$

where $w_g=\frac{\pi\left(\Theta^{(g)}\right)}{\pi^*\left(\Theta^{(g)}\right)}$, $\Theta^{(g)}\sim\pi^*(\Theta)$, and G is the number of simulations.

The challenge is to choose the right importance function. Following Kass and Raftery (1995), we define

$$\pi^*(\Theta) = p(\Theta|\mathbf{D}) = \frac{p(\mathbf{D}|\Theta)\,\pi(\Theta)}{p(\mathbf{D})}.$$

Consequently, we have

$$
\begin{aligned}
p(\mathbf{D}) &= \frac{\sum_{g=1}^{G} \frac{\pi\left(\Theta^{(g)}\right)}{p\left(\mathbf{D}|\Theta^{(g)}\right)\pi\left(\Theta^{(g)}\right)} p(\mathbf{D})\, p\left(\mathbf{D}|\Theta^{(g)}\right)}{\sum_{g=1}^{G} \frac{\pi\left(\Theta^{(g)}\right)}{p\left(\mathbf{D}|\Theta^{(g)}\right)\pi\left(\Theta^{(g)}\right)} p(\mathbf{D})} \\
&= \frac{Gp(\mathbf{D})}{p(\mathbf{D}) \sum_{g=1}^{G} p\left(\mathbf{D}|\Theta^{(g)}\right)^{-1}} = \left\{ \frac{1}{G} \sum_{g=1}^{G} p\left(\mathbf{D}|\Theta^{(g)}\right)^{-1} \right\}^{-1}.
\end{aligned}
$$

One advantage of this approach is that the posterior samples of $\Theta^{(g)}$ can be directly used to estimate $p(\mathbf{D})$.

A.2 Accuracy of the Approximation

Bond pricing under our regime-switching model depends on the linear approximation $e^{(B_j - B_i)^\top z} \approx 1 + z^\top (B_j - B_i)$. In this section, we illustrate the accuracy of the approximation given the specific data and models considered in our analysis.

In Figure 8, we first plot $A(\tau)$, $B_\pi(\tau)$, and $B_g(\tau)$ as a function of τ under the single-regime model M1. We see that the slope of $B_\pi(\tau)$ is much steeper than that of $A(\tau)$ and $B_g(\tau)$. This suggests that most of the discount is due to inflation. We also plot $A_i(\tau)$, $B_{i\pi}(\tau)$, and $B_{ig}(\tau)$ as a function of τ for $i = 1, 2$ under M3(π, g, δ). The slopes of $A_i(\tau)$ and $B_{i\pi}(\tau)$ are much steeper under the first regime than under the second regime, while the opposite is true for $B_{ig}(\tau)$. Due to the exponentially affine relation of bond yields to inflation and output gap, the slope of $B(\tau)$ measures the effect of inflation and output gap on the bond yields. Therefore, the above results show that higher inflation should lead to higher bond yields under the first regime than under the second regime. The reason is that under the first regime, the Fed is more aggressive in fighting inflation and thus increases the spot rate more for a given increase in inflation. On the other hand, higher output gap leads to higher bond yield under the second regime than under the first regime, because under the second regime, the Fed is more responsive to output gap.

The accuracy of the approximation depends on the magnitude of $(B_2(\tau) - B_1(\tau))^\top z$. From Figure 8 we see that the maximum difference between $B_{2\pi}(\tau) - B_{1\pi}(\tau)$ is about 3, while the maximum value of π (from Figure 1) is a little above 0.1. The approximation error for an exponential function at 0.3 by a first-order Taylor expansion is quite small. Similarly, the maximum difference between $B_{2g}(\tau) - B_{1g}(\tau)$ is about 1, while the maximum value of g (from Figure 1) is about 0.05, which again leads to small approximation errors.

Table 1: **Regression Analysis of Observed Yields on Contemporaneous Inflation and Output Gap**

This table provides the regression analysis of observed zero-coupon government bond yields on contemporaneous inflation and output gap. The regression equation is

$$\text{Observed Yields} = \gamma_0 + \gamma_1 \text{Inflation} + \gamma_2 \text{ Output Gap} + \text{error},$$

with standard errors reported in parentheses. The maturities of the bonds range from one quarter (1Q) to five years (20Q).

Bond Maturity	γ_0	γ_1	γ_2	$\mathbf{R^2}$
1Q	0.0214 (0.0025)	0.8504 (0.0604)	0.2132 (0.0898)	48.3%
4Q	0.0259 (0.0026)	0.8371 (0.0618)	0.1758 (0.0920)	46.1%
8Q	0.0290 (0.0026)	0.8049 (0.0628)	0.0981 (0.0934)	43.1%
12Q	0.0321 (0.0026)	0.7634 (0.0627)	0.0385 (0.0933)	40.4%
16Q	0.0340 (0.0026)	0.7429 (0.0629)	0.0059 (0.0936)	38.9%
20Q	0.0359 (0.0026)	0.7236 (0.0623)	-0.0185 (0.0927)	38.1%

Table 2: **Parameter Estimates for the Single-Regime Model M1**

This table provides the empirical estimates of the model parameters for the single-regime model. We run MCMC with 50,000 iterations and use the posterior mean (standard deviation) of the last 30,000 iterations as estimates of the model parameter (standard error, shown in parentheses).

Parameter	MCMC estimates	
$\sigma_{m=1,\cdots,6}$	0.0206	(0.0010)
	0.0211	(0.0010)
	0.0213	(0.0010)
	0.0213	(0.0010)
	0.0214	(0.0010)
	0.0212	(0.0010)
$\theta^{\mathbb{P}}$	0.0303	(0.0335)
	-0.0003	(0.0169)
σ_1	0.0071	(0.0003)
σ_2	0.0167	(0.0008)
ρ	0.0204	(0.0682)
$\kappa^{\mathbb{P}}$	0.0536	-0.4602
	(0.0356)	(0.0527)
	0.1341	0.7308
	(0.0787)	(0.1079)
δ	0.0187	(0.0026)
	0.9055	(0.0523)
	0.1863	(0.1097)
λ_0	-0.0121	(0.0103)
	0.0106	(0.1056)
λ	0.0160	0.4786
	(0.0413)	(0.1599)
	0.4514	0.2458
	(0.0614)	(0.0699)

Table 3: **Regression Analysis of Observed Yields on Model Yields for the Single-Regime Model M1**

This table provides the regression analysis of observed zero-coupon government bond yields on model-implied yields under the single-regime model at different maturities. The regression equation is

$$\text{Observed Yields} = \gamma_0 + \gamma_1 \text{Model Yields} + \text{error},$$

where the model yields are computed based on estimated parameters in the previous table. Standard errors are reported in parentheses. The maturities of the bonds range from one quarter (1Q) to five years (20Q).

Bond Maturity	γ_0	γ_1	Standard Deviation of Residuals	$\mathbf{R^2}$
1Q	0.0019 (0.0038)	0.9660 (0.068)	0.0207	48.2%
4Q	0.0015 (0.0042)	1.0053 (0.074)	0.0212	46.0%
8Q	-0.0011 (0.0048)	1.0206 (0.080)	0.0215	43.1%
12Q	-0.0027 (0.0053)	1.0115 (0.083)	0.0214	40.4%
16Q	-0.0060 (0.0058)	1.0230 (0.087)	0.0215	38.9%
20Q	-0.0095 (0.0063)	1.0318 (0.089)	0.0213	38.1%

Table 4: **Parameter Estimates of the Two-Regime Model M2(δ)**

This table provides the empirical estimates of the model parameters for the two-regime model M2(δ), where only monetary policies, i.e., δ, are regime-dependent. We run MCMC with 50,000 iterations and use the posterior mean (standard deviation) of the last 30,000 iterations as estimates of the model parameters (standard error, shown in parentheses).

Regime-Independent Parameters		
Parameter	MCMC estimates	
$\sigma_{m=1,\cdots,6}$	0.0122	(0.0006)
	0.0117	(0.0006)
	0.0113	(0.0005)
	0.0114	(0.0005)
	0.0115	(0.0005)
	0.0116	(0.0006)
$\theta^{\mathbb{P}}$	0.0461	(0.0253)
	-0.0008	(0.0027)
σ_1	0.0070	(0.0003)
σ_2	0.0165	(0.0008)
ρ	0.0376	(0.0675)
$\kappa^{\mathbb{P}}$	0.0566	-0.4691
	(0.0530)	(0.0641)
	0.0854	0.6956
	(0.0772)	(0.1263)
λ_0	-0.0042	(0.0026)
	0.0010	(0.0090)
λ	-0.0080	0.5424
	(0.0774)	(0.0848)
	0.1197	-0.4546
	(0.2311)	(0.1865)
Λ_{11}	3.7675	(0.4594)
Λ_{22}	3.2438	(0.3162)
q_{11}	-0.0556	(0.0244)
q_{22}	-0.0312	(0.0151)

Regime-Dependent Parameters				
Parameter	MCMC estimates			
	Regime 1		Regime 2	
δ	0.0246	(0.0026)	0.0106	(0.0019)
	1.5165	(0.0726)	0.8568	(0.0403)
	0.1250	(0.0962)	0.3629	(0.0550)

Table 5: **Regression Analysis of Observed Yields on Model Yields for the Two-Regime Model M2(δ)**

This table provides the regression analysis of observed zero-coupon government bond yields on model-implied yields under the two-regime model M2(δ) at different maturities. The regression equation is

$$\text{Observed Yields} = \gamma_0 + \gamma_1 \text{Model Yields} + \text{error},$$

where the model yields are computed based on the estimated parameters in the previous table. Standard errors are reported in parentheses. The maturities of the bonds range from one quarter (1Q) to five years (20Q).

Bond Maturity	γ_0	γ_1	Standard Deviation of Residuals	$\mathbf{R^2}$
1Q	0.0001 (0.0018)	0.9686 (0.0300)	0.0119	82.7%
4Q	0.0007 (0.0042)	0.9990 (0.0298)	0.0116	83.7%
8Q	-0.0005 (0.0019)	1.0136 (0.0296)	0.0113	84.4%
12Q	-0.0005 (0.0020)	1.0013 (0.0304)	0.0113	83.3%
16Q	-0.0017 (0.0021)	1.0022 (0.0313)	0.0115	82.5%
20Q	-0.0028 (0.0022)	0.9955 (0.0318)	0.0115	81.8%

Table 6: **Parameter Estimates of the Two-Regime Model M3**(π, g, δ)

This table provides the empirical estimates of the model parameters for the two-regime model M3($\boldsymbol{\pi}$, $\mathbf{g}$, δ), where both monetary policies and macro variables are regime-dependent. We run MCMC with 50,000 iterations and use the posterior mean (standard deviation) of the last 30,000 iterations as the estimates of model parameters (standard error, shown in parentheses).

Regime-Dependent Parameters				
Parameter	MCMC estimates			
	Regime 1		Regime 2	
$\theta^{\mathbb{P}}$	0.0216	(0.0032)	0.0658	(0.0272)
	0.0040	(0.0026)	-0.0028	(0.0033)
σ_1	0.0046	(0.0004)	0.0076	(0.0005)
σ_2	0.0126	(0.0013)	0.0183	(0.0010)
ρ	0.1678	(0.1300)	-0.0407	(0.0802)
$\kappa^{\mathbb{P}}$	0.3558	-0.1808	0.0166	-0.5245
	(0.0592)	(0.0763)	(0.0465)	(0.0795)
	0.5676	1.0773	0.0703	0.6457
	(0.1689)	(0.1433)	(0.0649)	(0.1866)
δ	0.0187	(0.0040)	0.0135	(0.0029)
	1.5060	(0.0938)	0.7286	(0.0820)
	0.2489	(0.1459)	0.4445	(0.1080)
λ_0	-0.0152	(0.0060)	0.0593	(0.0115)
	-0.1221	(0.0387)	-0.0141	(0.0294)
λ	-0.7408	0.4862	2.4259	-0.3400
	(0.1033)	(0.1562)	(0.2816)	(0.2004)
	0.6052	2.2004	-1.9997	0.8512
	(0.6966)	(0.7610)	(0.4413)	(0.4182)

Regime-Independent Parameters		
Parameter	MCMC estimates	
Λ_{11}	0.2573	(0.1140)
Λ_{22}	0.0587	(0.0280)
q_{11}	-1.0149	(0.2854)
q_{22}	-2.6626	(0.2804)
$\sigma_{m=1,\cdots,6}$	0.0120	(0.0006)
	0.0117	(0.0006)
	0.0113	(0.0005)
	0.0113	(0.0005)
	0.0115	(0.0005)
	0.0115	(0.0005)

Table 7: **Regression Analysis of Observed Yields on Model Yields for the Two-Regime Model M3(π, g, δ)**

This table provides the regression analysis of observed zero-coupon government bond yields on model-implied yields under the two-regime model M3($\boldsymbol{\pi}$, $\mathbf{g}$, δ) at different maturities. The regression equation is

$$\text{Observed Yields} = \gamma_0 + \gamma_1 \text{Model Yields} + \text{error},$$

where the model yields are computed based on the estimated parameters in the previous table. Standard errors are reported in parentheses. The maturities of the bonds range from one quarter (1Q) to five years (20Q).

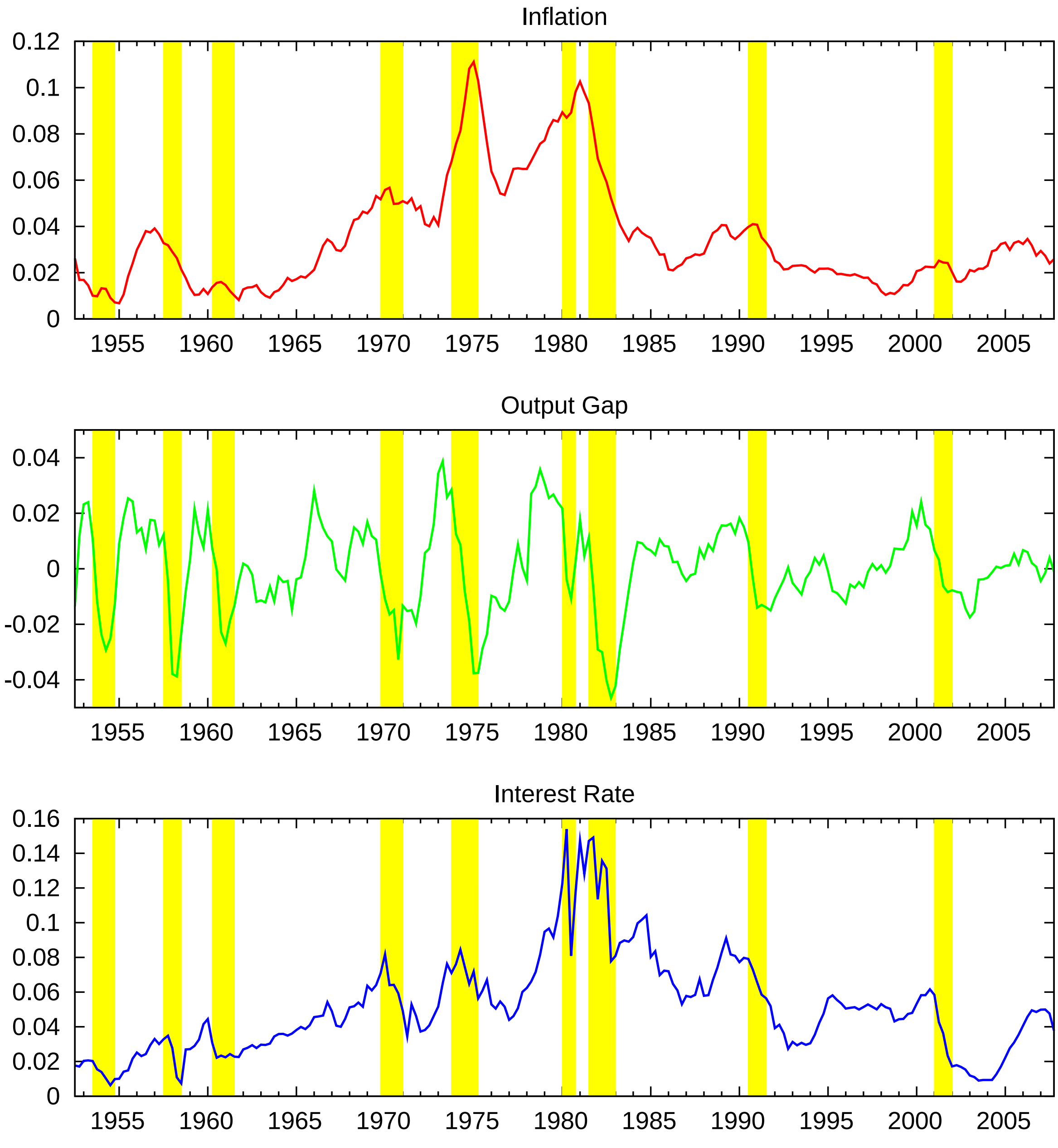
Bond Maturity	γ_0	γ_1	Standard Deviation of Residuals	$\mathbf{R^2}$
1Q	-0.0007 (0.0018)	1.0063 (0.0312)	0.0119	82.7%
4Q	0.0007 (0.0018)	0.9911 (0.0297)	0.0116	83.7%
8Q	-0.0003 (0.0018)	1.0021 (0.0293)	0.0113	84.3%
12Q	0.0003 (0.0019)	0.9968 (0.0303)	0.0113	83.3%
16Q	-0.0004 (0.0021)	1.0023 (0.0317)	0.0115	82.8%
20Q	-0.0011 (0.0021)	1.0278 (0.0330)	0.0116	81.7%

Table 8: **Model Comparison Using the Bayes Factor**

This table provides the results on the model comparison using the Bayes factor, which calculates the posterior odds ratio between two models. Below is the rule of thumb when interpreting the Bayes factor: 1-3.2 barely worth mentioning; 3.2-10 substantial; 10-100 strong; and >100 decisive.

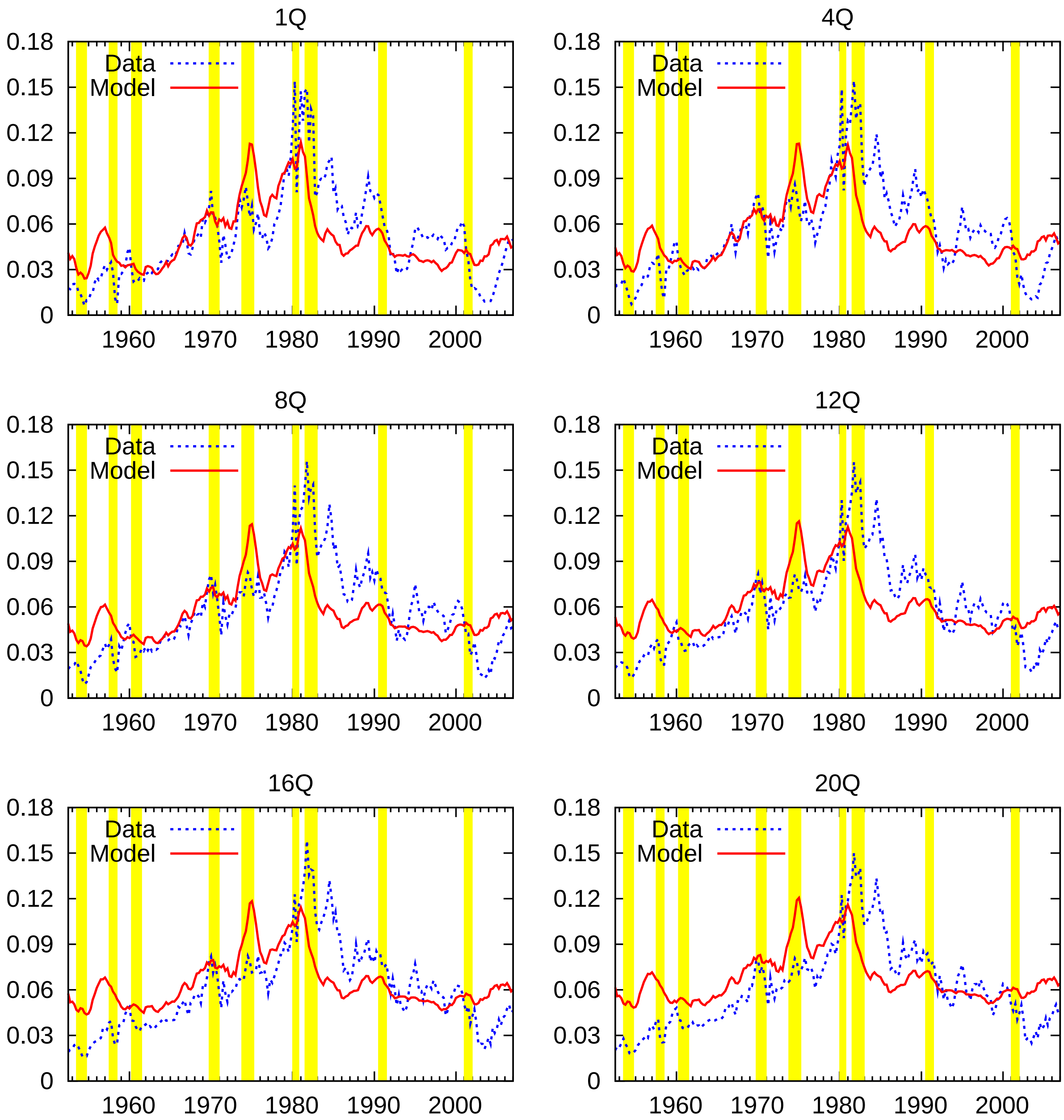
Pairs of Models	Bayes Factor
M2(δ) vs. M1	347.49
M3(π, g, δ) vs. M2(δ)	16.24

Figure 1: **Inflation, Output Gap, and Three-Month Government Bond Yields.**



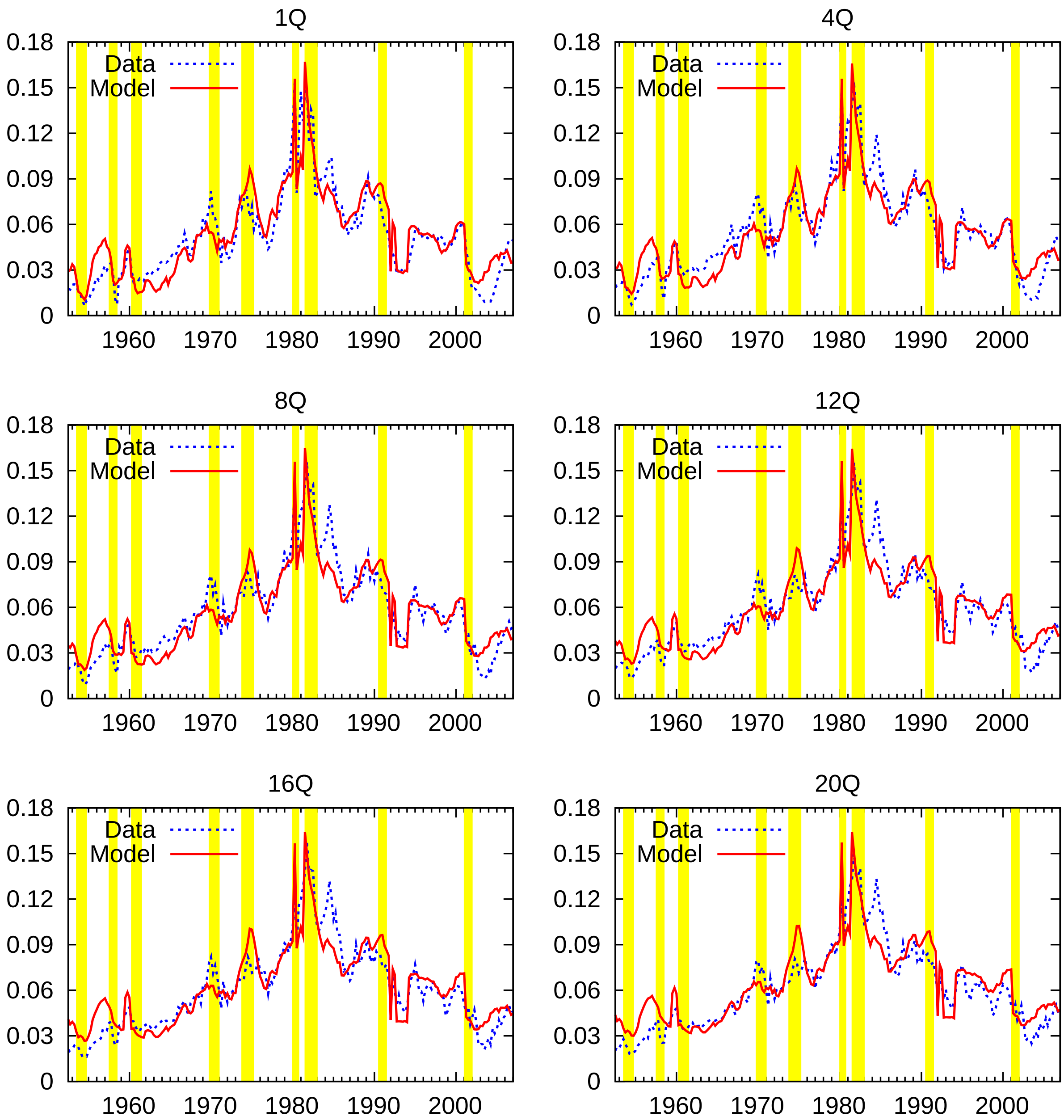
This figure provides the time series plots of inflation, output gap, and three-month government bond yields observed at a quarterly frequency from the second quarter of 1952 to the third quarter of 2007. The shaded areas represent the periods of NBER recessions.

Figure 2: **Observed and Model-Implied Yields under the Single-Regime Model M1.**



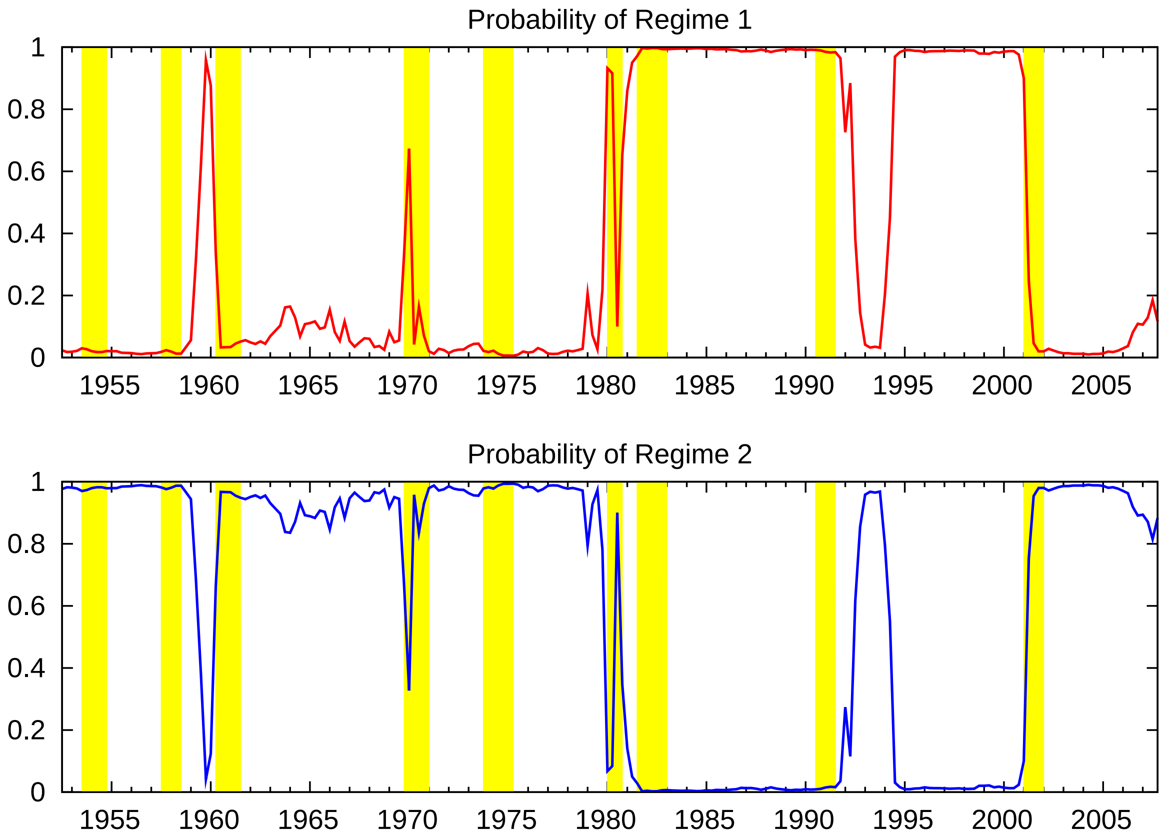
This figure provides the time series plots of observed yields on zero-coupon government bonds at six different maturities and model-implied yields under the single-regime model M1. Model yields are calculated given the estimated model parameters using the Bayesian MCMC methods. The shaded areas represent the periods of NBER recessions.

Figure 3: **Observed and Model-Implied Yields under the Two-Regime Model M2(δ).**



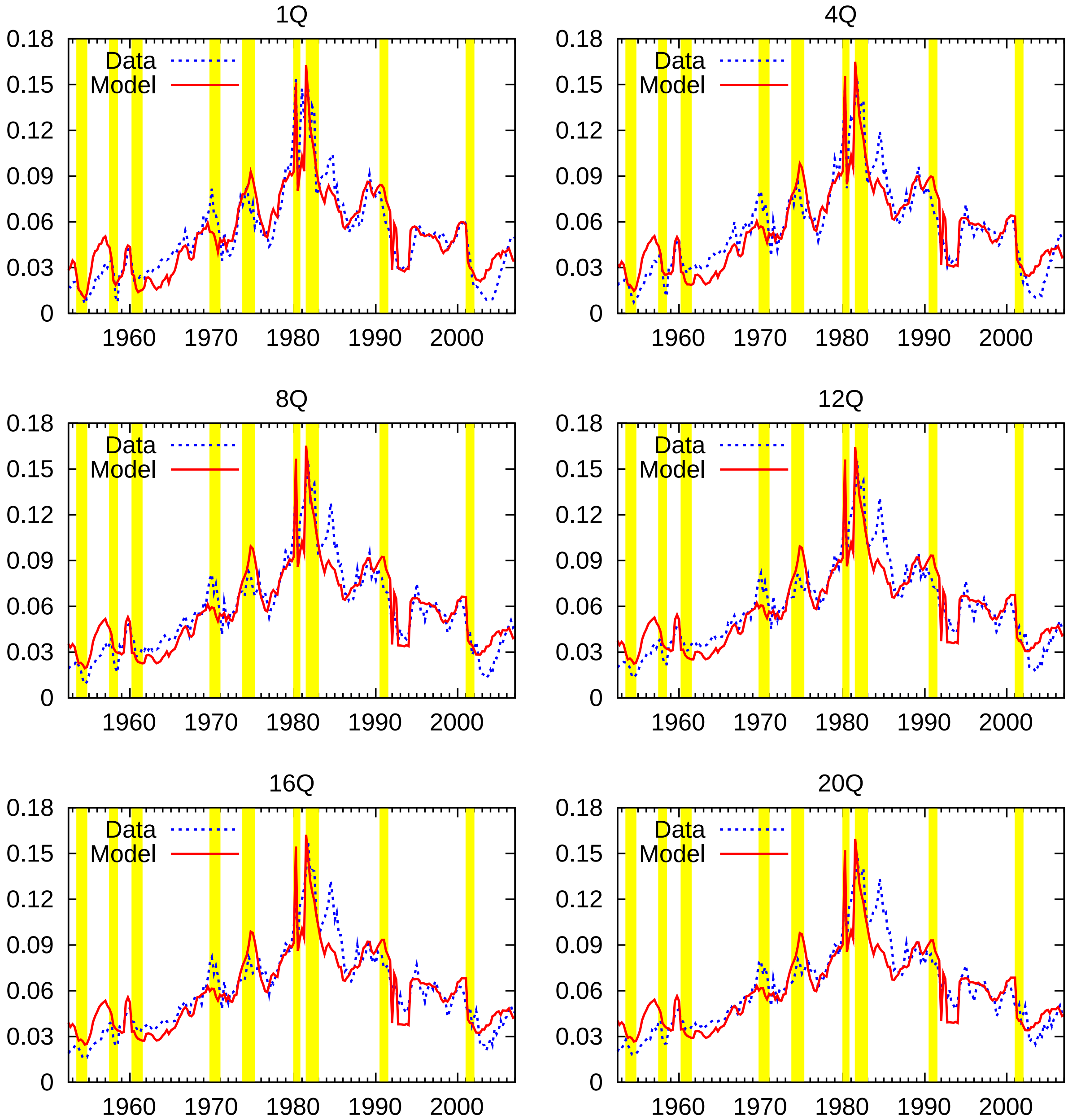
This figure provides the time series plots of observed yields on zero-coupon government bonds at six different maturities and model-implied yields under the two-regime model M2(δ), where only monetary policies are regime-dependent. Model yields are calculated given the estimated model parameters using the Bayesian MCMC methods. The shaded areas represent the periods of NBER recessions.

Figure 4: **Estimated Regime for the Two-Regime Model M2(δ).**



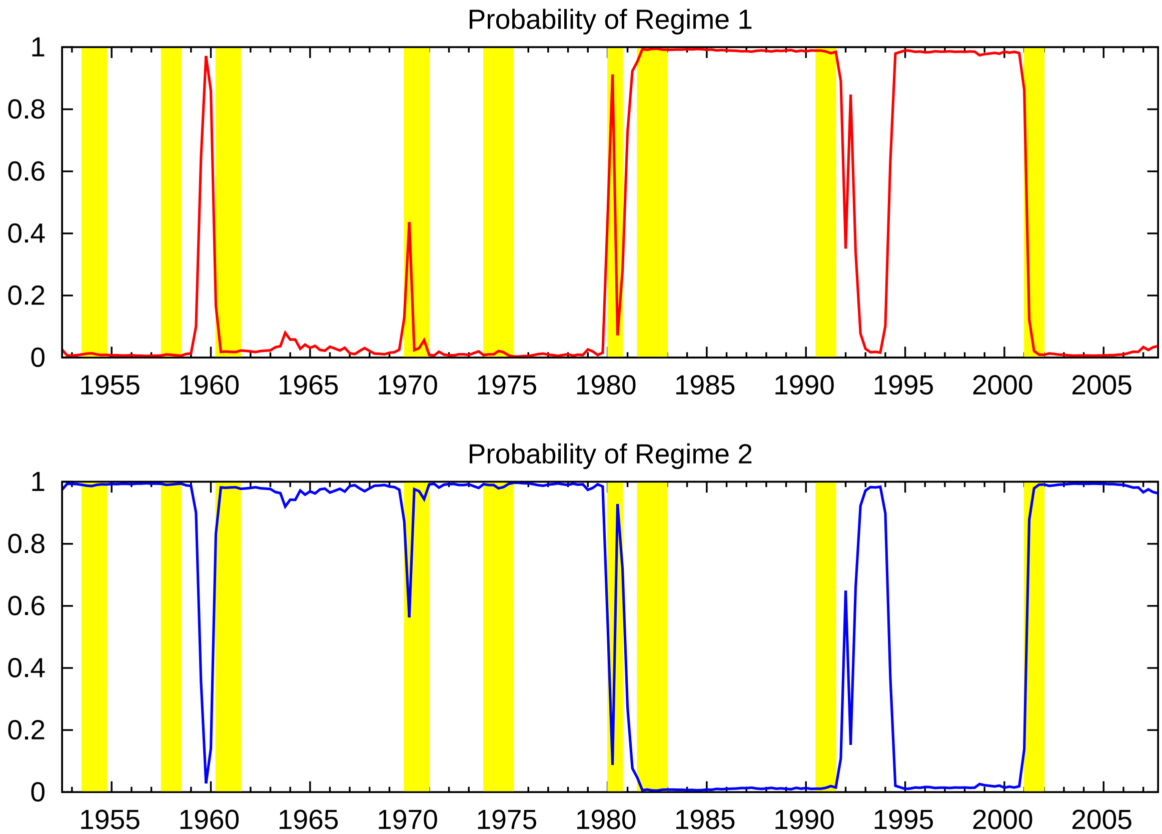
This figure provides the time series plots of the posterior probabilities that the economy is in regimes 1 and 2 under the two-regime model M2(δ), where only monetary policies are regime-dependent. The shaded areas represent the periods of NBER recessions.

Figure 5: **Observed and Model-Implied Yields under the Two-Regime Model M3(π, g, δ).**



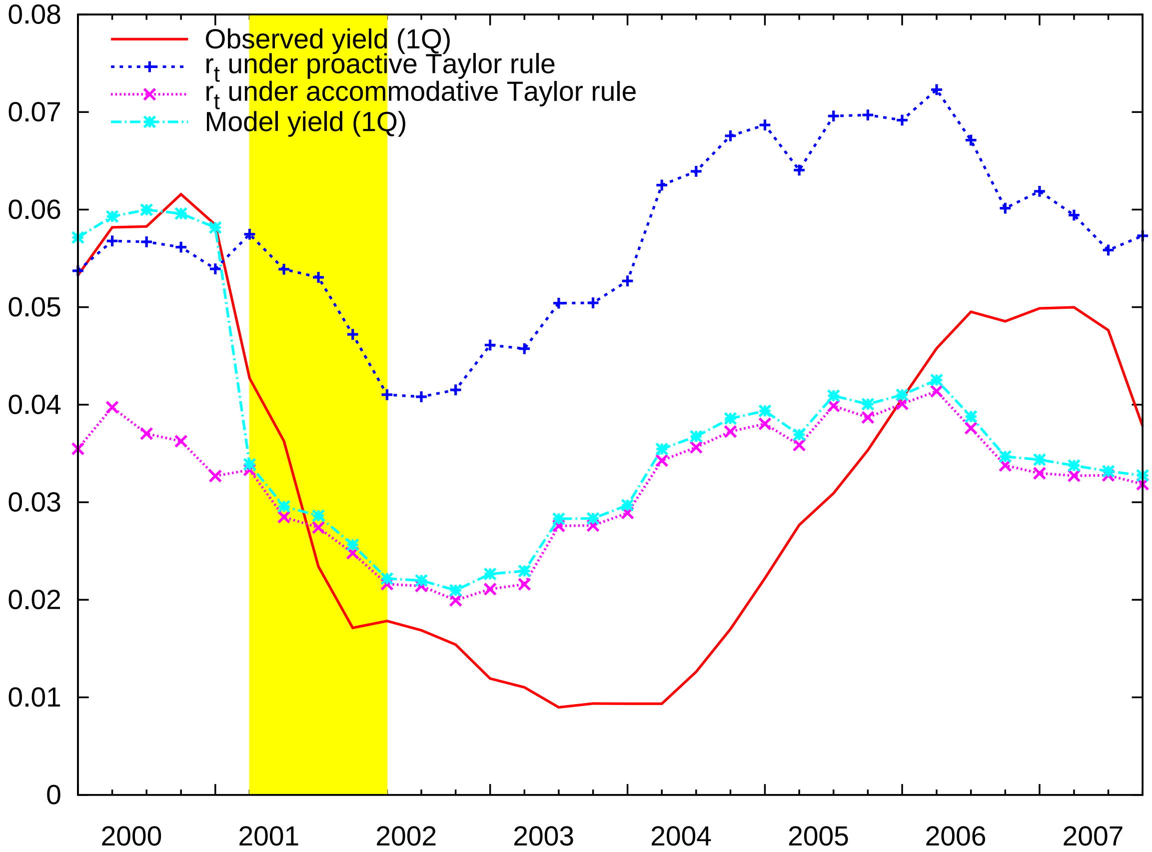
This figure provides the time series plots of observed yields on zero-coupon government bonds at six different maturities and model-implied yields under the two-regime model M3(π, g, δ), where both monetary policies and macro variables are regime-dependent. Model yields are calculated given the estimated model parameters using the Bayesian MCMC methods. The shaded areas represent the periods of NBER recessions.

Figure 6: **Estimated Regime for the Two-Regime Model M3(π, g, δ).**



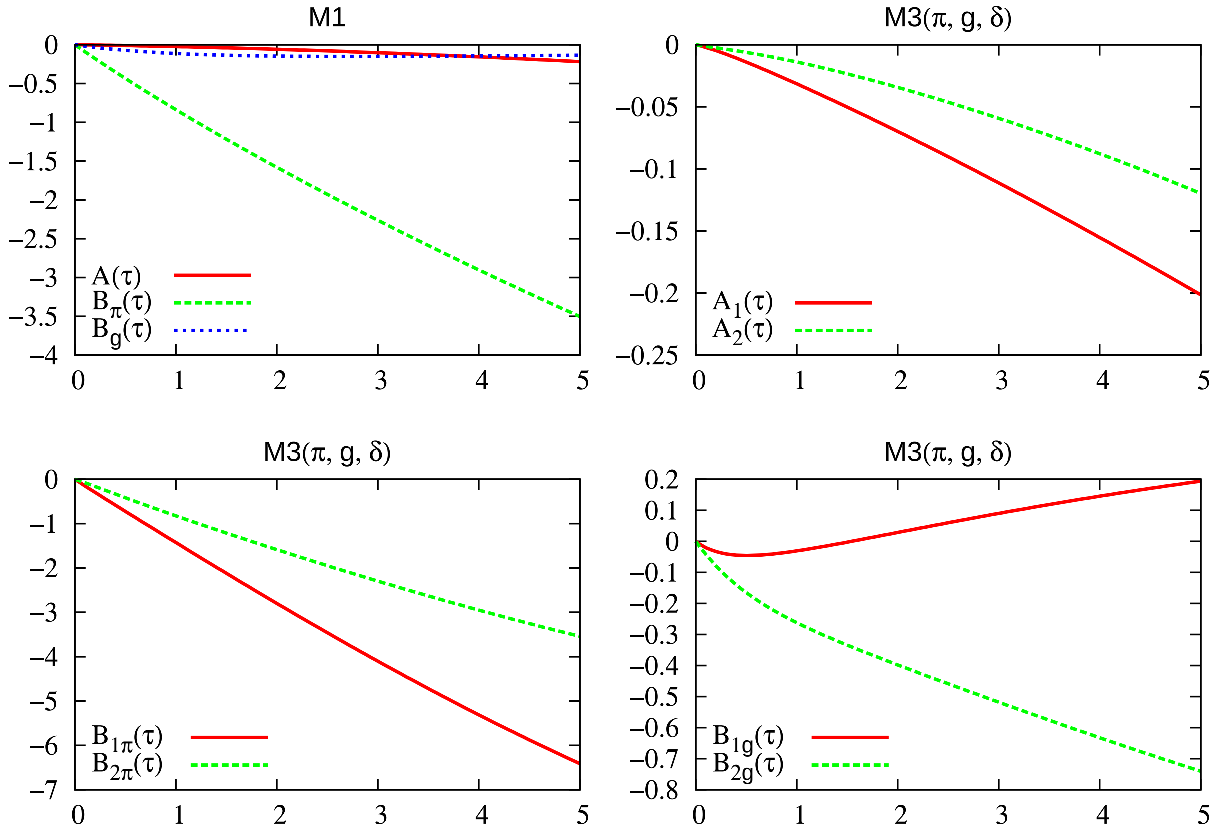
This figure provides the time series plots of the posterior probabilities that the economy is in regimes 1 and 2 under the two-regime model M3(π, g, δ), where both monetary policies and macro variables are regime-dependent. The shaded areas represent the periods of NBER recessions.

Figure 7: **Monetary policies after 2000.**



This figure plots the observed and model-implied three-month Treasury yields as well as the spot rate under aggressive and passive Taylor rules for the two-regime model M3(π, g, δ) after 2000. The shaded area represents the 2001 NBER recession.

Figure 8: **Accuracy of the Approximation.**



This figure plots $A(\tau)$, $B_\pi(\tau)$, and $B_g(\tau)$ under both the single-regime model M1 and the two-regime model M3(π, g, δ) with regime-dependent monetary policies and macroeconomic variables. Under the regime-switching model, we plot $A(\tau)$, $B_\pi(\tau)$, and $B_g(\tau)$ under two different regimes.