Dynamic Debt Runs*

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Abstract

We analyze a new source of debt runs generated by the coordination problem between creditors whose debt contracts with a firm mature at different times. In deciding whether to roll over his debt, each creditor faces the firm's future rollover risk with other creditors, i.e., the firm fundamental could fall during his contract period, causing other maturing creditors to run and thus forcing the firm to liquidate its asset at a fire sale price. We derive a unique monotone equilibrium, in which the creditors coordinate their asynchronous rollover decisions based on the publicly observable time-varying firm fundamental. Preemptive debt runs occur through a rat race among the creditors in choosing higher and higher fundamental thresholds for rolling over their debt contracts. Our model captures a central element in many crisis episodes–even in the absence of any fundamental deterioration, changes in the volatility and liquidation value of the firm asset could trigger preemptive runs by creditors on a solvent firm. Such preemptive runs originate from the lack of commitment from future maturing creditors to roll over their debt contracts, rather than the lack of communication between creditors in static models of runs.

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1 Introduction

Panic runs by creditors to flee ahead of others are commonly observed in financial crises. The collapse of investment bank Bear Stearns, a key event in the Wall Street crisis of 2008, provides a recent example. According to the former SEC chairman Christopher Cox, Bear Stearns experienced a panic run:[1]

> *"For the first time, a major investment bank that was well-capitalized and apparently fully liquid experienced a crisis of confidence that denied it not only unsecured financing, but short-term secured financing, even when the collateral consisted of agency securities with a market value in excess of the funds to be borrowed. Counterparties would not provide securities lending services and clearing services. Prime brokerage clients moved their cash balances elsewhere. These decisions by counterparties, clients, and lenders to no longer transact with Bear Stearns in turn influenced other counterparties, clients, and lenders to also reduce their exposure to Bear Stearns."*

The seed of this panic run was planted by Bear Stearns' reliance on rolling over short-term commercial paper and repo transactions to finance its investment in long-term risky assets such as mortgages. This type of rollover financing exposes Bear Stearns to rollover risk, i.e., the risk that a borrower may not be able to raise new funds to repay maturing short-term debt (Bernanke, 2009). In fact, the inability to roll over short-term debt has been described as one of the direct causes that had led to the collapse of the US investment banking system.[2] Interestingly, commercial banks also had similar problems, as revealed by the failure of UK bank Northern Rock, another high profile casualty of the financial crisis. In spite of the television images of long lines of depositors outside its branch offices, its demise was ultimately caused by the failure to roll over its short-term financing from institutional investors.[3] This type of panic runs caused by rollover risk is not a new phenomenon. The crisis of the hedge fund LTCM in 1998 also involved a run by its creditors and counterparties as it relied on repo transactions to finance its long-term arbitrage positions in various equity and interest rate securities. Even non-financial firms had experienced similar problems. The

[1] See the letter from Christopher Cox to the Basel Committee, which is available at http://www.sec.gov/news/press/2008/2008-48.htm.

[2] See Greenlaw et al (2008), Brunnermeier (2009), Gorton and Metrick (2009), and Krishnamurthy (2009) for comprehensive descriptions of the financial crisis of 2007-2008.

[3] See Shin (2009) for a vivid account of this episode.

Asian financial crisis in 1997 is widely attributed to the quick accumulation of short-term debt by the private sector of various Asian countries such as South Korea, Indonesia and Thailand, and to the panic runs by short-term creditors.[4]

To the extent that each of these crises had caused substantial premature liquidation of financial investment and misallocation of capital in the world economy, these crises must be driven by certain distortion in the financial system. What is the source of the distortion? The classic bank run model of Diamond and Dybvig (1983) attributes depositors' panic runs on a bank to the inability of depositors to coordinate their simultaneous withdrawal decisions. In this model, depositors hold demand deposits of a bank, which invests in a long-term asset. The long-term asset matures in two periods and can only be liquidated at a discount before maturity. The depositors, however, are free to withdraw their funds at the interim date, and their collective withdrawal can force a premature liquidation of the bank. As a result, two self-fulfulling equilibria emerge. In the good equilibrium, all depositors stay for long-term, while in the bad equilibrium, they simultaneously demand early withdrawal. This model nicely illustrates the coordination problem between the bank depositors generated by the duration mismatch between the bank's asset and liability.

The distortion could also arise from other channels. A potentially important channel is the coordination problem between creditors who have to make rollover decisions of their debt contracts with a firm at different times. In reality, firms are financed by debt contracts, which lock in the creditors during the contract periods. Different from bank depositors, a creditor can only choose to run after his contract matures. Furthermore, in practice, firms typically spread out their debt expirations over time to reduce liquidity risk.[5] In other words, a firm's debt contracts mature at different times. The lock-in effect of debt contracts together with the staggered debt structure imply that different creditors make rollover decisions at different times. As a result, each creditor in choosing whether to roll over his debt faces the risk that during his contract period, other maturing creditors could choose to run and eventually cause the firm to fail before he has another chance to get out. This concern about the firm's future rollover risk in turn motivates the creditor to run preemptively when he has the chance now. The rollover risk thus generates a realistic coordination problem between creditors across

[4] See, for example, Radelet and Sachs (1998) and Furman and Stiglitz (1998).

[5] For example, on February 10, 2009, the data from Bloomberg show that Morgan Stanley, one of the major U.S. investment banks, had short-term debt (with maturities less than 1.5 years) expiring on almost every day throughout February and March 2009. If we sum up the total value of Morgan Stanley's expiring short-term debt in each week, the values for the following five weeks are 62 million, 324 million, 339 million, 239 million, and 457 million, respectively. The Federal Reserve Release also shows that the commercial paper issued by financial firms in aggregate have maturities well spread out over time.

time. Despite its evident relevance to various crisis episodes, this coordination problem remains to be explored. Understanding this problem can help understand instability in the current financial system, as well as to design a better architecture in the future.

In this paper, we develop a parsimonious model in continuous time to analyze this problem. A firm, which shall be broadly interpreted as a financial or non-financial firm, finances its long-term investment by rolling over short-term debt. We assume that the capital markets are imperfect in three dimensions so that debt runs are a relevant concern for the firm. First, the firm cannot find a single creditor with "deep pockets" to finance all of its debt, and therefore has to rely on a continuum of small creditors. Second, when some of the creditors choose to run on the firm, the firm may not always find new funds to repay them and thus may be forced into a premature liquidation of its asset.[6] Finally, the secondary market for the firm asset is illiquid and the firm incurs a price discount by liquidating the asset prematurely.

Consistent with the staggered debt structure used by real-life firms, our model assumes that the firm's debt expirations are uniformly spread out across time. This staggered debt structure implies that the fraction of debt contracts that mature during a short period is small. As a result, the collective rollover decision of the creditors who hold these contracts is insignificant to affect the firm's survival. This feature thus insulates our model from the Diamond-Dybvig type simultaneous coordination problem between creditors whose contracts mature at the same time. Intead, our model focuses on the coordination problem between creditors whose contracts mature at different times, because the firm faces a significant bankruptcy probability after a series of creditors choose to run over time.

Our model also assumes that the firm's asset fundamental is time-varying and publicly observable. Time-varying fundamental is not only realistic, but also a crucial factor in driving each of the aforementioned debt-run episodes. The second part of this assumption on the public observability of firm fundamental is somewhat strong, but mainly serves to insulate our model from further complications generated by creditors' private information about firm fundamental.

[6]This feature implicitly assumes that the firm may not be able to raise more capital by issuing new equity. This assumption is consistent with another important source of distortion generated by the conflict of interest between debt and equity holders. When a firm faces liquidity problems in the debt market, equity holders could find it optimal not to inject more equity. This is because that by injecting equity they bear all the financial burden of keeping the firm from bankruptcy, but the benefit is shared by both debt and equity holders. See He and Xiong (2009) for an analysis of the effects of this distortion on short-term debt crises.

A nice feature of our model is that there is a unique monotone equilibrium. We derive this equilibrium in closed form. The underlying mechanism works as follows. When the firm fundamental is sufficiently high, the firm's liquidation value could be high enough to pay off all the creditors. As a result, each creditor's dominance strategy is rollover, in disregard of the other creditors' future decisions. In other words, the equilibrium is uniquely determined in this region, which is often referred to as the upper dominance region. Similarly, when the firm fundamental is sufficiently low (in the so called lower dominance region), each creditor's dominance strategy is run because even if all the future maturing creditors choose rollover, the firm fundamental is insufficient to pay off every one. When the firm fundamental is in the intermediate region between the upper and lower dominance regions, the Diamond-Dybvig type self-fulfilling multiple equilibria would arise if the fundamental is constant. However, when the fundamental is time-varying (either deterministically or stochastically) and is surely to hit at least one of the dominance regions in the future, each creditor can backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes at the two ends of the region as boundary conditions. This backward induction leads to a unique equilibrium in the intermediate region as well.

This unique equilibrium builds on the important economic insights and ingredients suggested by the game theory literature on coordination problems, e.g., Morris and Shin (2003) for the use of upper and lower dominance regions and Frankel and Pauzner (2000) for the insight that time-varying fundamental can act as a coordination device across agents who make decisions at different times. However, because creditors' payoffs in our model are derived from realistic debt contracts in debt run settings, we cannot directly apply the standard game theoretical frameworks to analyze the creditors' coordination problem across time. Instead, we use a guess-and-verify approach to derive the equilibrium.

Despite the absence of multiple equilibria, a preemptive debt run could occur through a "rat race" between the creditors in choosing higher and higher rollover thresholds. It is intuitive that each maturing creditor will choose to roll over his debt if and only if the current firm fundamental is higher than a threshold level so that there is a sufficient safety margin against the firm's future rollover risk with other creditors. Each creditor's optimal threshold choice depends on that of the others—if a creditor anticipates that other creditors are more likely to run (i.e., using a higher rollover threshold) when their contracts mature, he has a greater incentive to run ahead of them (i.e., using an even higher threshold) when he has the chance now. In this way, each creditor's anticipation that future maturing creditors would use a high threshold leads him to use a higher threshold, which in turn motivates the creditors before him to use an even higher one. In the equilibrium, each creditor's rollover threshold could be substantially higher than the firm's debt face value. That is, creditors choose to run on the firm even if it is fundamentally healthy.

The root cause of this rat race is the lack of commitment from future maturing creditors to roll over their debt. When the firm fundamental deteriorates, it is optimal for an individual maturing creditor to run to safety, even though his run exposes the remaining creditors to greater risk. This creditor's lack of commitment in turn makes it even more difficult for the maturing creditors before him to commit to a new contract period. The presence of time-varying firm fundamental thus makes the creditors' coordination problem across time different in nature from the Diamond-Dybvig type coordination problem in which creditors simultaneously choose their rollover decisions. The simultaneous coordination problem is essentially driven by lack of communication between agents and could be resolved by establishing a direct communication channel. However, resolving the creditors' coordination problem across time requires more than just a communication channel. It also requires each creditor to pre-commit to his future rollover strategy knowing that the firm fundamental could change. In reality, even if a creditor agrees to pre-commit to a certain rollover strategy, his credibility is questionable because he might also run into financial distress in the future and is thus forced to renege on his commitment. This difficulty lies at the heart of the challenges confronting the effect by the governments and central banks to restore stability to the world financial system during the 2008 Wall Street crisis.

Our model shows that the creditors' equilibrium rollover threshold is highly sensitive to the liquidation value of the firm asset and its fundamental volatility. Intuitively, a deeper discount of the firm asset in the illiquid secondary market exposes each creditor to a greater expected loss in the event of a forced firm liquidation. As a result, each creditor would choose a higher rollover threshold to protect himself, even if the other creditors' threshold stays the same. Because each creditor also needs to account for the increase in the other creditors' rollover threshold, the resulting rat race substantially amplifies the upward adjustment in each creditor's rollover threshold. Similarly, a higher volatility of the firm fundamental also exposes each creditor to a greater rollover risk because the firm fundamental is now more likely to hit below the other creditors' rollover threshold during the creditor's contract period. This effect, combined with the rat race mechanism illustrated above, motivates creditors to use a higher rollover threshold in equilibrium. Thus, through the rollover risk channel, our model captures the vulnerability of financial firms displayed in the 2008 Wall Street crisis to fluctuations in the external capital markets. That is, even in the absence of any fundamental deterioration, changes in the volatility and liquidation value of the assets held by the financial institutions could trigger preemptive runs by creditors.

Our model especially illustrates an alarming possibility that once the firm's asset volatility becomes sufficiently high, creditors would choose to run even when the firm's current liquidation value (i.e., the firm's asset value after taking the liquidation discount) is sufficient to cover all of its debt. The reason is as follows. Despite the firm's current strong fundamental, each creditor is still concerned that during his future contract period, volatility could cause the firm's liquidation value to drop below the debt value and other maturing creditors would run on the firm in those states. The possibility of such frantic debt runs contradicts the common sense argument that as long as a firm's liquidation value is sufficient to cover its debt, panic runs can be prevented. Our analysis also raises an intriguing question about whether the widely used staggered debt maturity structure is efficient in mitigating creditors' incentives to run.

Concerns about a firm's rollover risk could also lead to another rat race between the creditors in choosing shorter and shorter debt maturities. Our model shows that each individual creditor prefers a shorter debt maturity so that he has the option to pull out before the others when the fundamental is falling. Thus, in the absence of any commitment device like debt covenants or regulatory requirement, the firm would reduce the maturity of an atomless individual creditor without significantly affecting its overall risk. Since this argument applies to every creditor, it triggers a maturity rat race between the creditors. This maturity rat race explains why short-term financing becomes more and more pervasive, especially when the firm fundamental is deteriorating.

While the academic literature tends to treat the fundamental risk and liquidity risk of financial firms as two separate issues, our model shows that they are intertwined and operate jointly to determine a firm's credit risk. In particular, rollover risk is an additional source of credit risk. Our model suggests that after controlling for firm fundamentals, firms with shorter overall debt maturity and/or more illiquid asset holdings have greater credit risk, because they are more exposed to rollover risk.

The paper is organized as follows. In the next subsection, we review the related literature. Section 2 describes the model setup. We derive the unique monotone debt-run equilibrium in Section 3, and provide several comparative statics results in Section 4. Finally, Section

5 concludes the paper and provides some further discussions. All the technical proofs are given in the Appendix.

1.1 The Related Literature

Diamond and Dybvig (1983) provide a fundamental economic insight that banks, by granting demand deposits to depositors to insure against their preference shocks for early and late consumption, expose themselves to self-fulfilling runs. This insight had stimulated a large literature studying whether banks are inherently flowed institutions. See Gorton and Winton (2004) for a complete review of this literature. In particular, Goldstein and Pauzner (2005) adopt the global games framework (e.g., Morris and Shin, 2003) to derive a unique bank-run equilibrium in static settings. In their model, depositors coordinate their expectations of other depositors' withdrawal decisions through their private information about an unobservable bank fundamental. The unique equilibrium allows them to derive the probability of bank runs and relate it to the parameters of the deposit contracts. Rochet and Vives (2004) also provide a similar model. The ultimate source of bank runs in these models is the lack of efficient communication between depositors. Our model focuses on runs by creditors on firms, instead of depositors on banks. Because of firms' staggered debt maturity structure and time-varying fundamental, the inefficient debt runs in our model originate from the lack of commitment from future creditors to roll over their debt contracts when the firm fundamental deteriorates. This distortion differentiates our model from the existing models in the literature on panic runs.

Our model is also related to the growing literature on dynamic coordination problems related to broader economic issues, e.g., Abreu and Brunnermeier (2003), Chamley (2003), Angeletos, Hellwig, and Pavan (2007), Dasgupta (2007), and Toxvaerd (2008). A key theme of this literature is about strategic uncertainty generated by agents' higher order beliefs and the resulting inefficiencies of equilibrium outcomes. In our model, publicly observable firm fundamental eliminates the roles of creditors' higher order beliefs. Instead, the inefficient debt runs are caused by time-varying firm fundamental and the resulting lack of commitment from creditors to roll over their debt contracts in the future.

The unique equilibrium derived in our model builds on an important insight suggested by Frankel and Pauzner (2000) and Burdzy, Frankel and Pauzner (2001) that time-verying fundamental can act as a coordination device between agents who make decisions at different times. The same insight is also used by Guimaraes (2006) and Plantin and Shin (2008) to study coordinated currency attacks and speculative dynamics in carry trades. Because creditors' payoffs in our model are derived from realistic debt contracts, we cannot directly apply the standard approach based on iterated deletion of dominated strategies to derive the equilibrium. Instead, we use a guess-and-verify approach. More important, building on the unique equilibrium, our model highlights a new economic distortion–the commitment problem of creditors–caused by the time-varying firm fundamental. In our model, fundamental shocks are not just a technical tool for ensuring a unique equilibrium, they also play a key role in driving preemptive debt runs. Finally, the aforementioned models along this line rely on unspecified frictions to prevent agents from instantaneously changing their actions. In contrast, the frictions in our model emerge naturally from the lock-in effect of the creditors' debt contracts. Thus, our model directly links the creditors' incentives to run on a firm to the firm's staggered debt structure. We also show that such incentives in turn can lead to the use of shorter and shorter debt maturity.

Our paper complements several recent studies in analyzing instability of financial institutions as motivated by the recent financial crisis. Acharya, Gale, and Yorulmzer (2009) model the rollover risk faced by financial institutions, and show that under certain information structure the debt capacity of a given long-term asset can shrink to zero as rollover frequency increases to infinity. Brunnermeier and Oehmke (2009) study the rat race between creditors in choosing short debt maturity based on competitive pressures. Different from these models, our model focuses on the coordination problem among creditors and generates a set of implications for the instability of financial institutions, including rollover risk and maturity rat race.

*** More reviews to be added ***

2 Model

We consider a continuous-time model with an infinite time horizon. A firm invests in a long-term asset by rolling over short-term debt. We can broadly interpret this firm as a financial or non-financial firm, although some of our later discussion is motivated by the recent runs on financial firms. To make debt runs a relevant concern for the firm, we assume that the capital markets are imperfect in the following dimensions. First, the firm cannot find a single creditor with "deep pockets" to finance all of its debt and has to rely on a continuum of small creditors. Second, if some of the creditors choose not to roll over their debt, the firm might not always raise new capital to repay them and thus would have to liquidate its long-term asset prematurely. Third, the secondary market for the firm asset is illiquid and the firm incurs a price discount in the premature liquidation. We also impose two realistic features about the firm: The fundamental value of the firm asset changes randomly over time and is publicly observable; and the firm has a staggered debt structure.

2.1 Asset

We normalize the firm's asset holding to be 1 unit. The firm borrows \$1 at time 0 to acquire its asset. Once the asset is in place, it generates a constant stream of cash flow, i.e., rdt in the time interval $[t, t+dt]$. At a random time τ_ϕ, which arrives according to a Poisson process with parameter $\phi > 0$, the asset matures and provides a final payoff. An important advantage of assuming a random asset maturity with a Poisson process is that at any point before the maturity, the expected remaining maturity is always $1/\phi$.

The asset's final payoff is equal to the time-τ_ϕ value of a stochastic process y_t, which follows a geometric Brownian motion:

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t,$$

with constant drift μ and volatility $\sigma > 0$, where $\{Z_t\}$ is a standard Brownian motion. We assume that the value of the fundamental process is publicly observable at any time.

Taken together, the firm asset generates a constant cash flow of rdt before τ_ϕ and a final liquidation value of y_{τ_ϕ} at τ_ϕ. Then, by assuming that agents in this economy (including the firm creditors) are risk-neutral and have a discount rate of $\rho > 0$, we can compute the fundamental value of the firm asset as its expected discounted future cash flows:

$$F(y_t) = E_t\left[\int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho\left(\tau_\phi - t\right)} y_{\tau_\phi}\right] = \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t, \tag{1}$$

where the first component $\frac{r}{\rho+\phi}$ is the present value of the constant cash flows and the second component $\frac{\phi}{\rho+\phi-\mu} y_t$ is the expected present value of the asset's final payoff. Since the asset's fundamental value increases linearly with y_t, we will conveniently refer to y_t as the firm fundamental.

2.2 Debt Financing

The firm finances its asset holding by issuing short-term debt.[7] We emphasize an important feature of real-life firms' debt structure. A firm typically spreads out its debt expirations over time to reduce liquidity risk. That is, its debt contracts mature at different times. For example, on February 10, 2009, the data from Bloomberg show that Morgan Stanley, one of the major U.S. investment banks, had short-term debt (with maturities less than 1.5 years) expiring on almost every day throughout February and March 2009. If we sum up the total value of Morgan Stanley's expiring short-term debt in each week, the values for the following five weeks are 62 million, 324 million, 339 million, 239 million, and 457 million, respectively.[8] In this paper, we take this staggered debt structure as given and examine its implications for firms' rollover risk.

Specifically, we assume that the firm finances its asset holding by issuing one unit of short-term debt equally among a continuum of small creditors with measure 1. The promised interest rate is r so that the cash flow from the asset exactly pays off the interest payment until the asset matures or until the firm is forced to liquidate the asset prematurely. Once a creditor lends money to the firm, the debt contract lasts for a random period, which ends upon the arrival of an independent Poisson shock with parameter $\delta > 0$. In other words, the duration of each debt contract has an exponential distribution and the distribution is independent across different creditors. Once the contract expires, the creditor chooses whether to roll over the debt or to withdraw money (i.e., to run).

While the random duration assumption appears different from the standard debt contract with a predetermined maturity, it captures the aforementioned staggered debt structure of a typical financial firm—in aggregate, the firm has a fixed fraction δdt of its debt maturing over time, where the parameter δ represents the firm's rollover frequency. This random duration assumption simplifies the complication in dealing with the debt's maturity effect, because at any time before the debt maturity the expected remaining maturity is always $1/\delta$. By matching $1/\delta$ with the fixed maturity of a real-life debt contract, this assumption

[7] Short-term debt is a natural response of outside investors to a variety of agency problems inside the firms. By choosing short-term financing, investors keep the option to pull out if they discover the firm managers in pursuing value-destroying projects. See Kashyap, Rajan, and Stein (2008) for a recent review of this agency literature and capital regulation issues related to the recent financial crisis.

[8] The data released by the Federal Reserve Board also show that the commercial papers issued by financial firms in aggregate have maturities well spread out over time. Furthermore, our conversations with several bankers also confirm that financial institutions prefer to spread out the debt expirations so that institutions do not have to roll over a large fraction of their debts on a single day. Otherwise, they are overly exposed to the liquidity risk on that day.

captures the first order effect of debt maturity when a creditor makes his rollover decision,[9] although it may not be effective for valuing the debt contract that is already partially inside the contract period.

While we treat the rollover frequency as given for most of our analysis, we will analyze the creditors' preference over debt maturity in Section 4.3. To focus on the coordination problem between creditors, we also take the interest payment of the firm debt as given and leave a more elaborate analysis of the effects of endogenous interest payments for future research.

2.3 Runs and Liquidation

A key ingredient for capturing the firm's rollover risk is that when some of the creditors choose to run, the firm may not always be able to raise new fund to repay the running creditors even when the bank fundamental is healthy. This feature is a reflection of an illiquid capital market. If the firm were able to consistently find new fund to repay outgoing creditors, a debt run will never occur.

Of course, in practice every firm keeps cash reserves and acquires credit lines with other institutions to protect itself against such an adverse event. However, the experiences of many failed financial institutions during the recent financial crisis also indicate that none of these protections are perfect. Cash reserves cannot last long if a significant fraction of the creditors choose to run, and credit lines are not secure because the issuing institutions could be in financial distress at the same time. To explicitly model these protection mechanisms would significantly complicate our analysis and deflect our focus on the coordination problem among creditors. Instead, we adopt a reduced-form approach by assuming that when some creditors choose to run, the firm would fail with a certain probability.

More specifically, over a short time interval $[t, t + dt]$, δdt fraction of the firm's debt contracts expire. If these creditors choose to run, we assume that the probability of the firm failing is $\theta\delta dt$, where $\theta > 0$ is a parameter that measures the financial instability of the firm. The higher the value of θ, the more likely the firm will be forced into a liquidation given the same creditor outflow rate. It is intuitive that θ is higher for firms with less cash

[9] This assumption also generates an artificial second-order effect: If the debt contracts have a fixed maturity, a creditor, after rolling over his contract, will go to the end of the maturity queue. The random maturity assumption makes it possible for the creditor to be released early and therefore to run before other creditors when the asset fundamental deteriorates. This possibility makes the creditor less worried about the firm's rollover risk than he would if the debt contract has a fixed maturity. This in turn makes him more likely to roll over his debt. Thus, by assuming the random debt maturity, our model underestimates the firm's rollover risk.

reserves or credit lines. The random nature of the forced liquidation is also consistent with the fact that in reality creditors often face uncertainties about both quantity and reliability of a firm's credit line and cash reserves. These uncertainties make the exact timing of the firm's liquidation under runs to be random from creditors' perspectives. This assumption implies that if every maturing creditor chooses to run, the firm can survive on average for a period of $\frac{1}{\theta\delta}$.

2.4 Liquidation Value

Once the firm fails to raise new fund to pay off the running creditors, it is forced into bankruptcy and has to liquidate its asset. We broadly interpret the firm asset either as a long-term real investment position or as a long-term illiquid financial asset. If the bank has to liquidate the asset prematurely, the bank has to sell the asset on the secondary market to recover a fraction $\alpha \in (0,1)$ of the fundamental value. That is, the bank obtains a discounted price of

$$L(y_t) = \alpha F(y_t) = L + l y_t, \tag{2}$$

where

$$L = \frac{\alpha r}{\rho + \phi} \quad \text{and} \quad l = \frac{\alpha \phi}{\rho + \phi - \mu}. \tag{3}$$

In the case that the firm asset is a real asset, the price discount is caused by selling the asset to a second-best user; while in the case that the asset is a financial asset, the price discount is caused by illiquidity of the secondary markets. For simplicity, we rule out partial liquidations in this paper.

The liquidation value will then be used to pay off all creditors on an equal basis. In other words, both the running creditors and the creditors who are locked in by their current contracts, get the same payoff $\min(L(y), 1)$.[10]

[10] From the view of any running creditor, his expected payoff from choosing run is still 1 because the probability of the firm failure $\theta\delta dt$ is in a higher dt order. This observation implies that in our model the sharing rule in the event of bankruptcy is inconsequential. We can also assume that during bankruptcy those creditors who have expiring debt contracts and choose to run get a full pay 1, while the remaining creditors who are locked in by their current contracts get $\min(L(y), 1)$. This alternative assumption gives a greater incentive for maturing creditors to run. However, since the probability of the firm failure is $\theta\delta dt$, the difference in incentive is negligible.

2.5 Parameter Restrictions

To make our analysis meaningful, we impose several parameter restrictions. First, we bound the interest payment by

$$\rho < r < \rho + \phi. \tag{4}$$

The first part $r > \rho$ makes the interest payment attractive to the creditors, who have a discount rate of ρ. The second part $r < \rho + \phi$ rules out the scenario where the interest payment is so attractive that rollover becomes the dominant strategy even when the bank fundamental y_t is close to zero. Essentially, this condition ensures the existence of the lower dominance region in which each creditor's dominant strategy is to run if the firm fundamental y_t is sufficiently low.

Second, we limit the growth rate of the firm fundamental by

$$\mu < \rho + \phi. \tag{5}$$

Otherwise, the fundamental value of the firm asset in equation (1) would explode.

Third, we also limit the premature liquidation recovery rate of the firm asset:

$$\alpha < \frac{1}{\frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}}, \tag{6}$$

so that $L + l < 1$. Under this condition, the asset liquidation value is not enough to pay off all the creditors when $y_t = 1$. This condition is sufficient for ensuring that each creditor is concerned about the firm's future rollover risk when the firm fundamental y_t is in an intermediate region.

Finally, we assume that the parameter θ is sufficiently high:

$$\theta \geq \frac{\phi}{\delta\,(1 - L - l)}. \tag{7}$$

so that the firm faces a serious bankruptcy probability when some creditors choose to run.

3 The Debt-Run Equilibrium

Given the firm's financing structure described in the previous section, we now analyze the debt-run equilibrium. We limit our attention to monotone equilibria, that is, equilibria in which each creditor's rollover strategy is monotonic with respect to the firm fundamental y_t (i.e., to roll over the debt if and only if the firm fundamental is above a threshold). In making the rollover decision, a creditor rationally anticipates that once he rolls over the debt, he faces the firm's rollover risk. This is because during the following contract period, volatility may cause the firm fundamental to fall below the other creditors' rollover threshold. As a result, the creditor's optimal rollover threshold depends on the other creditors' threshold choice.

In this section, we first set up an individual creditor's optimization problem in choosing his optimal threshold. We then construct a unique monotone equilibrium in closed form. We also characterize the key ingredients that lead to the unique equilibrium. Finally, we discuss the rat race among creditors in choosing higher and higher thresholds. This rat race leads to a preemptive debt run.

3.1 An Individual Creditor's Problem

We first analyze the optimal rollover decision of an individual creditor who holds a small fraction of the firm's outstanding debts. In analyzing the individual creditor's problem, we take it as given that all other creditors use a monotone strategy with a rollover threshold y_* (i.e., other creditors will roll over their debts if and only if the firm fundamental is above y_* when their debt contracts mature). During the creditor's contract period, his value function depends directly on the firm fundamental y_t, and indirectly on the other creditors' rollover threshold y_*. We denote $V(y_t; y_*)$ as the creditor's value function normalized by the unit of debt he holds.

For each unit of debt, the creditor receives a stream of interest payments r until a random time τ,

$$\tau = \min\left(\tau_\phi, \tau_\delta, \tau_\theta\right)$$

which is the earliest of the following three events: the asset matures at a random time τ_ϕ, the creditor's own contract expires at τ_δ, or some of the other maturing creditors choose to run and eventually force the firm to fail at τ_θ.

Figure 1 illustrates these three possible outcomes at the end of three different fundamental paths. On the top path, the firm stays alive until its asset matures at τ_ϕ. At this time, the creditor gets a final payoff of $\min\left(1, y_{\tau_\phi}\right)$, i.e., the face value 1 if the asset's maturity payoff y_{τ_ϕ} is sufficient to pay all the debt, and y_{τ_ϕ} otherwise. On the bottom path, the firm fundamental drops below the creditors' rollover threshold and the firm is eventually forced to liquidate its asset prematurely at τ_θ before his contract expires. At this time, the creditor gets $\min\left(1, L + l y_{\tau_\theta}\right)$. On the middle path, the firm stays alive (although its fundamental dips below the other creditors' rollover threshold on the path) before τ_δ when the creditor's

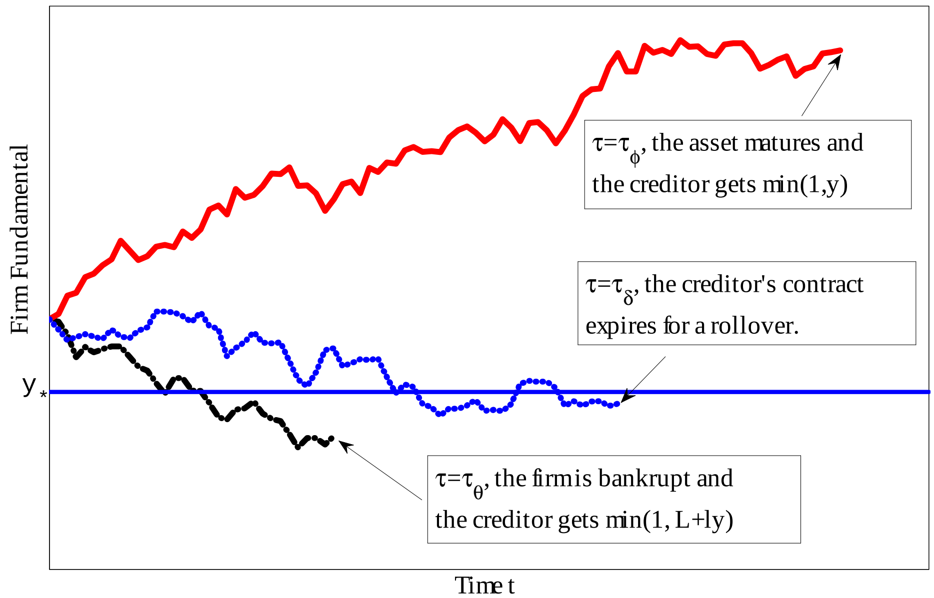

Figure 1: Three possible outcomes to a bank creditor.

contract expires. At this time, the creditor makes his rollover decision depending on whether the continuation value $V(y_{\tau_\delta}; y_*)$ is higher than getting the one dollar back or not.

Due to risk neutrality, the individual creditor's value function is given by

$$\begin{aligned} V(y_t; y_*) &= E_t \left\{ \int_t^\tau e^{-\rho(s-t)} r ds + e^{-\rho(\tau-t)} \left[\min(1, y_\tau) \mathbf{1}_{\{\tau=\tau_\phi\}} \right. \right. \\ & \left. \left. + \min(1, L + l y_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}} + \max_{\text{rollover or run}} \{V(y_\tau; y_*), 1\} \mathbf{1}_{\{\tau=\tau_\delta\}} \right] \right\} \end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function, which takes a value of 1 if the statement in the bracket is true and zero otherwise. The individual creditor's future payoff during his contract period depends on other creditors' rollover choices because other creditors' runs might force the firm to liquidate its asset prematurely, as illustrated by the bottom path of Figure 1. This dependence gives rise to the strategic complementarity in the creditors' rollover decisions, and therefore a coordination problem among the creditors who make rollover decisions at different times.[11]

[11] It is important to note that our model is substantially different from the standard game theoretical frameworks for analyzing dynamic coordiantion problems. For example, consider the framework in two closely related papers by Frankel and Pauzner (2000) and Burdzy, Frankel, and Pauzner (2001). This framework consists of a sequence of repeated stage games. In each period, each agent receives a flow payoff, which satisfies an exogenous form of strategic complementarity, i.e., the agent receives a higher flow payoff if his current-period strategy overlaps with that of a greater fraction of population. In contrast, each creditor's flow payoff in our model is endogenously determined by the debt contract (interest payment r and possible

Also note that when the firm fundamental y_t is sufficiently low (i.e., close to zero), an individual creditor's dominant strategy is run. This is because that even if all other creditors choose to roll over in the future, the expected asset payoff at the maturity plus the interest payments before the asset maturity are not as attractive as getting one dollar back now. On the other hand, when the firm fundamental y_t is sufficiently high (i.e., close to infinity), an individual creditor's dominant strategy is rollover. This is because that even if all other creditors choose to run in the future, the asset's liquidation value is sufficient to pay off the debt in the event of a forced liquidation. These two regions are often called the lower and upper dominance regions. Their existence is important for ensuring a unique equilibrium.

By considering the change of the creditor's value over a small time interval $[t, t+dt]$, we can derive his Bellman equation:

$$\begin{aligned} \rho V(y_t; y_*) &= \mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy} + r + \phi \left[\min(1, y_t) - V(y_t; y_*)\right] \\ &\quad + \theta\delta \mathbf{1}_{\{y_t < y_*\}} \left[\min(L + l y_t, 1) - V(y_t; y_*)\right] + \delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}. \end{aligned} \tag{8}$$

The left-hand side term $\rho V(y_t; y_*)$ represents the creditor's required return. This term should be equal to the expected increment in his value, as summarized by the terms on the right-hand side.

- The first two terms $\mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy}$ capture the expected change in the value function caused by the fluctuation in the firm fundamental y_t.

- The third term r is the interest payment per unit of time.

The next three terms capture the three events illustrated in Figure 1:

- The fourth term $\phi[\min(1, y_t) - V(y_t; y_*)]$ captures the possibility that the asset matures during the time interval, which occurs at a probability of ϕdt and generates an impact of $\min(1, y_t) - V(y_t; y_*)$ on the creditor's value function.

- The fifth term $\theta\delta \mathbf{1}_{\{y_t < y_*\}} [\min(L + l y_t, 1) - V(y_t; y_*)]$ represents the expected effect when the firm is forced into a premature liquidation by other creditors' runs, which

asset maturity payoff $\min(y, 1)$), which does not exhibit strategic complementarity. Instead, the strategic complementarity between creditors emerges from the implicit dependence of a creditor's continuation value function on other creditors' rollover decisions, as shown in Figure 1 and equation (8). This important difference in model frameworks prevents us from readily applying the method of iterated deletion of dominated strategies in Burdzy, Frankel, and Pauzner (2001) to our model. Instead, we derive the equilibrium by invoking a guess-and-verify approach detailed in Theorem 1.

occurs at a probability of $\theta\delta\mathbf{1}_{\{y_t<y_*\}}dt$ (the other maturing creditors will run only if $y_t < y_*$) and generates an impact of $\min(L + ly_t, 1) - V(y_t; y_*)$ on the creditor's value function. Here, once the forced liquidation occurs, all creditors have the same priority in dividing the firm's liquidation value.

- The last term $\delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}$ captures the expected effect from the expiration of the creditor's own contract, which arrives at a probability of δdt. Upon its arrival, the creditor chooses whether to rollover or to run: $\max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}$. Note that from any individual creditor's view, the probability of the event that his contract expires (and he runs) and the firm is forced into a premature liquidation is in the second order of $(dt)^2$.[12]

It is obvious that an individual creditor will choose to roll over his contract if and only if $V(y_t; y_*) > 1$, and to run otherwise. This implies that if the value function V only crosses 1 at a single point y', then y' is the creditor's optimal threshold. Later we will show that the equilibrium has to be symmetric; then we must have $y' = y_*$ so that

$$V(y_*; y_*) = 1.$$

This is the condition for determining the equilibrium threshold.

Externality on Other Creditors Each creditor's rollover decision not only affects his own payoff, but also other creditors'. In particular, each maturing creditor's decision to run adds to the firm's bankruptcy probability and thus imposes an implicit cost on other creditors. Since a creditor does not internalize the cost of his action on others, this externality effect is the ultimate source of debt runs in our model. To see this point precisely, we summarize the payoff or continuation value function of the current-period maturing creditors and other creditors depending on the choice of the maturing creditors.

Choice by maturing creditors[13]	Run		Rollover
Possible firm outcomes	failed ($\theta\delta dt$)	survived ($1 - \theta\delta dt$)	survived
Value function of maturing creditors	$L(y)$	1	$V(y)$
Value function of other creditors	$L(y)$	$V(y)$	$V(y)$

[12] As a result, whether the creditor gets 1 or the asset's premature liquidation value in such an event is inconsequential. See related discussion in footnote 10.

[13] To focus on the conflict between the maturing creditors and the remaining creditors, we treat all the maturing creditors as one identity.

The maturing creditors will choose run if $1 \cdot (1-\theta\delta dt) + L \cdot \theta\delta dt > V$, which is $V < 1$ after ignoring the higher order dt term. Their run reduces the remaining creditors' continuation value function by

$$V - [V \cdot (1-\theta\delta dt) + L \cdot \theta\delta dt] = (V - L)\,\theta\delta dt.$$

In expectation, this externality effect on the remaining creditors accumulates over time as more maturing creditors choose to run.

3.2 The Unique Monotone Equilibrium

We employ a guess-and-verify approach to derive a unique monotone equilibrium following four steps. First, we derive an individual creditor's value function $V(y_t; y_*)$ from the Bellman equation in (8) by assuming that every creditor (including the creditor under consideration) uses the same monotone strategy with a rollover threshold y_*. Second, based on the derived value function, we show that there exists a unique fixed point y_* such that $V(y_*; y_*) = 1$. Third, we prove the optimality of the threshold y_* for any individual creditor, i.e., $V(y; y_*)$ only crosses 1 with $V(y; y_*) > 1$ for $y > y_*$ and $V(y; y_*) < 1$ for $y < y_*$. Finally, we show that there cannot be any asymmetric monotone equilibrium.

We summarize the main results in the following theorem. Because the debt payoff is capped at its face value 1, there are three cases depending on whether the firm asset's final payoff and premature liquidation value at y_* are sufficient to pay off the debt.

Theorem 1 *There exists a unique monotone equilibrium, in which each creditor chooses to roll over his debt if y_t is above the threshold y_* and to run otherwise. The creditor's value function $V(y_t; y_*)$ is given by the following three cases:*

1. *If $y_* < 1$,*

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + A_1 y_t^{\eta_1} & \text{when } 0 < y_t \le y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_t + A_2 y_t^{-\gamma_2} + A_3 y_t^{\eta_2} & \text{when } y_* < y_t \le 1 \\ \frac{r+\phi}{\rho+\phi} + A_4 y_t^{-\gamma_2} & \text{when } y_t > 1 \end{cases};$$

2. *If $1 \le y_* < \frac{1-L}{l}$,*

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_1 y_t^{\eta_1} & \text{when } 0 < y_t \le 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + B_2 y_t^{-\gamma_1} + B_3 y_t^{\eta_1} & \text{when } 1 < y_t \le y_* \\ \frac{r+\phi}{\rho+\phi} + B_4 y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases};$$

3. If $y_* \geq \frac{1-L}{l}$,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_1 y^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_t + C_2 y_t^{-\gamma_1} + C_3 y_t^{\eta_1} & \text{when } 1 < y_t \leq \frac{1-L}{l} \\ \frac{r+\phi+\theta\delta+\delta}{\rho+\phi+(1+\theta)\delta} + C_4 y_t^{-\gamma_1} + C_5 y_t^{\eta_1} & \text{when } \frac{1-L}{l} < y_t \leq y_* \\ \frac{r+\phi}{\rho+\phi} + C_6 y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases}.$$

The coefficients η_1, η_2, γ_1, γ_2, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, C_5, and C_6 are given in the Appendix A.1 and are expressions of the model parameters and y_*. The equilibrium threshold y_* is uniquely determined by the condition that $V(y_*, y_*) = 1$.

Theorem 1 presents a unique dynamic monotone equilibrium—when each creditor's current contract expires, he will choose to run if the firm fundamental is below the equilibrium threshold y_*, thereby exposing the firm to the possibility of a forced liquidation. Note that the third case in Theorem 1 is particularly interesting as $L(y_*) = L + ly_* \geq 1$, i.e., creditors start to run on the firm even though the firm's liquidation value at y_* is sufficient to pay off all of the firm's debt. This frenzied debt run is a reflection of the creditors' preemptive motive of the firm's future rollover risk. We will examine this hyper run in more detail later.

3.3 Understanding the Uniqueness of the Equilibrium

In the classic bank run model of Diamond and Dybvig (1983), there exist two equilibria. While our model also features a coordination problem between the firm creditors as in their model, we are able to derive a unique monotone equilibrium in Theorem 1. What leads to the unique equilibrium? In this section, we discuss the role of two important ingredients of our model: staggered debt structure and time-varying firm fundamental.

3.3.1 Staggered Debt Structure

The staggered debt structure spreads out the creditors' rollover decisions over time. Since the fraction of contracts expiring over a small interval of time (say a day) is small, the collective choice of these creditors is insignificant to affect the firm. This feature thus avoids the coordination problem among the creditors whose contracts expire at the same time.

To highlight this role of the staggered debt structure, we consider the following thought experiment. Suppose that the firm's debt contracts all expire at the same time, say time 0, and the current firm fundamental is y_0. At this time, each creditor decides whether to run or to roll over into a perpetual debt contract until the firm asset matures at τ_ϕ. In this setting, the firm does not face any future rollover risk after time 0. However, at time 0, all creditors simultaneously choose their rollover decisions, leading to a Diamond-Dybvig type coordination problem. We formally characterize this coordination problem below.

Proposition 2 *There exist $y_h > y_l > 0$ such that if $y_0 > y_h$ (the upper dominance region), an individual creditor's dominant strategy is to roll over; if $y_0 < y_l$ (the lower dominance region), the creditor's dominant strategy is to run. However, if $y_0 \in [y_l, y_h]$, the creditor's optimal choice depends on the others', i.e., it is optimal to run if the others choose to run and it is optimal to roll over if the others choose to roll over.*

When the firm fundamental is between the two dominance regions, the firm fundamental is good enough to pay off the debt if the firm asset is kept to the maturity, but is insufficient after taking the price discount in a premature liquidation. Proposition 2 shows that in this case, an individual creditor's optimal rollover choice depends on the other creditors'. Put differently, when the firm fundamental is not strong enough to sustain the runs of the other creditors, an individual creditor is better off by going along with the other creditors. Like in Diamond and Dybvig (1983), there are two equilibria, in one of which all the creditors choose to roll over and in the other all choose to run. These equilibria emerge because the creditors' collective rollover/run decision at the same time is able to swing the survival of the bank. In reality, firm managers are well aware of the risk of having to roll over a significant fraction of their debt on a single day, and thus prefer to spread out the debt expirations over time. However, doing so leads to a different coordination problem between the creditors whose contracts expire at different times. This problem is exactly the focus of our paper.

Note that as $\delta \to \infty$, the maturity of each debt contract converges to zero. Then, each creditor effectively holds a demand deposit in the firm, as in Diamond and Dybvig (1983). Interestingly, the unique monotone equilibrium derived in Theorem 1 still holds, as shown in the following proposition:

Proposition 3 *When $\delta \to \infty$, the unique equilibrium rollover threshold y_* converges to $\frac{1-L}{l}$.*

This proposition suggests that the driver of the unique equilibrium in our model is not the finite maturity of the debt contract. Instead, it is the asynchronous timing of the creditors' rollover decisions caused by the staggered debt structure. As $\delta \to \infty$, the debt maturity goes down to zero, but the asynchronous timing of the creditors' rollover decisions still remains.

3.3.2 Time-Varying Fundamental

We now study the role of the time-varying fundamental. The following proposition shows that when the firm fundamental is constant, the coordination problem between the creditors whose contracts expire at different times can also lead to self-fulfilling multiple equilibria.

Proposition 4 *Suppose that $y_t = y$ is constant (i.e., $\sigma = 0$ and $\mu = 0$) and the creditors have staggered debt structure. There exist $y_h^c > y_l^c > 0$ such that when $y > y_h^c$ (the upper dominance region), an individual creditor's dominant strategy is to roll over; when $y < y_l^c$ (the lower dominance region), the creditor's dominance strategy is to run; and when $y \in [y_l^c, y_h^c]$, the creditor's optimal choice depends on the others', i.e., it is optimal to run if the others will choose to run in the future and it is optimal to roll over if the others will choose to roll over in the future.*

Proposition 4 shows that when the firm fundamental is constant and between the upper and lower dominance regions, the Diamond-Dybvig type self-fulfilling multiple equilibria could also emerge even if the firm's debt expirations are spread out over time. For example, for a given fundamental level in the intermediate region, once each individual creditor believes that other maturing creditors in the future will all choose to roll over, rollover is optimal for him now. This "no-future-rollover-risk" belief is in fact consistent with the equilibrium outcome because the firm fundamental is constant and thus always stays above the lower dominance region.

This self-fulfilling logic, however, breaks down if the firm fundamental changes over time and is expected to reach either one of the two dominance regions in the future. The creditors' anticipation of this occurrence would, instead, allow them to backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes at the two ends of the region as boundary conditions. A unique equilibrium thus arises in the intermediate region.

It is easy to see this mechanism in the case that the firm fundamental changes deterministically (i.e., $\sigma = 0$ and $\mu \neq 0$). Suppose that $\mu < 0$, i.e., the fundamental continues to deteriorate until the asset matures. Knowing that once the fundamental is in the lower dominance region other creditors will always choose run, each creditor will choose run right before the fundamental entering the region. This in turn motivates each creditor to choose run even earlier. This backward induction amplifies the creditors' incentive to run, and thus generating excessive rollover risk to the firm. Rollover is optimal only when the current firm fundamental is sufficiently high, i.e., above a threshold $y_{\mu-} > 1$, so that it provides enough cushion against the firm's future rollover risk. Otherwise, when $y \leq y_{\mu-}$ run is optimal for each creditor. A similar reasoning works in determining a unique equilibrium for the case $\mu > 0$. The following proposition formally derives this unique equilibrium.

Proposition 5 *Suppose that the firm fundamental is deterministic with a nonzero drift μ.*

1. *If $\mu > 0$, there is a unique monotone equilibrium, in which each creditor chooses rollover if the firm fundamental is above a threshold $y_{\mu+} < 1$, and run otherwise.*
2. *If $\mu < 0$, there is a similar unique monotone equilibrium with a threshold $y_{\mu-} > 1$.*

The same backward induction mechanism also applies to the case where the firm fundamental changes randomly over time (i.e., $\sigma > 0$). Consider the firm fundamental exactly at the boundary of the lower dominance region. At this point, a creditor is indifferent between rollover and run, *if other maturing creditors will always choose rollover in the future regardless of the fundamental.* However, the fundamental will stay inside the lower dominance region in the future for a significant portion of time. Knowing that the other maturing creditors *will choose run once they are inside the lower dominance region in the future,* an individual creditor will choose run at the boundary now. Then, knowing all the future maturing creditors will also update their strategies and choose run at this level, each creditor will choose run at an even higher fundamental level, and so on. Thus, random shocks can serve the same role as deterministic drifts, i.e., allowing the creditors to backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes in the two dominance regions. A similar insight has been previously pointed out by Frankel and Pauzner (2000). This mechanism leads to the unique equilibrium derived in Theorem 1.

3.4 The Rat Race in Choosing Thresholds

Despite the absence of self-fulfilling multiple equilibria in our model, a preemptive debt run could still occur through the interaction between creditors' rollover threshold choices. The Bellman equation in (8) shows that an individual creditor's optimal threshold choice y' depends on the other creditors' threshold choice y_*. Intuitively, if other creditors use a higher threshold, it is more likely that the firm fundamental would hit below their threshold during the individual creditor's contract period and force the firm into a premature liquidation. Consequently, the creditor would prefer a higher threshold to protect himself. This

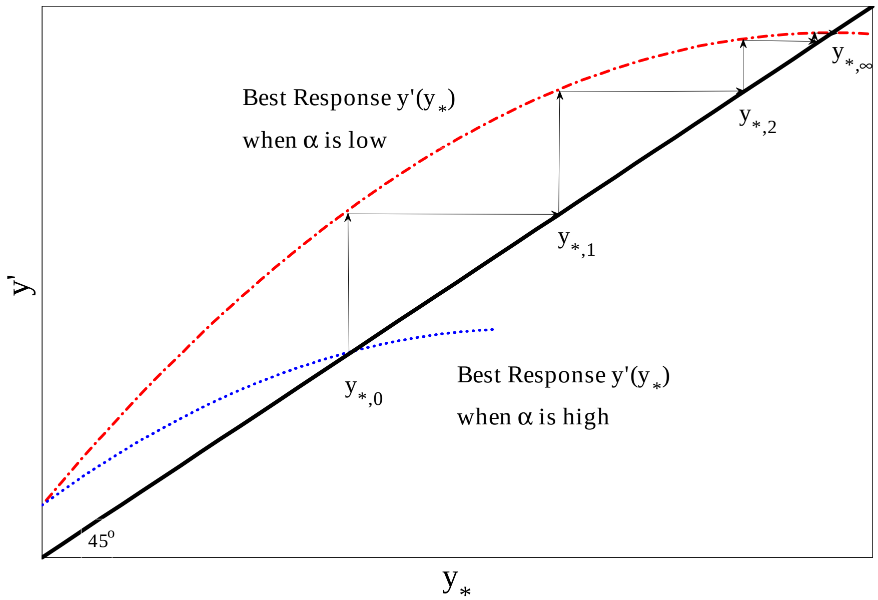

Figure 2: Rat race among the firm creditors in choosing rollover thresholds.

dependence in turn leads to a rat race among the creditors—when a creditor chooses a high rollover threshold, it motivates other creditors to choose an even higher threshold. This rat race can eventually lead each creditor to use a threshold substantially higher than the necessary fundamental level to justify the solvency of the firm.

We illustrate this rat race using a simple thought experiment. Suppose that initially the liquidation recovery rate of the firm asset is α_h, and, correspondingly, every creditor uses a threshold level $y_{*,0}$. Unexpectedly, at a certain time, all creditors find out that the liquidation recovery rate drops to a lower level $\alpha_l < \alpha_h$. What would the new equilibrium threshold be? Let's start with an individual creditor's threshold choice. Suppose that all the other creditors still use the original threshold $y_{*,0}$. Then, by solving the Bellman equation in (8), we can derive the creditor's optimal threshold $y_{*,1}$, which is higher than $y_{*,0}$ because the lower liquidation value generates a greater expected loss to the creditor in the event that the firm is forced into a premature liquidation during his contract period. Of course, each creditor will go through the same calculation and choose a new threshold. If all creditors choose a threshold $y_{*,1}$, then an individual creditor's optimal threshold would be $y_{*,2}$, another level even higher than $y_{*,1}$. If all creditors choose $y_{*,2}$, then each creditor would go through another round of threshold updating, and so on and so forth. Figure 2 illustrates this updating process until it eventually converges to a fixed point $y_{*,\infty}$, the new equilibrium threshold.

The difference between the threshold levels $y_{*,1}$ and $y_{*,0}$ represents the necessary safety margin a creditor would demand in response to the reduced asset liquidation value if the other creditors' rollover strategies stay the same. This increase in threshold is eventually magnified to a much larger increase $y_{*,\infty} - y_{*,0}$ through the rat race among the creditors. This amplification mechanism plays a key role in driving debt runs, which we discuss in the next section.

4 Comparative Statics

In this section, we provide several comparative statics results of our model. We focus on three key model parameters: the premature liquidation recovery rate α, the volatility of the firm asset σ, and the firm's rollover frequency δ. For illustration, we will use a set of baseline values for the model parameters:

$$\rho = 5\%,\ r = 10\%,\ \delta = 10,\ \phi = 0.2,\ \theta = 1,\ \mu = 5\%,\ \sigma = 10\%,\ \alpha = 70\%. \tag{9}$$

The creditors have a discount rate $\rho = 5\%$. The firm asset generates a constant stream of cash flow at a rate of 10% per annum, which is paid out to the creditors as interest payments. The interest payment is attractive since the interest rate r is much higher than the creditors' discount rate ρ. We choose the firm's rollover frequency δ to be 10, which implies an average debt maturity of about 37 days ($365/\delta$). This implied maturity matches the average maturity of outstanding asset-backed commercial paper in February 2009 (Federal Reserve Release). $\phi = 0.2$ implies that the firm asset on average lasts for 5 years ($1/\phi$), which is much longer than the debt maturity. $\theta = 1$ means that conditional on every maturing creditor choosing to run, the firm can survive on average for 37 days ($1/\theta\delta$). The firm fundamental y_t has a growth rate of $\mu = 5\%$ per annum and a volatility of $\sigma = 10\%$ per annum. Finally, when the firm liquidates its asset prematurely, it only recovers $\alpha = 70\%$ of the asset's fundamental value. This implies that $L = 0.28$ and $l = 0.7$ in equation (3).

4.1 Effects of Liquidation Value

The liquidation recovery rate α determines the firm's asset liquidation value $L(y)$, and thus plays an important role in determining the creditors' rollover threshold. To illustrate its effect, we examine the change in the equilibrium rollover threshold as we vary α from its baseline value of 0.7. We measure the threshold by the fundamental value of the firm asset at the point $F(y_*) = \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y_*$, because $F(y_*)$ is directly comparable to the firm's total

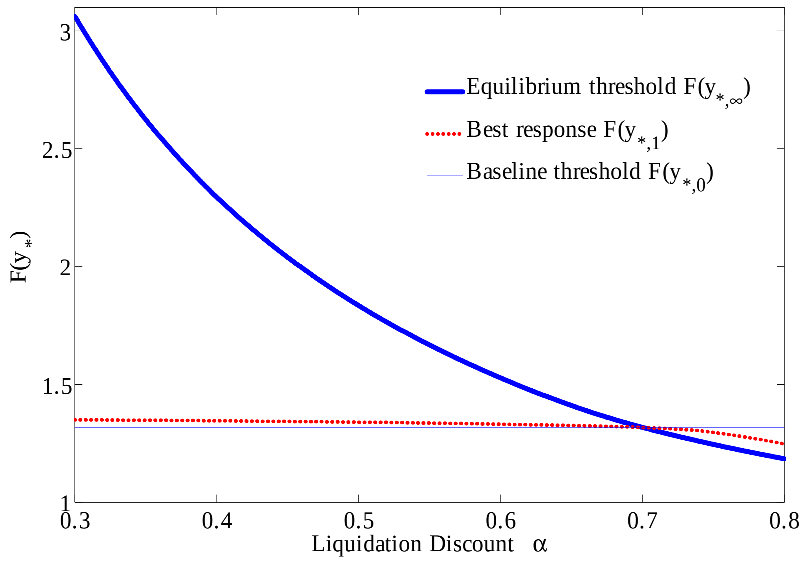

Figure 3: The equilibrium rollover threshold, measured in the firm asset's fundamental value $F(y_*)$, vs the liquidation recovery rate α. This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$. The thin solid line is the equilibrium threshold $F(y_{*,0})$ under the baseline parameters. The thick solid line plots the equilibrium threshold $F(y_{*,\infty})$ and the dashed line plots a creditor's best response $F(y_{*,1})$ to the change in α from its baseline value if the other creditors' threshold is fixed at the baseline level.

debt outstanding, 1. Based on the notation from Section 3.4, as α deviates from its baseline value, the equilibrium threshold $y_* = y_{*,\infty}$ is the fixed point in the threshold rat race among the creditors.

In Figure 3, the flat thin solid line represents the equilibrium threshold $F(y_{*,0}) = 1.32$ when α takes the baseline value. The thick solid line plots $F(y_{*,\infty})$ against α in the region between 0.3 to 0.8. This figure shows several interesting features. First, $F(y_{*,\infty})$ is always above 1—the creditors start to run on the firm when it is still solvent. This result is intuitive: The creditors only hold a partial stake in the firm. Therefore, it makes sense for each maturing creditor to run and get his money back before the firm's fundamental value drops below the outstanding debt.

Moreover, as the liquidation recovery rate decreases from 80% to 30%, the firm's fundamental value at the equilibrium rollover threshold rises sharply from 1.2 to 3.1. This is because a lower liquidation value increases the expected loss to each creditor in the event that during the creditor's contract period the firm is forced to liquidate its asset prematurely. We formally prove this result in the following proposition:

Proposition 6 *The equilibrium rollover threshold y_* decreases with the bank asset's premature liquidation recovery rate α.*

Our discussion in Section 3.4 suggests that creditors might engage in a rat race in choosing their rollover thresholds and that this rat race amplifies the effect of a reduction in the asset liquidation value on the equilibrium rollover threshold. To illustrate the magnitude of this amplification mechanism, we decompose the effect of a change in α on $F(y_*)$, which is $F(y_{*,\infty}) - F(y_{*,0})$, into two components. Figure 3 plots the best response of a creditor in the absence of the rat race, i.e., $F(y_{*,1})$, in the dashed line. Suppose α drops exogenously from its baseline level 70% to 50%. After the drop in α, an individual creditor will choose an optimal threshold $F(y_{*,1}) = 1.34$ (on the dashed line) if the other creditors' rollover threshold is fixed at the initial level $F(y_{*,0}) = 1.32$ (the thin solid line). The difference $F(y_{*,1}) - F(y_{*,0}) = 0.02$ represents the necessary safety margin to compensate the creditor for increased bankruptcy loss in the absence of the rat race among the creditors. Of course, once we take into account the rat race, each creditor ends up choosing a higher threshold of $F(y_{*,\infty}) = 1.8$ (on the thick solid line) in the equilibrium. The difference $F(y_{*,\infty}) - F(y_{*,1})$ represents the amplification effect of the rat race. In this example, it is 24 times of the effect without rat race.

The general pattern in Figure 3 suggests that as α decreases (increases) from the baseline value, the best response $F(y_{*,1})$ without the rat-race effect increases (decreases) only by a modest magnitude. Thus, the dramatic increase (decrease) in the equilibrium rollover threshold $F(y_{*,\infty})$ is mostly driven by the amplification effect caused by the rat race among the creditors.

The 2008 Wall Street Crisis Commentators often attribute the funding problems of many financial firms in the 2008 Wall Street crisis to one of two distinctive factors, either a liquidity breakdown in the capital markets or fundamental concerns about the firms' insolvency. Our model shows that these two factors are intertwined. A firm's possible future fundamental deterioration generates the concerns by its creditors that they might have to bear the cost of liquidating the firm asset in the illiquid secondary market. The deterioration of the secondary market liquidity, as dramatically occured during the crisis, can in turn motivate creditors to preemptively run on the firm to reduce the exposure to the worsened market liquidity.

Our model takes the illiquidity discount of the firm asset α as given. As demonstrated by the recent crisis, many financial firms hold similar assets. As one firm (for example, Lehman Brothers) runs into a financial distress, creditors of other firms will start to worry about the possible liquidation of this firm push down the market liquidity and liquidation value of their firms' assets. As a result, they might preemptively run on their firms. Thus, through the market liquidity channel, debt runs can spread from one firm to others. Our model can be extended to analyze this type of contagion mechanism.

The significant role played by the deteriorating liquidity on triggering preemptive runs provides a rationale for the wide range of lending facilities created by the Federal Reserve in the recent crisis period to boost the market liquidity. A good example is the Federal Reserve facility to buy high-quality commercial paper at a term of three months. Following a prominent money market mutual fund's "breaking the buck" (i.e., a decline of its net asset below par) in September 2008, investors started to withdraw money in large amounts from money market funds that invest in commercial paper. Created right at this time, the Federal Reserve facility provided a backstop on the funds' liquidation value of their commercial paper (i.e., a guarantee on α in our model). By soothing investors' concerns about the money market funds' future funding problems, this facility has been successful in preventing the adverse dynamic of investors trying to be the earlier ones to run from the funds.

4.2 Effects of Fundamental Volatility

Next, we discuss the effects of the firm asset's fundamental volatility σ. In Figure 4, the thick solid line plots the creditors' equilibrium rollover threshold $F(y_*)$ as σ deviates from the baseline value of 10% and takes different values between 3% and 50%. We also plot an individual creditor's best response $F(y_{*,1})$ to the change in σ (the dashed line) while fixing the other creditors' threshold at the original level $F(y_{*,0}) = 1.32$ when σ takes the baseline level 10%. The individual creditor's best response $F(y_{*,1})$ increases with σ. This pattern is intuitive. A higher volatility makes it more likely that the bank fundamental y might drop below the other creditors' rollover threshold during an individual creditor's contract period. The increase $F(y_{*,1}) - F(y_{*,0})$ represents the safety margin that the creditor would demand to protect himself against the increased rollover risk in the absence of the rat race among the creditors in choosing higher and higher thresholds. $F(y_{*,1})$ increases from 1.31 to 1.44 as σ varies from 3% to 50%.[14]

[14]Note that the change $F(y_{*,1}) - F(y_{*,0})$ has already incorporated the change in the firm's insolvency risk caused by the change in σ. As σ increases, it is now more likely for the fundamental value of the firm

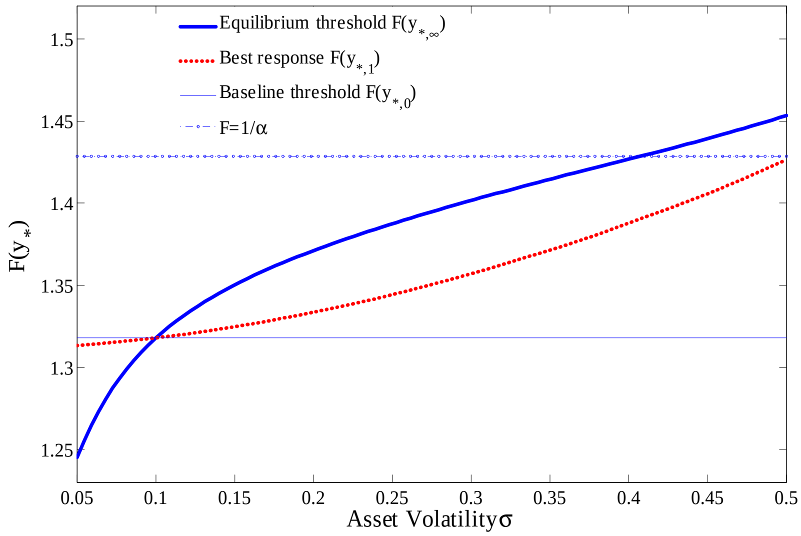

Figure 4: The equilibrium rollover threshold, measured in the firm asset's fundamental value $F(y_*)$, vs the asset volatility σ. This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$. The thin solid line is the equilibrium threshold $F(y_{*,0})$ under the baseline parameters. The thick solid line plots the equilibrium threshold $F(y_{*,\infty})$, while the dashed line plots a creditor's best repones $F(y_{*,1})$ to the change in α from its baseline value if the other creditors' threshold is fixed at the baseline level.

The thick solid line in Figure 4 shows that the range of the equilibrium threshold $F(y_{*,\infty})$ is wider. For instance, when we increase σ from 10% to 20%, an individual creditor will only raise his threshold by 0.01 from $F(y_{*,0}) = 1.32$ to $F(y_{*,1}) = 1.33$ by fixing the other creditors' threshold at 1.32. However, after taking into account the rat race among the creditors, each would use a new equilibrium threshold of 1.375, which implies that the rat race amplifies the effect of the volatility increase by 4.5 times. Overall, Figure 4 shows that as the asset volatility increases, a preemptive run by the creditors becomes much more imminent as each creditor dramatically increases his rollover threshold.

Frantic Runs from Volatility Figure 4 illustrates an alarming possibility that once the asset volatility σ rises above 40%, the equilibrium rollover threshold $F(y_{*,\infty})$ surpasses $1/\alpha$. That is, even if the firm is so well capitalized that the firm's asset value is sufficient to cover all the debt after taking the liquidation discount, creditors are still not assured and would choose to run. How could this happen? This type of frantic debt runs happens exactly because of the creditors' preemptive motives. Even though, the current liquidation value is

asset to drop below the firm's debt face value.

sufficient to pay off the debt now, volatility could cause the liquidation value to drop below the debt value during a creditor's new contract period. Concerns about the possible runs by other maturing creditors in these future states could be strong enough to cause the creditor to run despite the current strong fundamental. Such frantic debt runs also depend on the lock-in effect of the debt contract, i.e., a creditor cannot run to safety before his contract matures, and the staggered maturity structure, i.e., some other creditors are released earlier and would run when the firm fundamental deteriorates. Taken together, the existence of the frantic runs contradicts a common sense argument that as long as a firm's liquidation value is sufficient to cover its debt, debt runs can be prevented.

However, this argument applies to the static synchronous debt structure described in Section 3.3.1. Suppose that the firm's debt contracts all mature at the same time now and each creditor needs to simultaneously decide whether to roll over into a perpetual debt contract which only matures when the firm asset matures. In this setting, which is similar to the various extensions of the Diamond-Dybvig model (including Goldstein and Pauzner (2005)), the capacity of the firm's current liquidation value to cover the debt is sufficient to eliminate the need of each creditor to worry about the panic runs of other creditors (i.e., the state is in the upper dominance region). To the extent that this argument could fail in our setting, this suggests that the debt structure in our setting could be more unstable than the static synchronous structure. Two factors could have contributed to the instability, one is the short debt maturity and the other is the staggered maturity structure. The staggered maturity structure is widely used by firms in practice as it is perceived to reduce firms' liquidity risk. However, the frantic debt runs partially caused by the staggered maturity structure raises an intriguing question about the efficiency of this structure in mitigating liquidity risk. A complete examination of this question is beyond the scope of this paper, as it involves a more general model to nest both synchronous and staggered maturity structures. We leave such an analysis to future research.

4.3 Effects of Rollover Frequency

We now discuss the effects of the firm's rollover frequency δ, another key determinant of the rollover risk. As δ increases, each creditor's contract period, which has an expected duration of $1/\delta$, gets shorter. This generates two opposing effects on the equilibrium. First, each individual creditor is locked in for a shorter period. As a result, the creditor has more flexibility to pull out if the firm fundamental deteriorates. The increased flexibility makes the

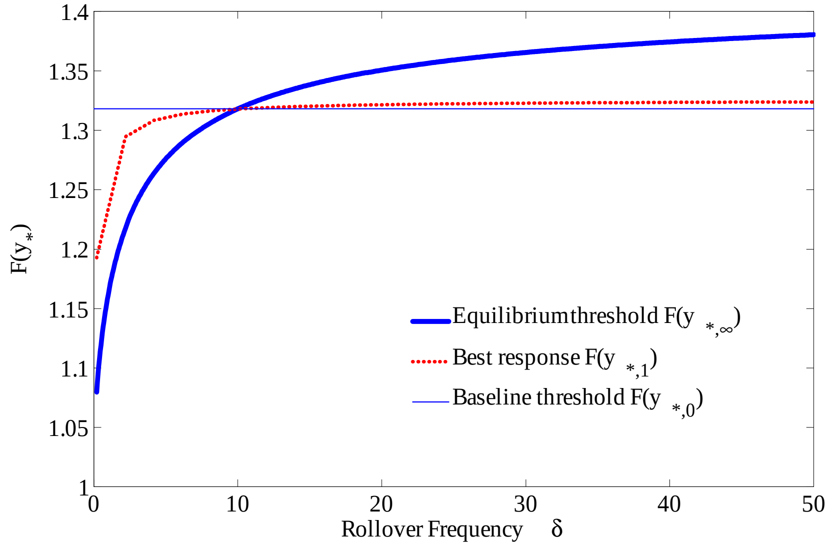

Figure 5: The equilibrium rollover threshold, measured in the firm asset's fundamental value $F(y_*)$, vs the rollover frequency δ. This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$. The thin solid line is the equilibrium threshold $F(y_{*,0})$ under the baseline parameters. The thick solid line plots the equilibrium threshold $F(y_{*,\infty})$, while the dashed line plots a creditor's best repones $F(y_{*,1})$ to the change in α from its baseline value if the other creditors' threshold is fixed at the baseline level.

creditor more willing to roll over his debt, i.e., to choose a lower rollover threshold. On the other hand, a higher δ also means that the other creditors are locked in for a shorter period. As a result, during the creditor's contract period, the firm is more susceptible to the rollover risk created by the other creditors. The increased rollover risk therefore motivates him to choose a higher rollover threshold. The equilibrium threshold y_* trades off the flexibility effect and the rollover risk effect.

Figure 5 plots the equilibrium rollover threshold (the thick solid line) as we vary δ from its baseline value of 10 to a range between 0.2 to 50, along with an individual creditor's best response (the dashed line) to the δ change while fixing other creditors' rollover threshold at the baseline level of 1.32. As δ increases from 0.2 to 50, the equilibrium rollover threshold $F(y_*)$ increases from 1.08 to 1.38. This monotonically increasing pattern in $F(y_*)$ suggests that the rollover risk effect dominates the flexibility effect in this illustration.[15] We again observe a dramatic amplification effect caused by the rat race among the creditors in choosing higher and higher thresholds. For instance, consider raising δ from the baseline level 10 to

[15] In unreported numerical analysis, we also find that the flexibility effect could dominates the rollover risk effect when θ is low, i.e., when the bank is sufficiently robust to the runs by the creditors.

50, which implies an average debt duration of about 1 week. An individual creditor would slightly increase his rollover threshold by 0.002 in the absence of the rat race, while the new equilibrium threshold is higher by 0.06, implying that the rat race amplifies the effect of the δ increase by about 30 times.

The Maturity Rat Race The important role played by the firm's rollover frequency motivates a natural question: What would happen if creditors are allowed to choose their rollover frequency? It is intuitive from our earlier discussion that each creditor would prefer a higher rollover frequency for himself so that he has more flexibility to pull out of a troubled firm. More formally, we can derive the following proposition:

Proposition 7 *Controlling for the other creditors' rollover frequency, each creditor's value function increases with his own rollover frequency.*

This proposition suggests that each individual creditor has the incentive to bribe the firm for a shorter debt maturity. In the absence of any commitment device like debt covenants or regulatory requirement, the firm would be willing to reduce the debt maturity of an atomless individual creditor because it does not affect on the overall probability of the firm failure. Since this argument applies to every creditor, it could trigger another rat race among the creditors in demanding shorter and shorter debt maturities, in addition to the one illustrated in Section 3.4 in choosing higher and higher rollover thresholds. As each creditor prefers to have the option to pull out before others, everyone wants a maturity shorter than the others'. This incentive is especially strong when the firm fundamental is falling. As a result, the equilibrium rollover frequency δ would diverge to infinity, which translates to ultra-short-term financing with zero maturity. This maturity rat race would, however, make the firm highly unstable and thus generate negative externality to other creditors. This mechanism explains why short-term financing becomes more and more pervasive—to some extent overly used—by financial institutions in the 2008 Wall Street Crisis, and thus calls for regulatory measures to force creditors to secure longer term financing in order to stabilize the financial system.[16]

Repo Runs The shortest debt contract in practice is overnight repo agreement with a maturity of one day. As we saw during the 2008 Wall Street Crisis, many financial firms

[16] For simplicity, we do not explicitly analyze issues related to endogenous interest payments, which would arise in a more formal analysis of the maturity rat race. See Brunnermeier and Oehmke (2009) for such an analysis in a model with two periods.

shifted a greater and greater fraction of their debt financing into overnight repos, e.g., Gorton and Metrick (2009). The conventional deadline of settling all repo agreements before 5pm deserves a special comment. It is important to note that practical reasons makes it impossible for a firm to wait until the last minute to process all of its repo transactions. Imagine if this is the case, it must be a super stressful one minute for the traders to simultaneously negotiate thousands of repo agreements for the firm. Thus, despite the simultaneous deadline of the overnight repos, practical considerations imply that some of the firm's repos will be negotiated at an earlier time during the day (although the physical transaction might occur at the end of the day), while some others later. As a result, when a creditor makes a decision on whether to roll over a repo agreement with the firm, he still faces the uncertainty that other creditors might choose to run on the firm, which is the key in our model. In some sense, the random maturity of an individual debt contract in our model well captures the randomness in the exact time that an individual repo agreement gets negotiated. Thus, our model provides a reasonable framework to analyze runs on repos once we set the firm's rollover frequency to be 250.[17]

4.4 Credit Risk

The standard credit modeling approach, following the classic structural model of Merton (1974), ignores firms' rollover risk by assuming that a solvent firm can always roll over its debt. Instead, it focuses on insolvency risk (i.e., the risk that the firm's asset value could fall below the debt face value) as the only source of credit risk. However, in an illiquid market environment, firms also face rollover risk caused by the coordination problem between their creditors. Our model provides a tractable framework to incorporate rollover risk as an additional source of credit risk. To illustrate this effect, we examine the credit spread of a hypothetical bond with face value 1 and fixed maturity T, issued by the firm analyzed in our model. Suppose that each bond provides the following payoff depending on three scenarios: 1) if the bank's asset matures before T and before a forced liquidation, the bond pays $\min\left(y_{\tau_\phi}, 1\right)$; 2) if a forced liquidation occurs before T and before the asset maturity, the bond pays $\min\left(L + l y_{\tau_\theta}, 1\right)$, the liquidation value of the bank asset; 3) otherwise, the bond pays 1. This payoff effectively captures the bank's credit risk before time T. The credit spread is the difference between its yield and the yield of a risk-free bond with the same

[17] As it is likely that more repos are negotiated near the end of the day, one can make our model more realistic by setting the firm's rollover frequency δ to be a smooth function of the time during the day.

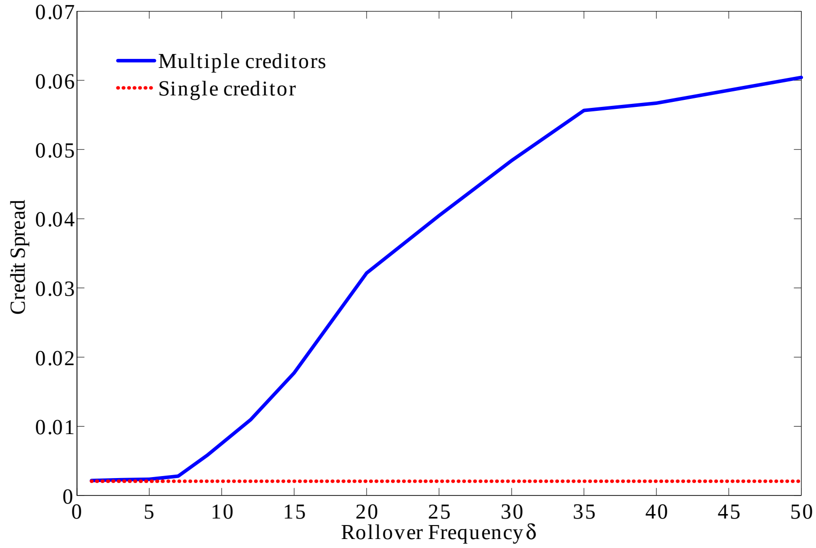

Figure 6: Credit spread vs debt rollover frequency. This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$, $y_0 = 1$, and $T = 0.25$.

maturity.[18] For comparison, we also introduce another firm identical in all other dimensions except that it is financed by a single creditor with deep pockets. Since the single creditor will internalize the cost of run, there is no rollover risk for this firm.

Figure 6 plots the credit spreads of the two firms with respect to their debt rollover frequency δ, based on the model parameters given in (9) and $y_0 = 1$, $T = 0.25$ (3 months). The difference between these two credit spreads measures the contribution of rollover risk to the credit risk of the firm with multiple creditors. The credit spread of the firm with a single creditor is independent of δ. However, the credit spread of the firm with multiple creditors increases sharply from less than 0.2% to over 6% as δ increases from 1 to 50 (i.e., from once every one year to once every week). This illustration shows that rollover risk could be a substantial part of firms' credit risk. Morris and Shin (2004, 2009) also emphasize that inefficiency in coordinating creditors' rollover decisions could increase financial institutions' credit risk. They model the coordination problems between creditors in two-period settings using global games with asymmetric information. Our continuous-time setting has a potential advantage in calibrating this effect.

[18] Our risky bond receives a payoff at a random time before the bond maturity T. For a fair comparison, we also impose the same random maturity on the risk-free bond, which has a value of $\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi} e^{-(\rho+\phi)T}$. Then we calculate the yield earned by the risk-free bond as $\beta_{riskfree} = -\frac{1}{T} \ln \left(\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi} e^{-(\rho+\phi)T} \right)$. The credit spread is measured relative to this yield.

5 Conclusion and Further Discussion

In this paper, we develop a dynamic model to debt runs generated by the coordination problem between creditors whose debt contracts with a firm mature at different times. In deciding whether to roll over his debt, each creditor faces the firm's future rollover risk with other creditors, i.e., the firm fundamental could fall during his contract period, causing other maturing creditors to run and thus forcing the firm to liquidate its asset at a fire sale price. Our model shows that even in the absence of any fundamental deterioration, changes in the volatility and liquidation value of the firm asset could trigger preemptive runs by creditors on a solvent firm. Such preemptive runs originate from the lack of commitment from future maturing creditors to roll over their debt contracts, rather than the lack of communication between creditors in static models of runs.

For simplicity, our model ignores several potentially important features relating to firms' liquidity management. In reality, firms can hold liquidity reserves against creditors' runs. The liquidity reserve can buffer some liquidity shocks, though typically not large enough to accommodate the runs by all creditors. Even though our model does not explicitly incorporate a cash reserve, the firm's ability to sustain the creditors' withdrawal for a period of time, which is inversely measured by the parameter θ, partially captures the role of a cash reserve inside the firm. The fact that in reality firms usually do not hold a sufficient amount of cash reserves against their short-term liabilities suggests a high opportunity cost of holding cash and/or liquid assets, and thus supports our simplified treatment. If we incorporate a cash reserve into the model, individual creditors' rollover decision would become reserve dependent. We do not expect such an extension to alter the key debt-run mechanism illustrated in the current model, although it could lead to richer implications about the dynamics of debt runs.

Another interesting issue is that as the fundamental deteriorates, the firm could raise interest payments to offset the creditors' incentives to run. However, to do so, the firm needs to have sufficient cash reserves to pay for the increased interest payments, which might not be realistic for a firm in the middle of debt runs. But, nevertheless, this consideration again points to the strategic importance of cash reserves. The bank could choose low interest payments and save some cash flows in normal periods when the fundamental is high, only to pay for the high interest payments in crisis times. We will leave this important and realistic issue for our future research.

A Appendix

A.1 Proof of Theorem 1

Using the Bellman equation in (8), we first construct an individual creditor's value function by assuming that he and all the other creditors use the same monotone strategy with a threshold y_*. This assumption implicitly imposes that $V(y;y_*)>1$ for $y>1$ and $V(y;y_*)<1$ for $y<1$. We will verify that this condition indeed holds in the equilibrium later. Under this assumption, the Bellman equation (8) becomes

- If $y<y_*$,

$$0=\frac{\sigma^2}{2}y^2V_{yy}+\mu yV_y-[\rho+\phi+(\theta+1)\delta]\,V(y;y_*)+\phi\min(1,y)+\theta\delta\min(L+ly,1)+r+\delta; \tag{10}$$

- If $y\geq y_*$,

$$0=\frac{\sigma^2}{2}y^2V_{yy}+\mu yV_y-(\rho+\phi)\,V(y;y_*)+\phi\min(1,y)+r. \tag{11}$$

The value function has to satisfy these two differential equations and be continuous and differentiable at the boundary point y_*. In solving these differential equations, we need to use the two solutions to the fundamental equation:

$$\frac{1}{2}\sigma^2x(x-1)+\mu x-[\rho+\phi+(1+\theta)\,\delta]=0,$$

which are

$$-\gamma_1=-\frac{\mu-\frac{1}{2}\sigma^2+\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2\left[\rho+\phi+(1+\theta)\,\delta\right]}}{\sigma^2}<0$$

and

$$\eta_1=-\frac{\mu-\frac{1}{2}\sigma^2-\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2\left[\rho+\phi+(1+\theta)\,\delta\right]}}{\sigma^2}>1,$$

and the two solutions to the fundamental equation:

$$\frac{1}{2}\sigma^2x(x-1)+\mu x-(\rho+\phi)=0, \tag{12}$$

which are

$$-\gamma_2=-\frac{\mu-\frac{1}{2}\sigma^2+\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2\left(\rho+\phi\right)}}{\sigma^2}<0$$

and

$$\eta_2=-\frac{\mu-\frac{1}{2}\sigma^2-\sqrt{\left(\frac{1}{2}\sigma^2-\mu\right)^2+2\sigma^2\left(\rho+\phi\right)}}{\sigma^2}>1.$$

We summarize the constructed value function below.

Lemma 8 *Suppose that every creditor uses a monotone strategy with a rollover threshold y_*. Then the value function of an individual creditor is given by the following three cases:*

1. *If $y_* < 1$,*

$$V(y;y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y + A_1 y^{\eta_1} & \text{when } 0 < y \le y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y + A_2 y^{-\gamma_2} + A_3 y^{\eta_2} & \text{when } y_* < y \le 1 \\ \frac{r+\phi}{\rho+\phi} + A_4 y^{-\gamma_2} & \text{when } 1 < y \end{cases} \tag{13}$$

The four coefficients A_1, A_2, A_3, and A_4 are given by

$$\begin{aligned} A_1 &= \frac{[H_3\gamma_2 + H_1] - y_*^{-\eta_2}(\gamma_2 H_4 + H_2 y_*)}{(\eta_1+\gamma_2)\, y_*^{\eta_1-\eta_2}} \\ A_2 &= \frac{y^{*\gamma_2}}{\eta_2+\gamma_2}\left[\eta_2 H_4 - H_2 y_* + A_1(\eta_2-\eta_1)\, y_*^{\eta_1}\right] \\ A_3 &= \frac{y_*^{-\eta_2}}{\eta_2+\gamma_2}\left[\gamma_2 H_4 + H_2 y_* + A_1(\eta_1+\gamma_2)\, y_*^{\eta_1}\right] \\ &= \frac{1}{\eta_2+\gamma_2}[H_3\gamma_2 + H_1] \\ A_4 &= A_2 - \frac{1}{\eta_2+\gamma_2}[H_3\eta_2 - H_1] \end{aligned}$$

where

$$\begin{aligned} H_1 &= -\frac{\phi}{\rho+\phi-\mu} \\ H_2 &= \frac{\theta\delta l(\rho+\phi-\mu) - \phi(1+\theta)\delta}{(\rho+\phi+(1+\theta)\delta-\mu)(\rho+\phi-\mu)} \\ H_3 &= -\frac{\phi\mu}{(\rho+\phi)(\rho+\phi-\mu)} \\ H_4 &= \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} - \frac{r}{\rho+\phi} + H_2 y_* \end{aligned}$$

2. *If $1 < y_* \le \frac{1-L}{l}$,*

$$V(y;y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y + B_1 y^{\eta_1} & \text{when } y \le 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y + B_2 y^{-\gamma_1} + B_3 y^{\eta_1} & \text{when } 1 < y \le y_* \\ \frac{r+\phi}{\rho+\phi} + B_4 y^{-\gamma_2} & \text{when } y_* < y \end{cases} . \tag{14}$$

The four coefficients B_1, B_2, B_3, and B_4 are given by

$$\begin{aligned}
B_1 &= B_3 - \frac{M_2\gamma_1 + M_1}{(\eta_1+\gamma_1)} \\
B_2 &= \frac{M_2\eta_1 - M_1}{(\eta_1+\gamma_1)} < 0 \\
B_3 &= \frac{(\gamma_1-\gamma_2) B_2 (y_*)^{-\gamma_1} + \gamma_2 M_3 - \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*}{(\eta_1+\gamma_2) y_*^{\eta_1}} \\
B_4 &= \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} B_2 y_*^{\gamma_2-\gamma_1} + \frac{\eta_1 - 1}{\eta_1+\gamma_2} \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*^{\gamma_2+1} \\
&\quad -\frac{\eta_1}{\eta_1+\gamma_2}\left[\frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}\right] y_*^{\gamma_2} \\
&= \frac{(\eta_1+\gamma_1) B_2 (y_*)^{-\gamma_1} - \eta_1 M_3 - \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*}{(\eta_1+\gamma_2) y_*^{-\gamma_2}}
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \frac{\phi}{\rho+\phi+(1+\theta)\delta-\mu} \\
M_2 &= \frac{\phi\mu}{(\rho+\phi+(1+\theta)\delta)(\rho+\phi+(1+\theta)\delta-\mu)} \\
M_3 &= \frac{r+\phi}{\rho+\phi} - \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} - \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_*
\end{aligned}$$

3. *If $y_* > \frac{1-L}{l}$,*

$$V(y; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y + C_1 y^{\eta_1} & \text{when } y \le 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y + C_2 y^{-\gamma_1} + C_3 y^{\eta_1} & \text{when } 1 < y \le \frac{1-L}{l} \\ \frac{r+\phi+\theta\delta+\delta}{\rho+\phi+(1+\theta)\delta} + C_4 y^{-\gamma_1} + C_5 y^{\eta_1} & \text{when } \frac{1-L}{l} < y \le y_* \\ \frac{r+\phi}{\rho+\phi} + C_6 y^{-\gamma_2} & \text{when } y > y_* \end{cases} \tag{15}$$

The six coefficients C_1, C_2, C_3, C_4, C_5 and C_6 are given by

$$\begin{aligned}
C_1 &= C_3 - \frac{K_4\gamma_1 + K_5}{\eta_1+\gamma_1} \\
C_2 &= \frac{K_4\eta_1 - K_5}{\eta_1+\gamma_1} \\
C_3 &= C_5 + \frac{K_2\gamma_1 - K_3\frac{1-L}{l}}{(\eta_1+\gamma_1)\left(\frac{1-L}{l}\right)^{\eta_1}} \\
C_4 &= C_2 - \frac{K_2\eta_1 + K_3\frac{1-L}{l}}{(\eta_1+\gamma_1)\left(\frac{1-L}{l}\right)^{-\gamma_1}} \\
C_5 &= \frac{(\gamma_1-\gamma_2) C_4 y_*^{-\gamma_1} - \gamma_2 K_1}{(\eta_1+\gamma_2) y_*^{\eta_1}} \\
C_6 &= \frac{(\eta_1+\gamma_1) C_4 y_*^{-\gamma_1} + \eta_1 K_1}{(\eta_1+\gamma_2) y_*^{-\gamma_2}}
\end{aligned}$$

where

$$
\begin{aligned}
K_1 &= \frac{r+\phi+\theta\delta+\delta}{\rho+\phi+(1+\theta)\,\delta} - \frac{r+\phi}{\rho+\phi} \\
K_2 &= \frac{\theta\delta\,(1-L)}{\rho+\phi+(1+\theta)\,\delta} - \frac{\theta\delta\,(1-L)}{\rho+\phi+(1+\theta)\,\delta-\mu} \\
K_3 &= \frac{\theta\delta l}{\rho+\phi+(1+\theta)\,\delta-\mu} \\
K_4 &= \frac{\phi}{\rho+\phi+(1+\theta)\,\delta-\mu} - \frac{\phi}{\rho+\phi+(1+\theta)\,\delta} \\
K_5 &= \frac{\phi}{\rho+\phi+(1+\theta)\,\delta-\mu}.
\end{aligned}
$$

Proof. We can derive the three cases listed above using the same method. Here we illustrate using the first case that $y_* < 1$. Depending on the value of y, we have the following three scenarios.

- If $0 < y \leq y_*$:

  $$\frac{\sigma^2}{2}y^2 V_{yy} + \mu y V_y - [\rho+\phi+(1+\theta)\,\delta]\,V(y) + (\phi+\theta\delta l)\,y + r + \theta\delta L + \delta = 0.$$

  The general solution of this differential equation is given in the first line of equation (13) with the coefficient A_1 to be determined by the boundary conditions. Note that to ensure the value of V to be finite as y approaches zero, we have ruled out another power solution of the equation $y^{-\gamma_1}$.

- If $y_* < y \leq 1$:

  $$\frac{\sigma^2}{2}y^2 V_{yy} + \mu y V_y - (\rho+\phi)\,V(y) + \phi y + r = 0.$$

  The general solution of this differential equation is given in the second line of equation (13) with the coefficients A_2 and A_3 to be determined by the boundary conditions.

- If $y > 1$:

  $$\frac{\sigma^2}{2}y^2 V_{yy} + \mu y V_y - (\rho+\phi)\,V(y) + r + \phi = 0.$$

  The general solution of this differential equation is given in the third line of equation (13) with the coefficient A_4 to be determined by the boundary conditions. Note that to ensure the value of V to be finite as y approaches infinity, we have ruled out another power solution of the equation y^{η_2}.

To determine the four coefficients A_1, A_2, A_3, and A_4, we have four boundary conditions at $y = y_*$ and 1, i.e., the value function $V(y)$ must be continuous and differentiable at these two points. Solving these boundary conditions leads to the coefficients given in Lemma 8. ■

Based on the value function derived in Lemma 8, we now show that there exists a unique threshold y_* for the equilibrium condition to hold.

Lemma 9 *There exists a unique y_* such that*

$$V(y_*; y_*) = 1.$$

Proof. Define

$$W(y) \equiv V(y; y).$$

We need to show that there is a unique y_* such that $W(y_*) = 1$.

We first show that $W(y)$ is monotonically increasing when $y < 1$. In this case, we can directly extract the value of $W(y)$ from equation (13), which, by neglecting terms independent of y, is given

$$\begin{aligned} W(y) &= \left[\frac{\phi + \theta\delta l}{\rho + \phi + (1+\theta)\delta - \mu} - \frac{1+\gamma_2}{\eta_1+\gamma_2}H_2\right] y + \frac{[H_3\gamma_2 + H_1]}{\eta_1+\gamma_2} y^{\eta_2} \\ &= \left[-H_1 + \frac{\eta_1 - 1}{\eta_1+\gamma_2}H_2\right] y + \frac{[H_3\gamma_2 + H_1]}{\eta_1+\gamma_2} y^{\eta_2}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{dW(y)}{dy} &= -H_1 + \frac{\eta_1 - 1}{\eta_1+\gamma_2}H_2 + \frac{[H_3\gamma_2 + H_1]}{\eta_1+\gamma_2}\eta_2 y^{\eta_2 - 1} \\ &> -H_1 + \frac{\eta_1 - 1}{\eta_1+\gamma_2}H_2 + \frac{[H_3\gamma_2 + H_1]}{\eta_1+\gamma_2}\eta_2 \\ &= -H_1 + \frac{\eta_1 - 1}{\eta_1+\gamma_2}H_2 + \frac{\eta_2}{\eta_1+\gamma_2}H_1 + \frac{\gamma_2\eta_2}{(\eta_1+\gamma_2)}H_3 \\ &= \frac{\eta_1 - 1}{\eta_1+\gamma_2}(H_2 - H_1) + \frac{\eta_2 - \gamma_2 - 1}{\eta_1+\gamma_2}H_1 + \frac{\gamma_2\eta_2}{(\eta_1+\gamma_2)}H_3 \end{aligned}$$

where the inequality is due to $H_3 < 0$ and $H_1 < 0$.

Note that in the first term above,

$$H_2 - H_1 = \frac{\theta\delta l + \phi}{\rho + \phi + (1+\theta)\delta - \mu}$$

is positive according to the parameter restriction in (5). For the second term, note that $\eta_2 - \gamma_2 - 1 = -2\frac{\mu}{\sigma^2}$. Then after some algebraic substitution (note that $\gamma_2\eta_2 = \frac{2(\rho+\phi)}{\sigma^2}$), the sum of the second and third terms is

$$-2\frac{\mu}{\sigma^2}\frac{1}{(\eta_1+\gamma_2)}H_1 + \frac{\gamma_2\eta_2}{(\eta_1+\gamma_2)}H_3 = 0$$

Thus, $\frac{dW(y)}{dy} > 0$.

We now show that $W(y)$ is monotonically increasing when $1 < y \leq \frac{1-L}{l}$. Equation (14) implies that

$$\begin{aligned} W(y) &= \frac{r+\phi}{\rho+\phi} + B_4 y^{-\gamma_2} \\ &= \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2}B_2 y^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1+\gamma_2}\frac{\theta\delta l}{\rho + \phi + (1+\theta)\delta - \mu} y \\ &\quad + \frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r+\phi}{\rho+\phi} + \frac{\eta_1}{\eta_1+\gamma_2}\frac{r + \phi + \theta\delta L + \delta}{\rho + \phi + (1+\theta)\delta} \end{aligned}$$

We now show $B_2 < 0$, which is equivalent to showing $\eta_1 < \frac{M_1}{M_2} = \frac{\rho+\phi+(1+\theta)\delta}{\mu}$. Plugging $x = \frac{\rho+\phi+(1+\theta)\delta}{\mu}$ into the fundamental equation, we find that the value is positive. This implies that $\eta_1 < \frac{M_1}{M_2}$. Now because $\eta_1 > 1$, $W(y)$ is increasing in y.

Similarly we can show that $W(y)$ is monotonically increasing when $y > \frac{1-L}{l}$. Equation (15) implies that

$$\begin{aligned}
W(y) &= \frac{r+\phi}{\rho+\phi} + C_6 y_*^{-\gamma_2} = \frac{r+\phi}{\rho+\phi} + \frac{(\eta_1+\gamma_1) C_4 y_*^{-\gamma_1} + \eta_1 K_1}{\eta_1+\gamma_2} \\
&= \frac{\gamma_2}{\eta_1+\gamma_2}\frac{r+\phi}{\rho+\phi} + \frac{\eta_1}{\eta_1+\gamma_2}\frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{(\eta_1+\gamma_1) C_4 y_*^{-\gamma_1}}{\eta_1+\gamma_2}.
\end{aligned}$$

Therefore, $W(y)$ is strictly increasing if and only if $C_4 < 0$. C_4 is given by

$$C_4 = C_2 - \frac{K_2\eta_1 + K_3\frac{1-L}{l}}{(\gamma_1+\eta_1)\left(\frac{1-L}{l}\right)^{-\gamma_1}} < C_2 - \frac{K_2\eta_1 + K_3\frac{1-L}{l}}{(\gamma_1+\eta_1)} = \frac{\frac{\mu\eta_1}{\rho+\phi+(1+\theta)\delta} - 1}{\gamma_1+\eta_1}\frac{\phi+\theta\delta(1-L)}{\rho+\phi+(1+\theta)\delta-\mu}.$$

As a result, $C_4 < 0$ if $\eta_1 < \frac{\rho+\phi+(1+\theta)\delta}{\mu}$, which we have shown in the case of $1 < y \leq \frac{1-L}{l}$.

Next, we need to ensure that $W(0) < 1$. Equation (13) implies that

$$W(0) = \frac{\eta_1}{(\eta_1+\gamma_2)}\frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r}{\rho+\phi}.$$

The parameter restriction in (4) insures that

$$\frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} < 1 \quad \text{and} \quad \frac{r}{\rho+\phi} < 1\ .$$

thus, $W(0) < 1$.

Finally note that under our parameter restrictions in (4) and (6) we have

$$W(\infty) = \frac{\gamma_2}{\eta_1+\gamma_2}\frac{r+\phi}{\rho+\phi} + \frac{\eta_1}{\eta_1+\gamma_2}\frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} > 1.$$

Because $W(y)$ is monotonically increasing with $W(0) < 1$ and $W(\infty) > 1$, there exists a unique y_* such that $W(y_*) = 1$. ■

Lemma 9 implies that there can be at most one symmetric monotone equilibrium. Next, we verify that a monotone strategy with the threshold level determined in Lemma 9 is indeed optimal for an individual creditor if every other creditor uses this threshold.

Lemma 10 *If every other creditor uses a monotone strategy with a threshold y_* identified in Lemma 9, then the same strategy is also optimal for an individual creditor.*

Proof. If every other creditor uses the monotone strategy with the threshold y_*, to show that the value function constructed from solving the differential equations (10) and (11) is indeed optimal for an individual creditor, we simply need to verify that $V(y) < 1$ for any $y < 1$, and $V(y) > 1$ for any $y > 1$, which directly implies that the value function solves the Bellman equation (8). By construction in Lemma 8, $V(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} < 1$ and $V(\infty) = \frac{r+\phi}{\rho+\phi} > 1$. We just need to show that $V(y)$ only crosses 1 once at y_*.

We first consider the case that $y_* < 1$.

We prove by contradiction. Suppose that $V(y)$ also crosses 1 at another point below y_*. Then, there exists $y_1 < y_*$ such that

$$V(y_1) > V(y_*) = 1,\ V'(y_1) = 0, \text{ and } V''(y_1) < 0.$$

Using the differential equation (10), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + \phi \min(1, y_1) + \theta\delta(L + l y_1) + r + \delta}{\rho + \phi + (\theta+1)\delta} \\ &< \frac{(\phi + \theta\delta l)\, y_1 + \theta\delta L + r + \delta}{\rho + \phi + (\theta+1)\delta} < \frac{\phi + \theta\delta l + \theta\delta L + r + \delta}{\rho + \phi + (\theta+1)\delta} < 1. \end{aligned}$$

The last inequality is implied by the parameter restrictions in (4) and (7). This is a contradiction with $V(y_1) > 1$. Thus, $V(y)$ cannot cross 1 at any y below y_*. This also implies that $V'(y_*) > 0$.

Next, we show that $V(y)$ is monotonic in the region $y > y_*$. Suppose that $V(y)$ is non-monotone, then there exist two points $y_1 < y_2$ such that

$$V(y_1) > V(y_2),\ V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

(If, say, y_1 happens to be on the break point 1 where the second derivative is not necessary continuous, then take the point as $1+$ as $V''(1+)$ has to be negative.) According to the differential equation (11), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1)}{\rho + \phi} \\ &> \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2)}{\rho + \phi} = V(y_2) \end{aligned}$$

which is a contradiction.

We next consider the case that $y_* \geq 1$. We do not separate the two cases of $1 < y_* \leq \frac{1-L}{l}$ and $y_* > \frac{1-L}{l}$, as the following proof applies to both.

The expression in equation (14) or (15) implies that $V(y)$ has to approach $\frac{r+\phi}{\rho+\phi}$ from below (because $\frac{r+\phi}{\rho+\phi}$ is the debt holder's highest payoff possible), thus B_4 or C_6 is strictly negative. This implies that $V(y)$ is increasing on $[y_*, \infty)$, and

$$V'(y_*) > 0.$$

Now consider the region $[0, y_*)$, it is easy to check that $V'(0) > 0$. Therefore, if $V(y)$ is not monotonic in $[0, y_*)$, there must exist two points $y_1 < y_2$ such that

$$V(y_1) > V(y_2),\ V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

According to the Bellman equation, we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1) + \delta[1 + \theta \min(L + ly_1, 1)]}{\rho + \phi + (1+\theta)\delta} \\ &< \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2) + \delta[1 + \theta \min(L + ly_2, 1)]}{\rho + \phi + (1+\theta)\delta} = V(y_2) \end{aligned}$$

which is a contradiction. Thus, $V(y)$ is also monotonically increasing in $[0, y_*)$.

To summarize, we have shown that $V(y)$ only crosses 1 once at y_*. Thus, it is optimal for an individual creditor to roll over his debt if $y > y_*$ and to run if $y < y_*$. ■

Finally, we prove that there is not any asymmetric monotone equilibrium.

Lemma 11 *There does not exist any asymmetric monotone equilibrium in which creditors choose different rollover thresholds.*

Proof. We prove by contradiction. Suppose that there exists an asymmetric monotone equilibrium. Then, there exist at least two groups of creditors who use two different monotone strategies with thresholds $y_1 < y_2$. For creditors who use the threshold y_i, we denote their value function as $V^i(y)$. At the corresponding threshold, we must have

$$V^1(y_1) = V^2(y_2) = 1.$$

Moreover, we must have

$$V^1(y_2) = V^2(y_1) = 1,$$

because each creditor is free to switch between these two strategies. Then for $y \in [y_1, y_2]$, we must have $V^1(y) = V^2(y) = 1$, because otherwise it violates the optimality of the threshold strategies for both types of creditors. This implies that each creditor is indifferent between choosing any threshold in $[y_1, y_2]$. Denote the $\zeta(y)$ as the measure of creditors who use a threshold lower than $y \in [y_1, y_2]$. Then, V^i has to satisfy the Bellman equation in this region:

$$\begin{aligned} \rho V^i(y) &= \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy} + r + \phi\left[\min(1, y) - V^i(y)\right] \\ &\quad + \theta\delta\zeta(y)\left[\min(L + ly, 1) - V^i(y)\right] + \delta \max\left\{1 - V^i(y), 0\right\} \end{aligned}$$

Since $V^i(y) = 1$ for any $y \in [y_1, y_2]$, we have

$$\rho = r + \phi[\min(1, y) - 1] + \theta\delta\zeta(y)[\min(L + ly, 1) - 1].$$

Note that $\zeta(y)$ is non-decreasing in y because it is a distribution function. Since both $\min(1, y)$ and $\min(L + ly, 1)$ are also non-decreasing in y, the only possibility that the above equation holds is that $L + ly > 1$ and $y > 1$ for $y \in [y_1, y_2]$. Then, $\rho = r$ has to hold. This contradicts the parameter restriction that $\rho > r$. ■

A.2 Proof of Proposition 2

We first derive an individual creditor's value function $U(y)$ if the bank survives the creditors' rollover decisions at time 0 and thus will be able to stay until the asset maturity at τ_ϕ. $U(y)$ satisfies the following differential equation:

$$\rho U = \mu y U_y + \frac{1}{2}\sigma^2 y^2 U_{yy} + \phi\left[\min(1, y) - U\right] + r.$$

It is direct to solve this differential equation:

$$U(y) = \begin{cases} \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y + D_1 y^{\eta_2} & \text{if } 0 < y < 1 \\ \frac{r+\phi}{\rho+\phi} + D_2 y^{-\gamma_2} & \text{if } y > 1 \end{cases}. \tag{16}$$

where

$$\begin{aligned} D_1 &= -\frac{\frac{\phi}{\rho+\phi-\mu} + \gamma_2 \frac{\phi\mu}{(\rho+\phi-\mu)(\rho+\phi)}}{\eta_2 + \gamma_2} \\ D_2 &= \frac{-\frac{\phi}{\rho+\phi-\mu} + \eta_2 \frac{\phi\mu}{(\rho+\phi-\mu)(\rho+\phi)}}{\eta_2 + \gamma_2}. \end{aligned}$$

D_1 and D_2 are constant and independent of the liquidation recovery parameter α. Because $U(y)$ is dominated by the fundamental value of the bank asset, $U(y) < \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu} y$. This implies that $D_1 < 0$. In addition, since $U(\infty) = \frac{r+\phi}{\rho+\phi}$, $D_2 < 0$ and $U(y)$ approaches $\frac{r+\phi}{\rho+\phi}$ from below. Therefore $U(y)$ is a monotonically increasing function with

$$U(0) = \frac{r}{r+\phi} < 1 \quad \text{and} \quad U(\infty) = \frac{r+\phi}{\rho+\phi} > 1.$$

Then the intermediate value theorem implies that there exists $y_l > 0$ such that $U(y_l) = 1$.

Define $y_h \equiv \frac{1-L}{l}$. According to the parameter restriction (6), $y_h > 1$. We impose the following condition for the cost of a premature liquidation to be sufficiently large, i.e., α is sufficiently small:

$$\alpha < \frac{\rho+\phi-\mu}{\phi\left[\left[-D_2 \frac{(\rho+\phi)}{r+\phi}\right]^{\frac{1}{\gamma_2}} + \frac{r(\rho+\phi-\mu)}{\phi(\rho+\phi)}\right]}. \tag{17}$$

This condition is analogous to the parameter restriction (6) in our main model. Given this condition and that $\frac{1-L}{l} = \frac{\rho+\phi-\mu}{\phi\alpha} - \frac{r(\rho+\phi-\mu)}{\phi(\rho+\phi)}$, we have

$$U\left(\frac{1-L}{l}\right) = \frac{r+\phi}{\rho+\phi} + D_2\left(\frac{1-L}{l}\right)^{-\gamma_2} > 1,$$

which further implies that $y_l < y_h = \frac{1-L}{l}$.

Next, we show that if $y_0 > y_h$, then it is optimal for an individual creditor to roll over even if all the other creditors choose to run. In this case, we assume that there is a probability of $\theta_s \in (0, 1)$ that the bank cannot find new creditors to replace the outgoing ones and is forced into a premature liquidation. Note that in this simultaneous rollover setting, the liquidation probability parameter θ_s has to be inside $(0, 1)$, while the liquidation probability parameter θ in the main model can be higher than 1 because the creditors' rollover decisions are spread out over time. Note that the liquidation value of the bank asset is sufficient to pay off all the creditors because $L + ly_0 > 1$. Thus, the creditor's expected payoff from choosing run is $\theta_s + (1-\theta_s) = 1$. His expected payoff from choosing rollover is $\theta_s + (1-\theta_s) U(y_0)$, which is higher than the expected payoff from choosing run.

Next, we show that if $y_0 < y_l$, then it is optimal for an individual creditor to run even if all the other creditors choose rollover. In this case, the bank will always survive no matter what the individual creditor's decision is. If he chooses to run, he gets a payoff of 1, while if he chooses to roll over, his continuation value function is $U(y_0)$. Thus, it is optimal for the creditor to run.

Finally, we consider the case when $y_0 \in [y_l, y_h]$. If all the other creditors choose to roll over, then an individual creditor's payoff from run is 1, while his continuation value function is $U(y_0)$. Thus it is optimal for him to roll over too. If all the other creditors choose to run, then his expected payoff from run is $\theta_s (L + ly_0) + (1-\theta_s)$. His expected payoff from choosing rollover is $(1-\theta_s) U(y_0)$, because once the bank is forced into a premature liquidation, the liquidation value of the bank asset is not sufficient to pay off the other outgoing creditors and the creditor who chooses to roll over gets zero. Analogous to the parameter restriction (7) of our main model, we impose a parameter restriction on θ_s so that it is sufficiently large:

$$\frac{\theta_s}{1-\theta_s} > \frac{1}{L}\frac{r-\rho}{\rho+\phi}.$$

Then, it is optimal for the creditor to run with other creditors.

A.3 Proof of Proposition 3

Consider any increasing sequence $\{\delta_n\}$ such that $\delta_n \to \infty$, and denote the corresponding equilibrium threshold sequence as $\{y_*(\delta_n)\}$ which satisfies $W(y_*(\delta_n); \delta_n) = 1$ with $W(y; \delta)$ defined in Lemma 9. If $\{y_*(\delta_n)\}$ does not converge to $\frac{1-L}{l}$, then for any $\varepsilon > 0$ and $\overline{\delta}$ there exists a $\delta_N > \overline{\delta}$ such that $y_*(\delta_N) \notin \left[\frac{1-L}{l} - \varepsilon, \frac{1-L}{l} + \varepsilon\right]$ and $W(y_*(\delta_N); \delta_N) = 1$. We have three cases to consider. In the following derivation, keep in mind that γ_1 and η_1 are in the order of $\sqrt{\delta_N}$, while γ_2 and η_2 are constant.

- Suppose that $y_*(\delta_N) > \frac{1-L}{l} + \varepsilon$. Then

$$W(y_*(\delta_N); \delta_N) = \frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r+\phi}{\rho+\phi} + \frac{\eta_1}{\eta_1+\gamma_2}\frac{r+\phi+\theta\delta_N L+\delta_N}{\rho+\phi+(1+\theta)\delta_N}$$
$$+\frac{(\eta_1+\gamma_1)\, y_*(\delta_N)^{-\gamma_1}}{\eta_1+\gamma_2}\left[\frac{K_4\eta_1 - K_5}{\gamma_1+\eta_1} - \frac{\theta\delta(1-L)\left(1-\frac{\eta_1}{\rho+\phi+(1+\theta)\delta_N}\right)}{(\gamma_1+\eta_1)(\rho+\phi+(1+\theta)\delta_N-\mu)}\left(\frac{1-L}{l}\right)^{\gamma_1}\right]$$

The first term goes to zero, the second term goes to $\frac{\theta L+1}{1+\theta}<1$, and the third term goes to zero as $y_*(\delta_N)^{-\gamma_1}$ dominates. Thus the sum of these terms contradicts with $W(y_*(\delta_N);\delta_N)=1$.

- Suppose that $1\le y_*(\delta_N)<\frac{1-L}{l}-\varepsilon$. Then

$$\begin{aligned} W(y_*(\delta_N);\delta_N) &= \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2}B_2 y_*(\delta_N)^{-\gamma_1}+\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{\theta\delta_N l}{\rho+\phi+(1+\theta)\delta_N-\mu}y_*(\delta_N) \\ &\quad +\frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r+\phi}{\rho+\phi}+\frac{\eta_1}{\eta_1+\gamma_2}\frac{r+\phi+\theta\delta_N L+\delta_N}{\rho+\phi+(1+\theta)\delta_N}. \end{aligned}$$

The first and third terms go to zero. The sum of second and fourth term converges to

$$\frac{\theta l}{1+\theta}y_*(\delta_N)+\frac{\theta L+1}{1+\theta}<1-\frac{\theta l}{1+\theta}\varepsilon,$$

which is again a contradiction with $W(y_*(\delta_N);\delta_N)=1$.

- Suppose that $y_*(\delta_N)<1$. Then

$$\begin{aligned} W(y_*) &= \frac{[H_3\gamma_2+H_1]}{(\eta_1+\gamma_2)}y_*(\delta_N)^{\eta_2}+\frac{\eta_1}{(\eta_1+\gamma_2)}\frac{r+\theta\delta_N L+\delta_N}{\rho+\phi+(1+\theta)\delta_N}+\frac{\gamma_2}{(\eta_1+\gamma_2)}\frac{r}{\rho+\phi} \\ &\quad +\left[\frac{\eta_1-1}{\eta_1+\gamma_2}\frac{\theta\delta_N l+\phi}{\rho+\phi+(1+\theta)\delta_N-\mu}+\frac{1+\gamma_2}{(\eta_1+\gamma_2)}\frac{\phi}{(\rho+\phi-\mu)}\right]y_*(\delta_N) \\ &\to \frac{\theta L+1}{1+\theta}+\frac{\theta l}{1+\theta}y_*(\delta_N)<\frac{1+\theta(L+l)}{1+\theta}<1 \end{aligned}$$

which is a contradiction. This concludes the proof.

A.4 Proof of Proposition 4

To be consistent with our main model, we restrict ourselves to monotone strategies based on the bank fundamental. Since the bank fundamental is constant, an individual creditor's strategy is to choose always rollover or run. Considering more flexible strategies would only make multiple equilibria more likely to emerge.

Suppose that all the other creditors always choose to run in the future. When an individual creditor needs to make his rollover decision, his payoff from run is 1, and his value function from always choosing rollover, based on the random debt maturity, is $\frac{r+\phi\min(y,1)+\theta\delta\min(L+ly,1)}{\rho+\phi+\theta\delta}$. Define

$$y_h^c\equiv\min\{y:r+\phi\min(y,1)+\theta\delta\min(L+ly,1)\ge\rho+\phi+\theta\delta\}.$$

Thus, if the other creditors always choose run in the future, rollover is optimal for the creditor if $y>y_h^c$, and run is optimal if $y\le y_h^c$.

Now suppose that all the other creditors always choose to roll over in the future. When an individual creditor needs to make his rollover decision, his payoff from run is 1, and his value function from always choosing rollover, based on the random debt maturity, is $\frac{r+\phi\min(y,1)}{\rho+\phi}$. Define

$$y_l^c\equiv\max\{y:r+\phi\min(y,1)\le\rho+\phi\}.$$

Thus, if the other creditors always choose to roll over in the future, run is optimal for an individual creditor if $y < y_l^c$, and rollover is optimal if $y \geq y_l^c$.

Next, we show that $y_h^c > y_l^c$. According to the definition of y_l^c, it suffices to show that

$$r + \phi \min(y_h^c, 1) > \rho + \phi.$$

Note that $y_h^c < \frac{1-L}{l}$, because

$$r + \phi \min\left(\frac{1-L}{l}, 1\right) + \theta\delta \min\left(L + l\frac{1-L}{l}, 1\right) = r + \phi + \theta\delta > \rho + \phi + \theta\delta.$$

Therefore according to the definition of y_h^c,

$$\begin{aligned} & r + \phi \min(y_h^c, 1) + \theta\delta(L + l y_h^c) = \rho + \phi + \theta\delta \\ \Rightarrow\quad & r + \phi \min(y_h^c, 1) = \rho + \phi + \theta\delta(1 - L - l y_h^c) > \rho + \phi, \end{aligned}$$

which implies that $y_h^c > y_l^c$. Therefore when $y \in [y_l^c, y_h^c]$, a creditor finds both rollover and run optimal depending on other creditors' strategy.

A.5 Proof of Proposition 5

1. $\mu > 0$ Case.

When $\mu > 0$, the bank fundamental will eventually travel to the upper dominance region, in which all creditors will always choose to roll over independent of other creditors' strategy. Let us first consider the value function of a creditor who is locked in by his current contract under the assumption that the other creditors in the future will always roll over:

$$V^R(y) \equiv E\left[\int_0^{\tau_\phi} e^{-\rho t} r dt + e^{-\rho\tau_\phi} \min\left(y_{\tau_\phi}, 1\right) \middle| y_0 = y\right] \tag{18}$$

It is easy to see that $V^R(y)$ is increasing with y and $V^R(1) = \frac{r+\phi}{\rho+\phi} > 1$. Define $y_{\mu+}$ as the solution to the unique equation

$$V^R(y) = 1.$$

It is clear that $y_{\mu+} < 1$. When $y > y_{\mu+}$, $V^R(y) > 1$. Thus, in this region, it is optimal for a maturing creditor to choose rollover knowing that every creditor after him will choose rollover. That is, the equilibrium is uniquely defined in the region $y > y_{\mu+}$, and the value function of an individual creditor who is currently in a debt contract is

$$V^{\mu+}(y) = V^R(y) \quad \text{if } \; y > y_{\mu+}.$$

However, when $y < y_{\mu+}$, it is optimal for a maturing creditor to run even if the other maturing creditors in the future will always choose rollover. Thus, it is reasonable to conjecture that in the equilibrium each maturing creditor indeed chooses run when $y \leq y_{\mu+} < 1$. We verify this in two steps: First, we construct the value function of a creditor under the assumption that every creditor (including himself) uses a monotone strategy with threshold $y_{\mu+}$; then, we show that $V^{\mu+}(y) < 1$ for $y < y_{\mu+}$.

Note that when $y < y_{\mu+}$, $V^{\mu+}$ satisfies

$$(\rho + \phi + (1+\theta)\delta) V^{\mu+} = \mu y V_y^{\mu+} + r + \phi y + \theta\delta (L + ly) + \delta, \tag{19}$$

with the boundary condition that $V^{\mu+}(y_{\mu+}) = 1$. Solving this equation provides that $V^R(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}$. Parameter restrictions (4) and (7) imply that

$$\frac{r + \phi + \theta\delta (L + l) + \delta}{\rho + \phi + (1+\theta)\delta} < 1,$$

which in turn provides that $V^{\mu+}(0) < 1$. Therefore, if $V^{\mu+}(y) > 1$ for some $y < y_{\mu+}$, then we must have some point $\widehat{y}$ such that $V^{\mu+}(\widehat{y}) > 1$ and $V_y^{\mu+}(\widehat{y}) = 0$. But then equation (19) implies that

$$V^{\mu+}(\widehat{y}) < \frac{r + \phi + \theta\delta (L + l) + \delta}{\rho + \phi + (1+\theta)\delta} < 1,$$

which contradicts with $V^{\mu+}(\widehat{y}) > 1$. Thus, $V^{\mu+}(\widehat{y}) < 1$ if $y < y_{\mu+}$. That is, it is optimal for a maturing creditor to choose run when $y \leq y_{\mu+}$.

This monotone equilibrium is unique, because there is only one $y_{\mu+}$ to satisfy the equilibrium condition of the threshold: $V^R(y_{\mu+}) = 1$.

2. $\mu < 0$ Case.

When $\mu < 0$, the bank fundamental will eventually travel to the lower dominance region, in which each maturing creditor will choose to run independent of other creditors' strategy. We first consider the value function $V^W(y)$ of a creditor who is locked in by his current contract, under the assumption that the other creditors will all choose run in the future. V^W satisfies

$$(\rho + \phi + (1+\theta)\delta) V^W = \mu y V_y^W + r + \phi \min(1, y) + \theta\delta (L + ly) + \delta, \tag{20}$$

with the boundary condition $V^W(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} < 1$. It is easy to show that V^W is increasing with y, therefore there exists a unique $y_{\mu-}$ such that

$$V^W(y_{\mu-}) = 1.$$

For $y < y_{\mu-}$, the general solution to equation (20) is

$$V^W(y) = \frac{r + \theta\delta L + \delta}{\rho + \phi + (1+\theta)\delta} + \frac{\phi + \theta\delta l}{\rho + \phi + (1+\theta)\delta - \mu} y + A y^{\frac{\rho+\phi+(1+\theta)\delta}{\mu}}$$

where A is constant. Because $\mu < 0$, A has to be zero because, otherwise, $V^W(0)$ diverges. Therefore, $V^W(1) < \frac{r+\phi+\theta\delta(L+l)+\delta}{\rho+\phi+(1+\theta)\delta} < 1$, which in turn implies that $y_{\mu-} > 1$. Thus, when $y < y_{\mu-}$, the equilibrium is uniquely determined and each maturing creditor chooses run knowing that other maturing creditors afterward will choose run. The value function of an individual creditor who is currently in a debt contract is

$$V^{\mu-}(y) = V^W(y) \quad \text{if} \;\; y < y_{\mu-}.$$

However, when $y > y_{\mu-}$, it is optimal for a maturing creditor to roll over even if other maturing creditors in the future will always choose run. Thus, it is reasonable to conjecture that in the equilibrium each maturing creditor indeed chooses rollover when $y > y_{\mu-} > 1$. We again verify this in two steps: First, we construct the value function of a creditor under the assumption that every creditor (including himself) uses a monotone strategy with threshold $y_{\mu-}$; then, we show that $V^{\mu-}(y) > 1$ for $y > y_{\mu-}$.

Note that if $y > y_{\mu-}$, $V^{\mu-}(y)$ satisfies

$$(\rho+\phi) V^{\mu-} = \mu y V_y^{\mu-} + r + \phi,$$

with the boundary condition $V^{\mu-}(y_{\mu-}) = 1$. The solution is $\frac{r+\phi}{\rho+\phi} + By^{\frac{\rho+\phi}{\mu}}$ where $B < 0$ is constant. This function is monotonically increasing. Thus, $V^{\mu-}(y) > 1$ if $y > y_{\mu-}$. In other words, rollover is optimal for a maturing creditor in the equilibrium if $y > y_{\mu-}$.

This monotone equilibrium is unique, because there is only one $y_{\mu-}$ to satisfy the equilibrium condition of the threshold: $V^W(y_{\mu-}) = 1$.

A.6 Proof of Proposition 6

Note that y_* is determined by the condition that $W(y_*) = V(y_*; y_*) = 1$. Theorem 1 implies that if $y_* > \frac{1-L}{l}$, it is determined by the following implicit function:

$$\begin{aligned} 1 \;\; = \;\; & W(y_*) = \frac{(\eta_1+\gamma_1) y_*^{-\gamma_1}}{\eta_1+\gamma_2} \left[\frac{K_4\eta_1 - K_5}{\gamma_1+\eta_1} - \frac{\theta\delta(1-L)\left(1 - \frac{\eta_1}{\rho+\phi+(1+\theta)\delta}\right)}{(\gamma_1+\eta_1)(\rho+\phi+(1+\theta)\delta-\mu)} \left(\frac{1-L}{l}\right)^{\gamma_1} \right] \\ & + \frac{\gamma_2}{(\eta_1+\gamma_2)} \frac{r+\phi}{\rho+\phi} + \frac{\eta_1}{\eta_1+\gamma_2} \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} \end{aligned}$$

where $L = \frac{\alpha r}{\rho+\phi}$ and $l = \frac{\alpha\phi}{\rho+\phi-\mu}$ increase with α, and K_4 and K_5 are independent of α. By the implicit function theorem, $\frac{dy_*}{d\alpha} = -\frac{\partial W/\partial\alpha}{\partial W/\partial y_*}$. Since we have shown that $\partial W/\partial y_* > 0$ in Lemma 9, to prove the claim we have to show that $\partial W/\partial\alpha > 0$. There are two terms in W involves α: 1) the second term in the first bracket is proportional to $-\frac{(1-L)^{1+\gamma_1}}{l^{\gamma_1}}$, which is increasing in α; and 2) the second term $\frac{\eta_1}{\eta_1+\gamma_2}\frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}$ in the second line is increasing in α. Therefore $\partial W/\partial\alpha > 0$, and $\frac{dy_*}{d\alpha} < 0$.

When $1 < y_* \leq \frac{1-L}{l}$, it is determined by the following implicit function:

$$\begin{aligned} 1 \;\; = \;\; & W(y_*) = \frac{\eta_1+\gamma_1}{\eta_1+\gamma_2} B_2 y_*^{-\gamma_1} + \frac{\eta_1-1}{\eta_1+\gamma_2} \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu} y_* \\ & + \frac{\gamma_2}{(\eta_1+\gamma_2)} \frac{r+\phi}{\rho+\phi} + \frac{\eta_1}{\eta_1+\gamma_2} \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} \end{aligned}$$

where B_2 is independent of α. Therefore

$$\partial W/\partial\alpha = \eta_1 \frac{\theta\delta\frac{r}{\rho+\phi}}{\rho+\phi+(1+\theta)\delta} + \frac{\eta_1 - 1}{\eta_1+\gamma_2} \frac{\theta\delta\frac{\phi}{\rho+\phi-\mu}}{\rho+\phi+(1+\theta)\delta-\mu} y_* > 0, \tag{21}$$

which implies $\frac{dy_*}{d\alpha} > 0$.

When $y_* < 1$, it is determined by the following implicit function:

$$\begin{aligned} W(y_*) &= \frac{\eta_1}{(\eta_1+\gamma_2)} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\gamma_2}{(\eta_1+\gamma_2)} \frac{r}{\rho+\phi} + \frac{[H_3\gamma_2 + H_1]}{(\eta_1+\gamma_2)} y_*^{\eta_2} \\ &+ \left[\frac{\eta_1 - 1}{\eta_1+\gamma_2} \frac{\theta\delta l+\phi}{\rho+\phi+(1+\theta)\delta-\mu} + \frac{1+\gamma_2}{(\eta_1+\gamma_2)} \frac{\phi}{(\rho+\phi-\mu)} \right] y_* = 1 \end{aligned}$$

where H_3 and H_1 are independent of α. Then

$$\partial W/\partial\alpha = \frac{\eta_1}{(\eta_1+\gamma_2)} \frac{\theta\delta\frac{r}{\rho+\phi}}{\rho+\phi+(1+\theta)\delta} + \frac{\eta_1 - 1}{\eta_1+\gamma_2} \frac{\theta\delta\frac{\phi}{\rho+\phi-\mu}}{\rho+\phi+(1+\theta)\delta-\mu} > 0. \tag{22}$$

Taken together, the equilibrium rollover threshold y_* decreases with α.

A.7 Proof of Proposition 7

We distinguish between an individual creditor i's rollover frequency δ_i and other creditors' rollover frequency δ_{-i}. We can rewrite the individual creditor's Bellman equation for his value function V^i:

$$\begin{aligned} \rho V^i(y_t; y_*) &= \mu y_t V_y^i + \frac{\sigma^2}{2} y_t^2 V_{yy}^i + r + \phi\left[\min(1, y_t) - V(y_t; y_*)\right] \\ &+ \theta\delta_{-i}\mathbf{1}_{\{y<y_*\}}\left[\min(L+ly_t, 1) - V(y_t; y_*)\right] + \delta_i \max_{\text{rollover or run}} \{1 - V(y_t; y_*), 0\}. \end{aligned} \tag{23}$$

Suppose that we increase δ_i from δ to $\delta' > \delta$. We need to show that the creditor i's value function with parameter δ' to that with parameter δ. To facilitate the comparison, we consider a new problem, in which while the creditor's contract expires with rate δ', he is only allowed to withdraw at his contract expiration if an independent random variable $X = 1$. This variable X can take values of 1 or 0 with probabilities of $\lambda = \delta/\delta' < 1$ and $1-\lambda$, respectively. This random variable effectively reduces the creditor's release rate to δ. Thus, in this constrained problem with parameter δ', the creditor has the same value function as in the unconstrained problem with parameter δ.

Next, consider the creditor's value function in the unconstrained problem with parameter δ', which should be strictly higher than that in the constrained problem. This is because if the creditor is allowed to withdraw when $X = 0$ and $y_t < y_*$, his value function is strictly increased even if he keeps the same threshold. Then, it is obvious that the creditor's value function in the unconstrained problem with parameter δ' is strictly higher than that in the same problem with parameter δ.

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