Macro factors and sovereign bond spreads: a quadratic no-arbitrage model*

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Abstract

We construct a quadratic, no-arbitrage model for credit risky sovereign bonds, based on macroeconomic factors, and show that it captures well both the dynamics and the cross-section of euro area yield spreads before and during the sovereign debt crisis. It is also capable of capturing key aspects of observed spread volatilities. The model's performance in forecasting 10-year spreads is at least comparable to that of professional forecasters. In the model, default intensities are closely related to macroeconomic factors: they increase during economic down-turns and when public debt increases. The recent rise of default intensities in Southern European countries can therefore be closely associated with domestic developments in macro fundamentals. Moreover, we provide evidence that risk-neutral default intensities, and hence also sovereign bond spreads, depend non-linearly on debt-to-GDP ratios. In all countries but Greece, however, the bulk of the increase in spreads on longer term sovereign bonds is not associated with higher default intensities, but with a surge in distress risk premia. Such premia are predominantly driven by a factor that is common across countries, and which could be the manifestation of self-fulfilling market dynamics.

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1 Introduction

From late 2009 onwards, yields on bonds issued by several euro area countries rose sharply above comparable yields on German government bonds. Sovereign credit spreads for Southern euro area countries – Greece, Italy, Portugal, Spain (and Ireland) – which had averaged only a few tens of basis points for most of the period since the introduction of the euro, surged to several hundred basis point (see Figure 1a). Conditional yield volatilities also increased sharply, in parallel with the increase in spreads (see Figure 1b). More than three years hence, and after a restructuring of Greek debt, bond yields in Southern euro area countries continue to hover around high levels.

The crisis has sparked a lively debate on the underlying causes of the observed high government bond yields. One viewpoint is that these high yields are simply a reflection of large increases in budget deficits and/or debts, which have eroded markets' confidence in countries' ability to repay their debt obligations. The opposite point of view is that high spreads are the result of self-fulfilling dynamics, i.e. the fact that sufficiently high real yields could ultimately trigger a default in any country with an outstanding stock of government debt.[1]

Our paper aims to identify some stylised facts to inform this debate. We therefore construct a model of sovereign spreads which is consistent with both the fundamental and the self-fulfilling explanations of the crisis. Specifically, we allow spreads to be related to both macroeconomic variables and an unobservable common factor.

The macro variables, namely expectations of public debt-to-GDP ratios and of rates of growth of GDP, aim to proxy the notion of sustainability of the intertemporal government budget constraint. Consistently with the recent theoretical literature – including Corsetti et al., 2012; Bi, 2012; Juessen at al. 2011 – we allow for yields' dependence on debt-to-GDP to be non-linear. Specifically, we rely on a quadratic model of countries' default intensities, that has well-known advantages in terms of tractability compared to alternative non-linear specifications. The model allows for the possibility of spreads increasing more than proportionally to debt, when debt levels become sufficiently high. Contrary to a simpler affine Gaussian specification, it also allows us to capture some of the time variation in conditional variances which is apparent from figure 1.

As already mentioned, we also allow an unobservable factor to influence spreads.

[1] See e.g. Calvo (1988), Jeanne (2012), Corsetti and Dedola (2012), Roch and Uhlig (2012).

This factor could be an indication that spread dynamics are unrelated to the relevant macroeconomic variables and reflect instead sunspot-like dynamics. We allow this single, common factor to affect yields in all countries, in order to capture the idea of cross-country contagion unrelated to common dynamics in fundamentals. Alternatively, the common factor may simply reflect additional relevant information affecting bonds across all sovereigns that is not incorporated in our observable macro variables.

The common factor and the macroeconomic variables are modelled as a vector autoregression (VAR). The VAR has the advantage of being empirically flexible, while incorporating, in reduced form, all relevant linkages between the variables of interest.

When confronted with euro area data over the EMU period, our model captures quite well both the dynamics and the cross-section of euro area sovereign bond spreads. It can also capture some key elements of observed spread volatilities. Both in sample and out of sample, the model's forecasting performance is at least comparable to that of professional forecasters, as surveyed by Consensus Economics.

Our estimated model is able to partly disentangle default probabilities from risk premia effects on spreads. Specifically, we can identify and estimate a component of spreads due to "distress" risk premia, i.e. compensation for unpredictable variation in default intensities, and an expected default component that is free of this premium.[2] Both components are related to the state variables of the model. To disentangle these components from the data we use jointly three sources of information: the time series of credit spreads, their cross section along the term structure, and their cross-country developments. Intuitively, our results can be understood as follows.

The time series information suggests that, since 2009, spreads have been exceptionally high by EMU standards. Since this high-spread episode is accompanied by high volatility, the persistence of the processes driving the increase in spreads is estimated to be moderate. Default intensities are expected to go back to lower levels as shocks are reabsorbed, and future default probabilities are therefore expected to fall over the medium term.

[2] As we discuss later on, this expected default component may still incorporate a risk premium to compensate investors for jump-at-default risk. This premium cannot be identified using the data at hand; see e.g. Singleton (2006). Remolona et al. (2008) suggest one way of estimating the jump-at-default risk premium component using credit rating data. We adopt the "distress risk premium" label of Longstaff et al. (2011) to distinguish it from the the jump-at-default premium.

In the absence of distress premia, the term structure of credit spreads should therefore be downward sloping. This is closer to the truth in the case of Greece. As a result, a relatively larger share of the high spread on longer-term Greek bonds is interpreted by the model as due to an increase in the expected default component. In the other countries, however, the term structure of credit spreads is relatively flatter. Given that the time series information suggests that default intensities are expected to fall over the future, this means that distress premia have played a key role in keeping long-term spreads high recently.

The cross-country information is crucial to identify the role of the unobservable factor in driving credit spreads, which it does mainly through its impact on distress risk premia. We assume that, in each country, the market prices of distress risk − and hence risk premia − can vary in relation to changes in both debt-to-GDP and GDP growth of that country. Moreover, we adopt a flexible specification which allows for such relationships to be country-specific. This implies that any common fluctuations in countries' distress risk premia which can be associated with changes in the country's macroeconomic and fiscal variables are identified as country-specific. Only the remaining, systematic variation in premia which is both common across countries, and not associated with observable macroeconomic fluctuations, is attributed to the common factor. Our results suggest that while much of the increase in distress risk premia during the sovereign debt crisis can be associated with rising debt-to-GDP, changes in the unobservable common factor accounts for a large portion of the surge in 2011.

The default intensities, net of distress risk premia, are instead more closely related to macroeconomic fundamentals: they increase during recessions and when the debt-to-GDP ratio rises. Specifically, they increase nonlinearly in the level of public debt. For example, according to our estimates, a further 1 percentage point increase in the debt-to-GDP ratio would have led to an immediate 20 basis points increase in 10-year spreads in Greece at the end of 2011, while the same debt increase would have been negligible for Greek spreads in, for example, January 2001.

Our modeling approach is related to a number of papers that study the price of credit risky securities and/or their relationship to macroeconomic variables.

A first group of papers analyses sovereign spreads or credit default swaps (CDS) in a dynamic, no-arbitrage setting – e.g. Duffie, Pedersen and Singleton (2003), Pan and Singleton (2008) and Longstaff, Pan, Pedersen and Singleton (2010). They estimate a single-factor model for each of the sovereigns they consider and then investigate how credit risk covaries across countries. Ang and Longstaff (2011) use a multi-factor specification to compare spreads on US States and on European countries, and find evidence that common ("systemic") credit risk plays a greater role for the euro area than for the US. All these models only use yield spreads or CDS data and make no attempt to relate yields to macroeconomic information.

Various papers, including Alesina et al. (1992), Ardagna et al. (2007), Bernoth et al. (2006), study the relationship between sovereign spreads at specific maturities and macroeconomic variables in a cross-section framework. These papers however, do not impose no arbitrage restrictions and focus on one specific yield maturity. As a result, they find only mild evidence for non-linearities in the relationship between yields and macro variables. They are also unable to distinguish between default risk and distress risk premia.

Finally, Borgy et al. (2011), who are closest to our study, include macro factors in a no-arbitrage model of credit spreads. However they rely on an affine framework and, as a result, do not allow for nonlinearities. Moreover, they explicitly rule out feedback effects from higher yields to the level of government debt. They also assume that default in any country is unrelated to the default events of other sovereigns.

In contrast to all these papers, we extend the sovereign bond pricing framework from a linear to a non-linear setting. Specifically, we allow the default intensity of a credit-risky country to depend on our fiscal forecast variable in a linear-quadratic way. As a result, we end up with a quadratic pricing framework, which we find to work well in capturing some of the extreme sovereign credit spread widening witnessed in recent months. We also explicitly allow for correlated default probabilities across countries. The correlation can be induced by the one factor driving sovereign spreads that is common across all countries in our analysis. This factor can potentially capture contagion effects. Finally, we adopt a relatively richer specification of the factor dynamics in each country, which allows for feedback effects from the other factors – GDP growth and the common factor – to the debt/GDP level. We show that this is important for the model to be consistent, to first order, with the possibility of feed-back effects from higher yields to the level of government debt.

The paper is organised as follows. Section 2 presents and motivates our modelling approach. It argues that a quadratic specification is consistent with the results of the theoretical literature and with empirically observed patterns. We describe the data in section 3, which also provides some information on the unscented Kalman filter that we use to estimate the model. We present our results in three separate sections. Section 4 focuses on in sample fit and on the model's decomposition of expected default intensities and distress risk premia. In Section 5 we show our model's ability to forecast yields, both in sample and out of sample, and to match the changes in conditional volatilities apparent from the data. We conclude that the empirical performance of the model is good. We therefore proceed to compute its implications in terms of estimated default probabilities for the countries in our sample. We also compute impulse responses to illustrate the model's ability to account for interactions between the state variables. Finally, Section 7 offers some concluding remarks.

2 Model and estimation

Our empirical specification builds on the class of reduced-form credit pricing models in which assumptions are made about the process for default intensity, as in Lando (1998) and Duffie and Singleton (1999). In this framework, default is assumed to be doubly stochastic, meaning that default arrives randomly according to a Poisson process with some intensity and that, in addition, this intensity process varies randomly over time. The advantage of this approach is that it gives rise to tractable pricing formulas. Specifically, in discrete time and assuming zero recovery , for a given risk-neutral default intensity process Λ and a given risk-free interest rate process r, the price at t of a zero-coupon defaultable bond with n periods to maturity is (see e.g. Duffie and Singleton, 2003):

$$B_t^{t+n} = E_t^{\mathbb{Q}} \left[\exp \left(- \sum_{j=1}^{n} \left(r_{t+j-1} + \Lambda_{t+j} \right) \right) \right], \tag{1}$$

where $E_t^{\mathbb{Q}}[\cdot]$ denotes the expected value under the risk-neutral probability measure. In case of non-zero (possibly stochastic) recovery, Lando (1998) shows that the price can be written as in (1), with an added term that captures the risk-neutral expectation of the recovery value in case of default.

In some cases, notably under the assumption of fractional recovery of the market value (RMV) of the bond, it is possible to obtain closed-form solutions for defaultable bonds (Duffie and Singleton, 1999). We therefore assume RMV and proceed by setting up our empirical specification in discrete time. This involves specifying (i) the relevant state variables and their dynamics; (ii) the relationship between default intensities and the state variables; and (iii) the pricing kernel.

2.1 The state vector

We specify our model directly in terms of sovereign yield spreads.[3] In each country i, the risk-neutral default intensities of risky bonds, Λ_t, are assumed to depend on a vector of state variables X_t^i which in turn follows a vector AR(1) process[4]

$$X_t^i = \Phi^i X_{t-1}^i + \Sigma^i \varepsilon_t^i. \tag{2}$$

In each country issuing risky bonds, X_t^i comprises three elements. The first one is an unobservable factor C_t, which is common across countries. The second element is the country's expected rate of growth of real GDP, g_t^i. The final element is the expected debt-to-GDP ratio d_t^i.[5] We therefore have $X_t^i = \left[C_t, g_t^i, d_t^i\right]'$, and we assume that the covariance matrix is diagonal with elements $\text{diag}\left(\Sigma^i\right)' = \left[1, \sigma_g^i, \sigma_d^i\right]$.

The Φ^i matrix is specified as follows

$$\Phi^i = \begin{bmatrix} \phi_{1,1} & 0 & 0 \\ 0 & \phi_{2,2}^i & 0 \\ \phi_{3,1}^i & \phi_{3,2}^i & \phi_{3,3}^i \end{bmatrix}.$$

This structure for Φ^i is motivated by the following considerations. For the common factor, we adopt an identification assumption. Specifically, we assume that C_t is a simple autoregressive process whose level is not affected by any of the other state variables in any country, i.e. $C_t = \phi_{1,1} C_{t-1} + \varepsilon_t^C$. We wish to think of C_t as a factor capturing effects that are directly unrelated to macroeconomic developments. These effects could be the product of self-fulfilling expectations, which have an impact on yields independently of a country's fiscal stance or growth performance. This is in line with results in Calvo (1988) and Cole and Kehoe (2000) – and more recently Cooper (2011), Corsetti and Dedola (2012), Jeanne (2012), Roch and Uhlig (2012). These papers show that multiple equilibria can characterise sovereign bond markets due to expectations coordination problems. Self-fulfilling developments in other countries may act as an area-wide coordination device and therefore lead to cross-country contagion. It is this type of contagion, which is unrelated to fundamentals, which we wish to capture with the C_t factor. We therefore assume that the first

[3] Here, we follow much of the literature and assume that the risk-neutral default intensity of each country is independent from the risk-free short-term interest rate; see e.g. Pan and Singleton (2008) and Longstaff et al. (2011).

[4] The state variables are here specified in deviation from their respective mean values.

[5] We discuss the exact definition of these macro factors in the next section.

row of Φ^i has zeros everywhere except for the first element and that its innovations are uncorrelated with those in the macro variables. The assumption of a unitary standard deviation is a normalisation to allow econometric identification.

By contrast, we allow for non-zero values of all elements in the last row of Φ^i. This ensures that developments in both the common factor and the rate of growth of GDP can feed back on the debt-to-GDP level. Allowing for non-zero coefficients $\phi^i_{3,1}$ and $\phi^i_{3,2}$ is important to capture, to first order, feedback effects from higher yields to higher debt through the increase in the costs of servicing the debt. This point can be illustrated through a simple government budget constraint $D_t = -S_t + \tilde{r}_{t-1} D_{t-1}$, where S_t is the nominal primary surplus, D_t is the nominal debt at t, which has to be financed at some one-period interest rate $\tilde{r}_t$ (assuming that the debt is rolled over each period). Assume for simplicity that the main driver of fluctuations in risky yields is the spread relative to the risk-free yield, which in turn is driven by the default risk, Λ^i_t. Assume further that the spread is a given function of some state variables X_t: $\Lambda_t = F(X_t)$. Then, in reduced form, the debt accumulation equation will be $D_t = -S_t + F(X_{t-1}) D_{t-1}$, which to first order can be approximated as

$$\widehat{D}_t = a_1 X_{t-1} + a_2 \widehat{D}_{t-1} + u_t \tag{3}$$

where $a_1 \equiv F'(X)$ and $a_2 \equiv F(X)$ are constant parameters vectors and $u_t \equiv -(S/D)\widehat{S}_t$. Equation (3) demonstrates that debt will in general react to all the state variables which affect sovereign spreads. It is therefore important to allow for non-zero elements $\phi^i_{3,1}$ and $\phi^i_{3,2}$ in the Φ^i matrix.

In principle we could also allow for feedback of debt and the common factor on growth. In a simple preliminary bivariate VAR regressions of GDP growth and debt-to-GDP ratios, however, we find that the feedback coefficient of debt levels on growth are quantitatively negligible. We therefore assume that GDP growth can be described by a simple autoregression to reduce the overall number of parameters to be estimated.

2.2 Default intensity

We next need an assumption for the risk-neutral default intensity in each country i, Λ^i_t. The literature has largely adopted an affine specification, which has well know advantages for tractability. However, there are theoretical and empirical results in the literature which suggest that an affine model may not be the right choice when specifying sovereign default intensities.

Specifically, a few recent structural models suggest that sovereign yields spreads over a default-free benchmark are likely to be nonlinear functions of fiscal conditions – see e.g. Bi (2012), Juessen et al. (2011) and Corsetti et al. (2012). These models explicitly take into account the fact that governments can only repay bonds up to a level given by the expected present discounted value of government net surpluses over all future dates. Such level, denoted as the *fiscal limit*, will fluctuate over time in reaction to changes in the state of the economy, e.g. due to productivity shocks that can change the economy's growth potential. From the private sector's perspective, the possibility that the government will default on its debt once the fiscal limit is reached generates a non-linearity on bond prices as a function of government debt. When the economy is far from the fiscal limit, there is no reason to expect a default on government debt and sovereign spreads will be low. However, as government debt increases and the economy is hit by recessionary shocks, the probability of default increases rapidly as the fiscal limit is approached. Beyond some value, bond yields therefore become very steep, nonlinear functions of the state variables that drive the economy towards the fiscal limit.

There is also some stylised empirical evidence of a nonlinear relationship between fiscal fundamentals and either the level of yields or proxies for credit risk. Based on quarterly data for 12 OECD countries over the period 1974-1989, Alesina et al. (1992) find evidence of a threshold effect. They separate the countries into two groups, depending on whether their public debt, relative to GDP, is below or above a certain level. They then regress the spread between public and private returns on public debt-to-GDP levels and find positive and highly significant coefficients only for the countries with high debt levels. Ardagna et al. (2007) focus on yearly data on 16 OECD countries over a maximum time span from 1960 to 2002 and test explicitly for non-linearity by including the squared term of the debt-to-GDP ratio in a regression of long term interest rates. The coefficient on the square term is significant over different subsamples and estimation methods. Bernoth et al. (2004) also find significant nonlinear effects for fiscal variables as determinants of bond yield differentials on EU eurobonds issued between 1991 and 2002.

Motivated by the aforementioned results, we allow for a nonlinear relationship between spreads and the debt-to-GDP level in each country. The advantage of casting such a nonlinear relationship within an explicit, no-arbitrage framework is that we will be able to exploit the cross-sectional information provided by the term structure of credit spreads. No-arbitrage restrictions also have the advantage of helping discipline inference and allowing us to partially disentangle credit risk premia from default probabilities.

To allow for a nonlinear relationship between spreads and debt levels without losing the tractability of the general affine world,[6] we adopt an affine-quadratic specification for the risk-neutral default intensities, such that

$$\Lambda_t^i = \lambda_0^i + \lambda^i X_t^i + \left(X_t^i\right)' \Xi^i X_t^i. \tag{4}$$

Here, motivated by the aforementioned empirical evidence, we assume that the quadratic term in Λ_t^i is only a function of debt-to-GDP. This means that Ξ^i will only include one non-zero element, λ_{dd}, corresponding to squared debt-to-GDP.

Beyond capturing the theoretical nonlinearities mentioned above, a quadratic specification for the default intensities has the advantage of generating time variation in the conditional variance of yields, even if the state vector is Gaussian. As we have already highlighted in the introduction, time-variation in spread variances is clearly a stylised fact in the crisis – a fact that can potentially be captured by a quadratic term structure model. Any improvement in capturing time-variation in spread volatilities would be useful when pricing derivative contracts or for risk management considerations. Allowing for heteroskedastic second moments is also important to attain reliable estimates of the uncertainty surrounding forecasts of future interest rate levels.

Finally, it is important to note that our model allows for cross-country correlation in default intensities. Specifically, movements in the common factor C_t have the potential to drive default intensities in the same direction in all countries. When observing common cross-country dynamics in credit spreads, we can therefore disentangle whether they are due to correlated developments in country-specific fundamentals, or instead to some form of financial contagion. We talk about the first type of correlation when spreads are affected by domestic, macroeconomic determinants, which move in the same direction in all countries. We talk about contagion when spreads are all driven by the common factor.

[6] In general, a Gaussian quadratic term structure model can be rewritten as an affine term structure model with heteroskedastic innovations.

2.3 Bond prices

In a discrete-time setting, we can write the price of a credit-risky bond at t as the expected value of the product of the pricing kernel, $m_{t,t+1}$, and the value of the bond one period ahead. Specifically, the price at t of a risky bond maturing at $t+n$ can be written as

$$B_t^{t+n} = E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\mathbf{1}_{\tau>t+1} + Z_{t+1}\mathbf{1}_{\tau<t+1}\right)\right],$$

with boundary condition

$$B_{t+n-1}^{t+n} = E_{t+n-1}\left[m_{t+n-1,t+n}\left(\mathbf{1}_{\tau>t+n} + Z_{t+n}\mathbf{1}_{\tau<t+n}\right)\right],$$

where Z_t is the recovery payment, τ denotes the time of default and $\mathbf{1}_{\tau>t+1}$ is an indicator variable that takes the value one if $\tau > t+1$. In general, the expectation $E_t\left[\mathbf{1}_{\tau>t+k}\right]$ is the probability of survival until $t+k$:[7]

$$E_t\left[\mathbf{1}_{\tau>t+k}\right] = E_t\left[\exp\left(-\sum_{i=1}^{k}\Lambda_{t+i}\right)\right].$$

Under a RMV assumption, the expected recovery payment is a fraction of the bond price at $t+1$, conditional on no default, i.e. for an n-period bond

$$E_t\left[Z_{t+1}\right] = E_t\left[\left(1 - L_{t+1}\right)B_{t+1}^{t+n}\right],$$

where L_{t+1} is the (risk-neutral) fractional loss rate.

Assuming that the loss rate is a constant L, we have, under RMV,

$$B_t^{t+n} = E_t\left[m_{t,t+1}\left(1 - L\left(1 - \exp\left(-\Lambda_{t+1}\right)\right)\right)B_{t+1}^{t+n}\right].$$

In discrete time, we can make the following approximation

$$1 - L\left(1 - \exp\left(-\Lambda_{t+1}\right)\right) \approx \exp\left(-\Lambda_{t+1}\right).$$

This approximation holds exactly for $L = 1$. For L different from 1, we should view Λ as reflecting adjusted default intensities, rather than actual intensities. This is

[7] This would be the true, objective survival probability only if Λ denoted the objective default intensities. Instead, as mentioned earlier, Λ denotes the risk-neutral arrival intensity of default.

analogous to the use of "recovery-adjusted default intensities" in continuous time models with RMV (e.g. Duffie and Singleton, 1999). Given this approximation, we can write

$$B_t^{t+n} = E_t \left[m_{t,t+1} \exp\left(-\Lambda_{t+1}\right) B_{t+1}^{t+n} \right].$$

The pricing kernel $m_{t,t+1}$ is assumed to depend on the state variables X_t. Specifically, $m_{t,t+1} = \exp\left(-r_t\right) \kappa_{t+1}/\kappa_t$, where κ_{t+1} is assumed to follow the log-normal process $\kappa_{t+1} = \kappa_t \exp\left(-\frac{1}{2}\psi_t'\psi_t - \psi_t'\varepsilon_{t+1}\right)$, which results in[8]

$$m_{t,t+1} = \exp\left(-r_t - \frac{1}{2}\psi_t'\psi_t - \psi_t'\varepsilon_{t+1}\right).$$

At this point we only need to specify the market prices of risk, denoted as ψ_t^i for country i. We rely on the Duffee (2002) essentially affine specification and assume

$$\psi_t^i = \begin{bmatrix} \psi_0^{C,i} \\ \psi_0^{g,i} \\ \psi_0^{d,i} \end{bmatrix} + \begin{bmatrix} \psi_{C,C}^i & 0 & 0 \\ 0 & \psi_{g,g}^i & 0 \\ \psi_{d,C}^i & \psi_{d,g}^i & \psi_{d,d}^i \end{bmatrix} X_t^i. \tag{5}$$

where the zeros are imposed for symmetry with the Φ^i matrix. It is important to note that premia resulting form non-zero market prices of risk represent compensation for the risk of unpredictable changes in the default intensities, over and above the possible compensation required for the risk associated with a drop in the bond price in the event of default. Consistent with the terminology introduced in Pan and Singleton (2008), we therefore refer to premia due to default intensity risk as "distress premia".

Given the aforementioned assumptions, the price of an n-period bond can be written as

$$B_t^{t+n} = \exp\left(A_n + B_n X_t + X_t' C_n X_t\right). \tag{6}$$

for constants A_n, B_n and C_n that are defined in the appendix and that can be obtained using simple recursions.[9]

[8] Under our assumption that the default intensities are independent of the factors driving the risk-free interest rate r, this rate will drop out later on when we focus on bond spreads relative to a safe benchmark.

[9] We will be working in spread-space rather than yield or price space, but the spread expressions are analogous.

3 Data and estimation method

3.1 Data

Our data is monthly and covers the period from the introduction of the euro, January 1999, to end-November 2011. We consider government bonds of five euro area countries: Greece, Portugal, Spain, France and Italy, and we regard German government bonds as proxies for credit risk-free euro-denominated bonds.

In order to construct sovereign spreads, we first estimate zero-coupon yields for these countries based on end-of-month prices of all available government bonds, as reported by Bloomberg, using the Nelson-Siegel model.[10] We select six maturities to be used in subsequent estimations, namely 2, 3, 4, 5, 7 and 10 years. For these maturities, we take the estimated zero-coupon yields for each of our five countries and subtract the corresponding German yield to obtain zero-coupon sovereign spreads.

Concerning the macroeconomic variables, we follow Laubach (2009) and rely on forecasts, rather than official published data. The problem with using official data is that both GDP growth and public debt data are subject to considerable revisions over time, and it would therefore be important to keep track of the different vintages of data releases. While these are not readily available for all the countries we consider, we do have access to the different vintages of macroeconomic forecasts prepared twice a year by the European Commission.

Such forecasts have the additional advantage of relating to a medium-term horizon – roughly one and two years ahead. They should therefore represent better proxies for the sustainability of government finances than current debt-to-GDP data. And their forward-looking nature should be closer to what investors care about when pricing sovereign bonds than official data, which is always released with a considerable lag.

More specifically, we use data from both the Spring and Fall forecasts. For the date released in the Spring, the forecasts cover the current and next year, i.e. until the end of the current year and until the end of the following year. For the Fall forecast, the horizons extend through the next and the following years. By including this data, we are implicitly assuming that the forecasts by the European Commission are close to those made by the private sector when taking their pricing decisions.

[10] We obviously exclude bonds that are not euro denominated or inflation-index linked, and those that pay floating rates or have other non-standard features.

There are two final choices we need to make when using these data. The first one has to do with their frequency. Our yields are sampled at monthly frequency, while the forecasts are only available twice per year. To simplify the estimation of our term structure model, we pre-filter monthly data from the European Commission forecasts using the Kalman filter and a simple autoregressive law of motion. Here, we take as input data expected debt-to-GDP and GDP growth roughly one year ahead, which is constructed using the two published forecasts on either side of the one-year horizon. At this stage, we do not use any yield information at all. We use the resulting filtered monthly series of one-year ahead expectations as our macro data in the subsequent estimation of the term structure model.

The second choice is related to the long run means of debt-to-GDP and GDP growth. A key issue regards the sustainable debt-to-GDP ratio in each country. Debt could be stabilised around different levels, each requiring different primary surpluses. In turn, nonlinear effects of debt should kick in only when debt deviates significantly from the sustainable level. As a result, a certain debt-to-GDP ratio could be perceived as sustainable, or unsustainable, in different countries. To allow for this possibility, we use debt-to-GDP ratios in deviation from the historical pre-crisis mean, i.e. from 1999 to 2006.[11] GDP growth is simply measured in deviation from the sample mean.

The resulting macro data are shown in Figure 2.

3.2 Estimation method

In our setup, yields on credit risky bonds are non-linear functions of the state variables. As a result, we cannot use the standard Kalman filter approach to estimate the model. We therefore rely on the unscented Kalman filter of Julier and Uhlmann (1997, 2004) to construct the likelihood function. The unscented Kalman filter relies on a deterministic sampling technique to pick "sigma points" around the mean of some underlying random variable. The sigma points are then propagated through the non-linear functions of interest, in order to recover the first two moments of the non-linear system. These can subsequently be used in the updating step of the filter.

In our application, the transition equation is the state variable VAR,

$$X_t = \Phi X_{t-1} + \Sigma \varepsilon_t, \tag{7}$$

[11] Including 2007-2011 in the calculation of the mean values would, in our view, skew these values towards unsustainable levels.

while the observation equation can be written as

$$z_t = \Theta(X_t) + \xi_t, \tag{8}$$

where z_t is a vector of observables, $\Theta(\cdot)$ is a non-linear function, and where the observation error vector ξ_t is assumed to have zero mean and a diagonal covariance matrix $\tilde{R}$. In our case, the observation vector consists of n_s zero-coupon spreads for each country i, stacked in $\mathbf{s}_t^i$, and a vector f_t^i that contains data on the expected fiscal position and expected GDP growth rate of country i, based on forecasts of the debt to GDP ratio and GDP growth as described above. Given data for m countries, we can define the observation vector as[12]

$$z_t \equiv \begin{bmatrix} \mathbf{s}_t^1 \\ \vdots \\ \mathbf{s}_t^m \\ f_t^1 \\ \vdots \\ f_t^m \end{bmatrix}.$$

The function $\Theta(X_t)$ will then contain the non-linear model expressions for the spreads, $-\frac{1}{n}\ln B_t^{t+n}$ (less the risk-free yield), with the bond price B_t^{t+n} given by (6), and 0/1 vectors selecting the appropriate elements in X_t corresponding to the observable macro variables.

Similar to the standard Kalman filter, the unscented filter relies on a linear updating rule according to

$$\hat{X}_{t|t} = \hat{X}_{t|t-1} + \tilde{K}_t\left(z_t - \hat{z}_{t|t-1}\right), \tag{9}$$

where

$$\begin{aligned} \hat{X}_{t|t-1} &= \Phi \hat{X}_{t-1|t-1}, \\ \tilde{K}_t &= P_{xz(t|t-1)} P_{zz(t|t-1)}^{-1}, \\ \hat{z}_{t|t-1} &= \mathrm{E}\left[\Theta\left(\hat{X}_{t|t-1}\right)\right], \end{aligned}$$

and where P_{zz} is the innovation covariance matrix and P_{xz} is the cross covariance

[12] Greece adopted the euro in 2001, and therefore enters the data set only at this point.

matrix. The updated state is associated with updated covariance[13]

$$P_{xx(t|t)} = P_{xx(t|t-1)} - \tilde{K}_t P_{xx(t|t-1)} \tilde{K}_t', \tag{10}$$

where

$$P_{xx(t|t-1)} = \Phi P_{xx(t-1|t-1)} \Phi' + \Sigma\Sigma'.$$

For an n_x-dimensional state vector X, a set of $2n_x + 1$ sigma points $\varkappa_0$, $\varkappa_1, ...,$ $\varkappa_{n_x}$ with associated weights ϖ_0, $\varpi_1, ...,$ ϖ_{n_x} are chosen (see the Appendix [to come] for details). For each sigma point i, the nonlinear transformation in (8) is applied

$$\mathcal{Z}_i = \Theta\left(\varkappa_i\right).$$

The covariance matrices P_{xx}, P_{zz} and P_{xz} are then approximated using $\varkappa_i$ and the transformed points $\mathcal{Z}_i$.

Based on the obtained forecasts of the states and the associated covariances, we define the log-likelihood function in the usual way and proceed to estimate the model using the maximum likelihood method.

4 Estimation results

In this section we present the main results on our model's ability to fit sovereign spreads data. Table 1 reports the estimated parameter values.

Figure 3 shows actual and fitted yields for the five countries in our estimation sample.[14] All in all, our model can fit the data well. The standard deviations of the measurement errors on spreads vary between around 5 and 12 basis points for France, Italy and Spain and between 18 and 26 basis points for Portugal. Unsurprisingly, measurement errors tend to be larger when spreads surge over the most recent part of the sample. In the case of Greece, where the 2-year spread reached levels of around 7000 basis points towards the end of the sample, the corresponding measurement error standard deviation is 95 basis points. The values for longer-maturity Greek spreads range from 45 to 60 basis points.

In our model, part of the spreads are explained by the estimated common

[13] In practice, we rely on the square-root verion of the unscented Kalman filter by van der Merwe and Wan (2001), which guarantees positive semi-definiteness of the state covariances and improves numerical stability during the estimation.

[14] In this figure, and most of the subsequent ones, we show only the period from 2004 onwards instead of the full sample period, in order to provide a less compressed picture of the crisis period.

factor, which is displayed in Figure 4. We discuss the role of this factor in more detail below. We also note that the non-linear features of the model seem crucial in capturing spread dynamics during the sovereign debt crisis. Figure 5 displays the difference between the full estimated non-linear model, and a version of the model that relies on a linear approximation around the mean values of the state variables. While an estimated linear model may improve on the result of the linearized model, it would probably have a hard time capturing the most recent surge in sovereign spreads unless country-specific latent variables were introduced.

Figure 6 shows snapshots of the term structure of credit spreads before (December 2006) and during (November 2011) the sovereign debt crisis. All term structures tended to be flat around zero or slightly upward sloping in 2006, prior to the sovereign debt crisis. At the end of 2011, credit spreads were substantially higher across all maturities and in all countries. From the maturity perspective, however, the striking development in countries under stress is that their term structure of credit spreads becomes downward sloping. This property is only mildly apparent in Spain and Italy, but more clear in the case of Portugal, and striking for Greece.

This feature of the data helps explain our findings on the importance of distress risk premia in the various countries. Since default intensities are stationary, any increase in their levels is eventually expected to be reabsorbed. In the absence of distress premia, this implies that the term structure of credit spreads should be downward sloping. When the observed spreads term structure is also downward sloping, the spread on longer-term bonds can be easily consistent with the expected future path of default intensities. There is no need for distress risk premia. If, however, the observed term structure is relatively flat, it is much harder to explain long term spreads without distress risk premia. Based on the slope of the term structures of credit spreads during the crisis, therefore, one can expect distress risk premia to play a relatively larger role in Spain and Italy, compared to Portugal and, especially, Greece.

This intuition is confirmed by Figure 7, which shows 10-year sovereign spreads including and excluding distress risk premia. These premia (i.e. the difference between the two curves in Figure 7) are negative but relatively small everywhere in the early years of EMU – a result consistent with the view of an under-pricing of sovereign credit risks in the euro area in those years. In 2010 and especially 2011, they increase dramatically in all countries. In all countries but Greece, their increase accounts for over 70% of the total increase in spreads at the end of the sample. In

Greece, however, they account for about 50% of the increase in total spreads.

According to our estimates, sovereign spreads in November 2011 would have hovered around 1 percentage point in Spain and Italy, around 3 percentage points in Portugal, and around 10 percentage points in Greece, had distress premia been zero.

Figure 8 presents a decompositions of the expected default component – i.e. the part of the spread that is not due to the distress risk premium – for 10-year bonds, to disentangle the role of the various factors.[15] At each point in time, and for each country, the decomposition plots four components: the constant, the component due to the common factor, the component due to GDP growth and the component due to debt-to-GDP.

A notable feature of the decomposition is that the common component plays a relatively minor role in explaining default risk. Its contribution reaches a maximum level of just over 3 percentage point in Greece at the end of the sample, when the overall expected default component exceeds 10 percentage points, and in Portugal it explains about a quarter of the rise in expected default. In all other countries, the contribution of the common component is almost negligible.

The bulk of the deviation of the expected default component from its mean is instead explained, primarily, by variations in the debt-to-GDP component and, to some extent, by changes in the rate of growth of GDP. The level of debt sustains the increase in spreads since late 2009. This is striking not only in Greece, where the debt-to-GDP ratio increased dramatically over the crisis period; it is also true in the other countries, France included.

The expected default component also increases during recessions. It therefore increases significantly everywhere at the time of the Great recession, when public debts soar and growth tanks. In France, Italy and Spain, default intensities are subsequently brought down somewhat by the (partial) economic rebound; in Greece, however, the prolonged, deep recession contributes to keeping the expected default component high throughout the sovereign crisis period.

Figure 9 presents a corresponding decomposition for distress risk premia. In contrast to the expected default component, the unobservable common factor plays an important role in explaining the recent surge in distress risk premia. Specifically, it accounts for more than 50% of the increase in premia in Italy and in Spain at the

[15] Again, it should be noted that this corresponds to expectations under the objective probability measure $\mathbb{P}$ for the risk-neutral default intensities, which we obtain by setting all market price of risk parameters to zero.

end of the sample.

The level of public debt, however, continues to play an important role in all countries. With the exception of Portugal, risk premia also tend to increase during recessions, which is consistent with the evidence for term premia on default risk-free bonds (e.g. Cochrane and Piazzesi, 2005).

5 Spread forecasts and volatilities

We have imposed no-arbitrage restrictions on our model of sovereign spreads. In our setup, the evolution of the state variables is specified in reduced form, and the specification of the prices of risk is very flexible. It is therefore not too surprising that the model can fit the data well. A much tougher specification test is to check the model's forecasting ability, as shown by Duffee (2002). In this section, we therefore report results of a forecasting test. We conclude the section with a test of the model's ability to match the volatilities of spreads.

5.1 Forecasts

One key difficulty in assessing the model's forecasting ability is that most of the information used to estimate our parameters comes from the crisis period. The sample would be too short and too little informative, if we attempted to estimate the model on pre-crisis data only. To perform a forecasting exercise, we therefore proceed as follows.

We start by looking at in-sample forecasts. Specifically, starting in January 2009, we take the model as given but update our macroeconomic information only when it arrives – that is, only twice per year, when the Commission forecasts are released. We then forecast yields 1 year ahead over the crisis period. A relatively good forecasting performance would indicate that the model captures well the persistence of yields data, when the model nonlinearities become important.

We use two benchmarks in this test. The first is a random walk model. The second are forecasts by professional forecasters, as reported by Consensus Economics. We focus on France, Italy and Spain, as these are the only countries in for which Consensus forecasts are available.

Our second exercise is a truly out-of-sample forecast. We compute 1 year ahead forecasts over the December 2011 - November 2012 period, which was not included in the information set for estimation. In this case, realised one-year ahead data are not available. We therefore only compare our model to Consensus forecasts.

Figure 10 compares 1-year ahead forecasts for our model and Consensus. In the cases of Consensus, the mean, the maximum and the minimum of the cross-sectional forecast distribution are reported. Average root mean squared errors over the 2009-2011 period are reported in table 2 [to be added].

In-sample our model forecasts as well as, or better than Consensus. Its performance is especially good for Italy in the early phases of the sovereign debt crisis, until the summer of 2010; 1-year ahead forecasts are almost on top of realised data. Over the same period, the model also does quite well for France and clearly better than Consensus for Spain.

In the subsequent phase of the crisis, 1 year ahead forecast errors increase dramatically. Nevertheless, our model continues doing better than the average Consensus forecast most of the time, especially so in the case of Italy.

Our of sample forecasts for 2013 tend to be aligned with Consensus. They are somewhat above the Consensus mean for Spain and Italy, slightly below the mean for France. Since professional forecasts are often a hard benchmark to beat, we tentatively conclude that our model has a reasonably good out-of-sample forecasting performance.

5.2 Volatilities

We have already emphasised that, contrary to a simpler affine Gaussian specification, our model also allows for time variation in conditional variances. The ability of a quadratic specification to match changes in conditional second moments is, however, tightly constrained: variances can only increase when yields do – more precisely, when the quadratic component in the yields equation becomes large.

The correlation between yield levels and conditional volatilities is obviously a feature of the sovereign bond crisis. In Figure 11 and 12, however, we explore more formally our model's ability to capture the dynamics of conditional second moments of yields. As in Jacobs and Karoui (2009), volatilities are defined as the sum of the conditional standard deviations of the state vector and the measurement error shocks. Specifically, Figure 11 shows the term structure of average 1-step ahead conditional volatilities and Figure 12 displays the time series of 1-step ahead conditional volatilities for the 5-year maturity. In both cases, we compare our estimates to those obtained through a GARCH(1,1) model.

The figures suggest that our quadratic specification can capture some key features of the GARCH estimates. We capture the overall, downward sloping shape of the average term structure of volatilities in most countries (Figure 11). Our model does especially well for Greece and, at short and medium maturities, Portugal and Spain. Only in France does it suggest, counterfactually, a downward-sloping term structure of volatilities, although the magnitudes are tiny.

These results are broadly confirmed along the time series dimension. Consistently with the GARCH estimates, conditional volatilities measured through our model increase dramatically for the countries hardest hit during the sovereign bond crisis years at the end of the sample. The increase, however, is always smaller than in the GARCH estimates.

All in all, we view the results on variances as a confirmation that a quadratic model is strictly preferable to an affine Gaussian specification, in terms of modelling the dynamics of euro sovereign yields spreads over recent years. Of course, there are other options available to allow for nonlinear effects. One example which permits more flexibility in the specification of volatilities is a regime switching model as in Monfort and Renne (2011).

6 A few implications of the model

6.1 Default probabilities

Given our estimates, we can derive (risk neutral) probabilities that a particular country may default over a certain future horizon. Since our intensities Λ_t are "recovery-adjusted default intensities," we first need make an explicit assumption on the recovery value in case of default. Under the RMV assumption, the adjusted intensities relate to the true (risk-neutral) intensities Λ_t^* by

$$\exp\left(-\Lambda_{t+1}\right) = \exp\left(-\Lambda_{t+1}^*\right) + (1-L)\left(1-\exp\left(-\Lambda_{t+1}^*\right)\right).$$

A first-order approximation gives

$$\Lambda_{t+1}^* \approx \frac{1}{L}\Lambda_{t+1}.$$

Hence, by making an explicit assumption on L and scaling the adjusted default intensities accordingly, we can obtain default probabilities for any given horizon k

in the same way as we price bonds (see the appendix for details):

$$PD\left(t,t+k\right)=1-E_{t}\left[\exp\left(-\frac{1}{L}\sum_{i=1}^{k}\Lambda_{t+i}\right)\right].$$

Here, the expectation $E_t[\cdot]$ is taken under the objective probability measure $\mathbb{P}$, obtained by setting all risk price parameters in (5) to zero.

Figure 13 displays one-year ahead default probabilities under the assumption that the recovery value is equal to 50% of the market value. Consistently with our estimates of a large distress risk premium component in the wide spreads of Italy and Spain, we find that 1-year ahead default probabilities in these countries are relatively small even at the peak of the crisis. In our sample, these probabilities are estimated not to exceed 7% in Italy, 5% in Spain and 2% in France.

At the opposite side of the spectrum is Greece, where distress risk premia were proportionately smaller. For this country, the 1-year ahead default probability is estimated to reach almost 80% in November 2011. A restructuring of the Greek debt was eventually agreed in March 2012.

It is important to keep in mind that uncertainty surrounding such estimates increases markedly during the crisis. Error bands are especially large in Greece. Secondly, as already pointed out, while we remove the distress risk premium component to obtain these default probabilities, they are still not objective probabilities of default. Instead, they are the probabilities that would be observed if investors were not requiring any compensation for unexpected losses due to default.

6.2 Impulse responses

Figures 14 and 15 present nonlinear impulse responses from our model. Impulse responses are computed as the difference of two conditional forecasts: one including a selected shock, the other not including the shock. Since yields are nonlinear functions of the states, their impulse responses will be state dependent. To explore this property of the model, we compare impulse responses early in the sample – in Figure 14 – and the end of it –in Figure 15. Note that only the impulse responses of yields change, since the state variables follow a linear process.

We focus on adverse shocks, namely shocks which are likely to lead to an increase in yield spreads. Specifically, we consider an increase in the debt to GDP ratio, a fall in GDP growth, and – given our estimates of the common factor – a downward shock to this factor. In terms of size, we look at a 1 standard deviation shock for the common factor, a 1 percentage point shock for the debt-to-GDP ratio and the GDP growth variables. The graphs compare impulse responses across all countries in our sample.

The shock to the common component, displayed in the first row of Figures 14 and 15, is useful to highlight the model's properties in terms of feedback effects of spreads on the state variables. Focusing on Greece for illustrative purposes, the change in the common factor would have led to an 8 basis points increase in the spread in January 2001. In November 2011 the increase in the spread would instead have exceeded 30 basis points. As a result of the shock, the debt-to-GDP ratio would have increased by almost a percentage point after 9 months. This increase is larger than in other countries, but more short lived. The cross country differences may be related to the heterogeneity in the maturity of public debt. As already emphasised above, we can capture this effect only to a first order approximation. The impulse responses of the public debt are invariant to the initial state of the economy.

The second rows of Figures 14 and 15 show impulse responses to a 1 percentage point fall in GDP growth. The shock leads to an increase in the debt-to-GDP ratio by approximately 1 percentage points in all countries except France and Portugal, where the increase in debt is more contained. In January 2001, the combined effect of the economic slowdown and of the resulting higher public debt would have had almost no effect on sovereign spreads. In November 2011, however, the effects on spreads of the same economic slowdown would have been much larger, especially in Greece where 10-year spreads would have gone up by over 2 percentage points.

Finally, the last rows of Figures 14 and 15 show impulse responses to a 1 percentage point increase in the debt-to-GDP ratio. This variable is estimated to be more highly persistent in France and in Italy, relatively less persistent in Greece. Once again the implications of the shock on spreads are strongly state-dependent. Higher debt in Greece would have had essentially no impact on spreads in January 2001, while spreads would have increased by almost 20 basis points at the end of 2011. A similar pattern applies to Portugal, where the increase in spreads in 2011 would have reached almost 40 basis points, whereas it would have been 30 basis points for Italy. In the other two countries the change in impulses responses is less striking: the differential increase in spreads between 2011 and 2001 is approximately 10 basis points. Our results suggest that, under periods of stress, the sensitivity of spreads to the debt-to-GDP ratio can be quite larger than the estimates by e.g. Laubach (2009) of 4 basis points based on US data.

7 Concluding remarks

We have shown that a quadratic, no-arbitrage term structure model of sovereign spreads can capture well both the dynamics and the cross-section of euro area data over the whole EMU period, i.e. before and during the sovereign debt crisis.

The model can capture developments of the levels and conditional volatilities of spreads as a function of country specific macro-economic factors – debt-to-GDP and GDP growth – and an area-wide unobservable common factor. The model's performance in forecasting 10-year spreads is at least comparable to that of professional forecasters.

Our results suggest that (risk-neutral) expected default intensities are closely related to macro factors in all countries: they increase during economic slowdowns and when public debt grows. This relationship displays strong nonlinear features during the crisis.

In all countries but Greece, however, the bulk of the increase in spreads on longer term sovereign bonds is not associated with a higher expected default component, but with a surge in distress risk premia. Such premia appear to be predominantly driven by developments in the common factor, although the debt-to-GDP level also plays an important role. Our model allows explicitly for correlated increases in spreads due to a parallel cross-country deterioration of fiscal and macro fundamentals. As a result, the correlation in spreads induced by the common factor could be the manifestation of self-fulfilling market dynamics.

A Appendix:

A.1 Credit-risky bond prices

The price at t of a risky bond maturing at $t+n$ can be written as

$$B_t^{t+n} = E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\mathbf{1}_{\tau>t+1} + Z_{t+1}\mathbf{1}_{\tau<t+1}\right)\right],$$

with boundary condition

$$B_{t+n-1}^{t+n} = E_{t+n-1}\left[m_{t+n-1,t+n}\left(B_{t+n-1}^{t+n}\mathbf{1}_{\tau>t+n} + Z_{t+n}\mathbf{1}_{\tau<t+n}\right)\right],$$

where Z_t is the recovery payment, τ denotes the time of default and $\mathbf{1}_{\tau>t+1}$ is an indicator variable that takes the value one if $\tau > t+1$. In general, the expectation $E_t\left[\mathbf{1}_{\tau>t+k}\right]$ is the probability of survival until $t+k$:

$$E_t\left[\mathbf{1}_{\tau>t+k}\right] = E_t\left[\exp\left(-\sum_{i=1}^{k}\Lambda_{t+i}\right)\right].$$

Under a RMV assumption, the expected recovery payment is a fraction of the bond price at $t+1$, conditional on no default, i.e. for an n-period bond

$$E_t\left[Z_{t+1}\right] = E_t\left[\left(1-L_{t+1}\right)B_{t+1}^{t+n}\right],$$

where L_{t+1} is the fractional loss rate.

Assuming that the loss rate is a constant L, we have, under RMV,

$$B_t^{t+n} = E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\mathbf{1}_{\tau>t+1} + (1-L)\,B_{t+1}^{t+n}\mathbf{1}_{\tau<t+1}\right)\right],$$

which can be written

$$\begin{aligned}
B_t^{t+n} &= E_t\left[m_{t,t+1}\left(B_{t+1}^{t+n}\exp\left(-\Lambda_{t+1}\right) + B_{t+1}^{t+n}\left(1-L\right)\left(1-\exp\left(-\Lambda_{t+1}\right)\right)\right)\right] \\
&= E_t\left[m_{t,t+1}\left(\exp\left(-\Lambda_{t+1}\right) + \left(1-L\right)\left(1-\exp\left(-\Lambda_{t+1}\right)\right)\right)B_{t+1}^{t+n}\right] \\
&= E_t\left[m_{t,t+1}\left(1-L\left(1-\exp\left(-\Lambda_{t+1}\right)\right)\right)B_{t+1}^{t+n}\right].
\end{aligned}$$

Assume that we can make the following approximation

$$1-L\left(1-\exp\left(-\Lambda_{t+1}\right)\right) \approx \exp\left(-\Lambda_{t+1}\right).$$

This approximation holds exactly for $L=1$. For L different from 1, we should view Λ as reflecting adjusted default intensities, rather than actual intensities. This analogous to the use of "recovery-adjusted default intensities" in continuous time models with RMV (e.g. Duffie and Singleton, 1999). Given this assumption, we can write

$$B_t^{t+n} = E_t\left[m_{t,t+1}\exp\left(-\Lambda_{t+1}\right)B_{t+1}^{t+n}\right].$$

We will assume that the (adjusted) default intensity of country i is a quadratic function of the states:

$$\Lambda_t^i = \lambda_0^i + \lambda^i X_t + X_t'\Xi^i X_t.$$

The price of an n-period bond is therefore (suppressing superscripts i)

$$\begin{aligned}
B_t^{t+n} &= E_t\left[m_{t,t+1}\exp\left(-\Lambda_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= E_t\left[\exp\left(-r_t-\frac{1}{2}\psi_t'\psi_t-\psi_t'\varepsilon_{t+1}\right)\exp\left(-\lambda_0-\lambda X_{t+1}-X_{t+1}'\Xi X_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= E_t\left[\exp\left(\begin{array}{c} -\delta_0-\delta X_t-\frac{1}{2}\left(\psi_0'\psi_0+2\psi_0'\psi_1X_t+X_t'\psi_1'\psi_1X_t\right)-\left(\psi_0'+X_t'\psi_1'\right)\varepsilon_{t+1} \\ -\lambda_0-\lambda\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)-\left(X_t'\Phi'+\varepsilon_{t+1}'\Sigma'\right)\Xi\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)\end{array}\right)B_{t+1}^{t+n}\right] \\
&= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t\right) \\
&\quad\times E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma\right)\varepsilon_{t+1}-X_t'\Phi'\Xi\Phi X_t-2X_t'\Phi'\Xi\Sigma\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)B_{t+1}^{t+n}\right] \\
&= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t-X_t'\Phi'\Xi\Phi X_t\right) \\
&\quad\times E_t\left[\exp\left(-\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)B_{t+1}^{t+n}\right]
\end{aligned}$$

We know that we can write the price of a bond as

$$B_t^{t+n}=\exp\left(A_n+B_nX_t+X_t'C_nX_t\right).$$

We plug in

$$\begin{aligned}
B_{t+1}^{t+n} &= \exp\left(A_{n-1}+B_{n-1}X_{t+1}+X_{t+1}'C_{n-1}X_{t+1}\right) \\
&= \exp\left(A_{n-1}+B_{n-1}\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)+\left(X_t'\Phi'+\varepsilon_{t+1}'\Sigma'\right)C_{n-1}\left(\Phi X_t+\Sigma\varepsilon_{t+1}\right)\right) \\
&= \exp\left(\begin{array}{c} A_{n-1}+B_{n-1}\Phi X_t+B_{n-1}\Sigma\varepsilon_{t+1} \\ +X_t'\Phi'C_{n-1}\Phi X_t+2X_t'\Phi'C_{n-1}\Sigma\varepsilon_{t+1}+\varepsilon_{t+1}'\Sigma'C_{n-1}\Sigma\varepsilon_{t+1}\end{array}\right)
\end{aligned}$$

into the bond price above to get

$$\begin{aligned}
B_t^{t+n} &= \exp\left(-\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0-\delta X_t-\lambda\Phi X_t-\psi_0'\psi_1X_t-\frac{1}{2}X_t'\psi_1'\psi_1X_t-X_t'\Phi'\Xi\Phi X_t\right) \\
&\quad\times E_t\left[\exp\left(\begin{array}{c} -\left(\psi_0'+X_t'\psi_1'+\lambda\Sigma+2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1}-\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1} \\ +A_{n-1}+B_{n-1}\Phi X_t+B_{n-1}\Sigma\varepsilon_{t+1} \\ +X_t'\Phi'C_{n-1}\Phi X_t+2X_t'\Phi'C_{n-1}\Sigma\varepsilon_{t+1}+\varepsilon_{t+1}'\Sigma'C_{n-1}\Sigma\varepsilon_{t+1}\end{array}\right)\right] \\
&= \exp\left(\begin{array}{c} -\delta_0-\lambda_0-\frac{1}{2}\psi_0'\psi_0+A_{n-1}+\left(B_{n-1}\Phi-\delta-\lambda\Phi-\psi_0'\psi_1\right)X_t \\ +X_t'\left(\Phi'C_{n-1}\Phi-\Phi'\Xi\Phi-\frac{1}{2}\psi_1'\psi_1\right)X_t\end{array}\right) \\
&\quad\times E_t\left[\exp\left(\begin{array}{c} \left(B_{n-1}\Sigma+2X_t'\Phi'C_{n-1}\Sigma-\psi_0'-X_t'\psi_1'-\lambda\Sigma-2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right].
\end{aligned}$$

Rewrite the expectation as

$$\begin{aligned}
&E_t\left[\exp\left(\begin{array}{c} \left(B_{n-1}\Sigma+2X_t'\Phi'C_{n-1}\Sigma-\psi_0'-X_t'\psi_1'-\lambda\Sigma-2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right] \\
=\ &E_t\left[\exp\left(\begin{array}{c} \left(B_{n-1}+2X_t'\Phi'C_{n-1}-\psi_0'\Sigma^{-1}-X_t'\psi_1'\Sigma^{-1}-\lambda-2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} \\ +\varepsilon_{t+1}'\Sigma'\left(C_{n-1}-\Xi\right)\Sigma\varepsilon_{t+1}\end{array}\right)\right] \\
=\ &E_t\left[\exp\left(aw_{t+1}+w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right],
\end{aligned}$$

where

$$\begin{aligned} w_{t+1} &\equiv \Sigma\varepsilon_{t+1} \\ a &\equiv B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi, \\ \bar{C}_{n-1} &\equiv C_{n-1} - \Xi. \end{aligned}$$

To evaluate the expectation we follow Realdon (2006), who demonstrates that (if γ is of full rank)

$$E_t\left[\exp\left(aw_{t+1} + w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right] = \frac{|\gamma|}{\text{abs}\,|\Sigma|}\prod_{i=1}^{N}\exp\left(\frac{(a\gamma_i)^2}{2}\right)$$

where $\gamma \equiv \left((\Sigma\Sigma')^{-1} - 2\bar{C}_{n-1}\right)^{-1/2}$, γ_i denotes the i-th column of γ, $|\gamma|$ denotes the determinant of γ and abs$|\Sigma|$ denotes the absolute value of the determinant of Σ. We therefore get

$$\begin{aligned} & E_t\left[\exp\left(aw_{t+1} + w_{t+1}'\bar{C}_{n-1}w_{t+1}\right)\right] \\ = & \frac{|\gamma|}{\text{abs}\,|\Sigma|}\prod_{i=1}^{N}\exp\left(\frac{\left(\left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i\right)^2}{2}\right), \end{aligned}$$

so that

$$\begin{aligned} \ln B_t^{t+n} = & -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + A_{n-1} + \left(B_{n-1}\Phi - \delta - \lambda\Phi - \psi_0'\psi_1\right)X_t \\ & + X_t'\left(\Phi' C_{n-1}\Phi - \Phi'\Xi\Phi - \frac{1}{2}\psi_1'\psi_1\right)X_t \\ & + \ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i\right)^2. \end{aligned}$$

Evaluating the squared term:

$$\begin{aligned} & \left(\left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i\right)^2 \\ = & \left(B_{n-1} + 2X_t'\Phi' C_{n-1} - \psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i \\ & \times\gamma_i'\left(B_{n-1}' + 2C_{n-1}'\Phi X_t - \Sigma^{-1'}\psi_0 - \Sigma^{-1'}\psi_1 X_t - \lambda' - 2\Xi'\Phi X_t\right) \\ = & \left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(B_{n-1}' - \Sigma^{-1'}\psi_0 - \lambda'\right) \\ & + 2\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t \\ & + X_t'\left(2\Phi' C_{n-1} - \psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t, \end{aligned}$$

we get

$$\begin{aligned}
\ln B_t^{t+n} &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + A_{n-1} + \left(B_{n-1}\Phi - \delta - \lambda\Phi - \psi_0'\psi_1\right) X_t \\
&\quad + X_t'\left(\Phi' C_{n-1}\Phi - \Phi'\Xi\Phi - \frac{1}{2}\psi_1'\psi_1\right) X_t \\
&\quad + \ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(B_{n-1}' - \Sigma^{-1'}\psi_0 - \lambda'\right) \\
&\quad + \sum_{i=1}^{N}\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right) X_t \\
&\quad + \frac{1}{2}\sum_{i=1}^{N} X_t'\left(2\Phi' C_{n-1} - \psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right) X_t.
\end{aligned}$$

We can therefore identify the recusive factor loadings of the bond price $B_t^{t+n} = \exp\left(A_n + B_n X_t + X_t' C_n X_t\right)$ as

$$\begin{aligned}
A_n &= A_{n-1} - \delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + \ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(B_{n-1}' - \Sigma^{-1'}\psi_0 - \lambda'\right), \\
B_n &= B_{n-1}\Phi - \delta - \lambda\Phi - \psi_0'\psi_1 \\
&\quad + \sum_{i=1}^{N}\left(B_{n-1} - \psi_0'\Sigma^{-1} - \lambda\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right), \\
C_n &= \Phi' C_{n-1}\Phi - \Phi'\Xi\Phi - \frac{1}{2}\psi_1'\psi_1 \\
&\quad + \frac{1}{2}\sum_{i=1}^{N}\left(2\Phi' C_{n-1} - \psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i\gamma_i'\left(2C_{n-1}'\Phi - \Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right).
\end{aligned}$$

To get the initial conditions, consider the price of a 1-period bond:

$$\begin{aligned}
B_t^{t+1} &= \exp\left(-\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \delta X_t - \lambda\Phi X_t - \psi_0'\psi_1 X_t - \frac{1}{2}X_t'\psi_1'\psi_1 X_t - X_t'\Phi'\Xi\Phi X_t\right) \\
&\quad \times E_t\left[\exp\left(-\left(\psi_0' + X_t'\psi_1' + \lambda\Sigma + 2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} - \varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right].
\end{aligned}$$

Rewrite the expectation as

$$\begin{aligned}
& E_t\left[\exp\left(-\left(\psi_0' + X_t'\psi_1' + \lambda\Sigma + 2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} - \varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right] \\
=\; & E_t\left[\exp\left(\left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} - \varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right] \\
=\; & E_t\left[\exp\left(a_1 w_{t+1} - w_{t+1}'\Xi w_{t+1}\right)\right],
\end{aligned}$$

where

$$a_1 \equiv -\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi,$$

so that

$$E_t\left[\exp\left(a_1 w_{t+1} - w_{t+1}'\Xi w_{t+1}\right)\right] = \frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|}\prod_{i=1}^{N}\exp\left(\frac{\left(a_1\gamma_i^1\right)^2}{2}\right)$$

where $\gamma^1 \equiv \left(\left(\Sigma\Sigma'\right)^{-1} + 2\Xi\right)^{-1/2}$. We get

$$\begin{aligned} B_t^{t+1} &= \exp\left(-\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \delta X_t - \lambda\Phi X_t - \psi_0'\psi_1 X_t - \frac{1}{2}X_t'\psi_1'\psi_1 X_t - X_t'\Phi'\Xi\Phi X_t\right) \\ &\quad \times E_t\left[\exp\left(-\left(\psi_0' + X_t'\psi_1' + \lambda\Sigma + 2X_t'\Phi'\Xi\Sigma\right)\varepsilon_{t+1} - \varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right]. \end{aligned}$$

$$\begin{aligned} \ln B_t^{t+1} &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \left(\delta + \lambda\Phi + \psi_0'\psi_1\right)X_t - X_t'\left(\frac{1}{2}\psi_1'\psi_1 + \Phi'\Xi\Phi\right)X_t \\ &\quad + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i^1\right)^2. \end{aligned}$$

The squared term is:

$$\begin{aligned} &\left(\left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i^1\right)^2 \\ =\ &\left(-\psi_0'\Sigma^{-1} - X_t'\psi_1'\Sigma^{-1} - \lambda - 2X_t'\Phi'\Xi\right)\gamma_i^1 \\ &\times\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \Sigma^{-1'}\psi_1 X_t - \lambda' - 2\Xi'\Phi X_t\right) \\ =\ &\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \lambda'\right) \\ &+2\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t \\ &+X_t'\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t, \end{aligned}$$

so that

$$\begin{aligned} \ln B_t^{t+1} &= -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 - \left(\delta + \lambda\Phi + \psi_0'\psi_1\right)X_t - X_t'\left(\frac{1}{2}\psi_1'\psi_1 + \Phi'\Xi\Phi\right)X_t \\ &\quad + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_0 - \lambda'\right) \\ &\quad + \sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t \\ &\quad + \frac{1}{2}\sum_{i=1}^{N}X_t'\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1'}\left(-\Sigma^{-1'}\psi_1 - 2\Xi'\Phi\right)X_t, \end{aligned}$$

which gives

$$A_1 = -\delta_0 - \lambda_0 - \frac{1}{2}\psi_0'\psi_0 + \ln\frac{|\gamma^1|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1\prime}\left(-\Sigma^{-1\prime}\psi_0 - \lambda'\right),$$

$$B_1 = -\delta - \lambda\Phi - \psi_0'\psi_1 + \sum_{i=1}^{N}\left(-\psi_0'\Sigma^{-1} - \lambda\right)\gamma_i^1\gamma_i^{1\prime}\left(-\Sigma^{-1\prime}\psi_1 - 2\Xi'\Phi\right)$$

$$C_1 = -\frac{1}{2}\psi_1'\psi_1 - \Phi'\Xi\Phi + \frac{1}{2}\sum_{i=1}^{N}\left(-\psi_1'\Sigma^{-1} - 2\Phi'\Xi\right)\gamma_i^1\gamma_i^{1\prime}\left(-\Sigma^{-1\prime}\psi_1 - 2\Xi'\Phi\right).$$

B Default probabilities

The "recovery-adjusted default intensities" are denoted Λ_t. We let τ denote the time of default and $\mathbf{1}_{\tau>t+k}$ an indicator variable that takes the value one if $\tau > t+k$. Based on the adjusted intensities, the expectation $E_t\left[\mathbf{1}_{\tau>t+k}\right]$ is the probability of survival from t until $t+k$ (conditional on no default up to t),

$$E_t\left[\mathbf{1}_{\tau>t+k}\right] = E_t\left[\exp\left(-\sum_{i=1}^{k}\Lambda_{t+i}\right)\right],$$

and the corresponding probability of default is

$$E_t\left[\mathbf{1}_{\tau\leq t+k}\right] = 1 - E_t\left[\exp\left(-\sum_{i=1}^{k}\Lambda_{t+i}\right)\right].$$

The standard assumption on the recovery-adjusted default intensities (RMV), which we have also adopted, is that they relate to the true (risk-neutral) intensities Λ_t^* by

$$\exp\left(-\Lambda_{t+1}\right) = \exp\left(-\Lambda_{t+1}^*\right) + (1-L)\left(1 - \exp\left(-\Lambda_{t+1}^*\right)\right).$$

A first-order approximation gives

$$\Lambda_{t+1}^* \approx \frac{1}{L}\Lambda_{t+1}.$$

Hence, by making an explicit assumption on L and scaling the adjusted default intensities accordingly, we can obtain default probabilities for any given horizon k in the same way as we price bonds:

$$PD\left(t, t+k\right) = 1 - E_t\left[\exp\left(-\frac{1}{L}\sum_{i=1}^{k}\Lambda_{t+i}\right)\right].$$

Let $h = \frac{1}{L}$. For $k = 1$, we then have

$$PD\left(t, t+1\right) = 1 - E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right],$$

where

$$\begin{aligned} E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right] &= E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right] \\ &= E_t\left[\exp\left(-h\lambda_0 - h\lambda X_{t+1} - hX_{t+1}'\Xi X_{t+1}\right)\right] \\ &= \exp\left(-h\lambda_0\right) E_t\left[\exp\left(-h\lambda X_{t+1} - hX_{t+1}'\Xi X_{t+1}\right)\right], \end{aligned}$$

where

$$X_{t+1} = \Phi X_t + \Sigma\varepsilon_{t+1}.$$

We get

$$\begin{aligned} E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right] &= \exp\left(-h\lambda_0 - h\lambda\Phi X_t - hX_t'\Phi'\Xi\Phi X_t\right) \\ &\quad \times E_t\left[\exp\left(-h\lambda\Sigma\varepsilon_{t+1} - 2hX_t'\Phi'\Xi\Sigma\varepsilon_{t+1} - h\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right] \\ &= \exp\left(-h\lambda_0 - h\lambda\Phi X_t - hX_t'\Phi'\Xi\Phi X_t\right) \\ &\quad \times E_t\left[\exp\left(-h\left(\lambda + 2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} - h\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right]. \end{aligned}$$

Write the expectation as

$$\begin{aligned} & E_t\left[\exp\left(-h\left(\lambda + 2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} - h\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right] \\ = \ & E_t\left[\exp\left(\alpha_1 w_{t+1} - w_{t+1}'h\Xi w_{t+1}\right)\right], \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &\equiv -h\left(\lambda + 2X_t'\Phi'\Xi\right), \\ w_{t+1} &\equiv \Sigma\varepsilon_{t+1}, \end{aligned}$$

so that

$$E_t\left[\exp\left(\alpha_1 w_{t+1} - w_{t+1}'h\Xi w_{t+1}\right)\right] = \frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|}\prod_{i=1}^{N}\exp\left(\frac{\left(\alpha_1\gamma_i^1\right)^2}{2}\right)$$

where

$$\gamma^1 \equiv \left(\left(\Sigma\Sigma'\right)^{-1} + 2h\Xi\right)^{-1/2}.$$

The log survival probability is then

$$\begin{aligned} \ln\left(E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right]\right) &= -h\lambda_0 - h\lambda\Phi X_t - hX_t'\Phi'\Xi\Phi X_t \\ &\quad + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} + \frac{1}{2}\sum_{i=1}^{N}\left(-h\left(\lambda + 2X_t'\Phi'\Xi\right)\gamma_i^1\right)^2. \end{aligned}$$

The squared term is

$$\begin{aligned} \left(-h\left(\lambda + 2X_t'\Phi'\Xi\right)\gamma_i^1\right)^2 &= h\left(\lambda + 2X_t'\Phi'\Xi\right)\gamma_i^1\gamma_i^{1\prime}h\left(\lambda' + 2\Xi'\Phi X_t\right) \\ &= h^2\lambda\gamma_i^1\gamma_i^{1\prime}\lambda' + 4h^2\lambda\gamma_i^1\gamma_i^{1\prime}\Xi'\Phi X_t + 4h^2X_t'\Phi'\Xi\gamma_i^1\gamma_i^{1\prime}\Xi'\Phi X_t, \end{aligned}$$

so that

$$\begin{aligned}
\ln\left(E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right]\right) &= -h\lambda_0 - h\lambda\Phi X_t - hX_t'\Phi'\Xi\Phi X_t + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} \\
&\quad + \frac{1}{2}h^2\sum_{i=1}^{N}\lambda\gamma_i^1\gamma_i^{1\prime}\lambda' + 2h^2\sum_{i=1}^{N}\lambda\gamma_i^1\gamma_i^{1\prime}\Xi'\Phi X_t \\
&\quad + 2h^2\sum_{i=1}^{N}X_t'\Phi'\Xi\gamma_i^1\gamma_i^{1\prime}\Xi'\Phi X_t.
\end{aligned}$$

Letting

$$\begin{aligned}
a_1 &= -h\lambda_0 + \ln\frac{\left|\gamma^1\right|}{\text{abs}\left|\Sigma\right|} + \frac{1}{2}h^2\sum_{i=1}^{N}\lambda\gamma_i^1\gamma_i^{1\prime}\lambda', \\
b_1 &= -h\lambda\Phi + 2h^2\sum_{i=1}^{N}\lambda\gamma_i^1\gamma_i^{1\prime}\Xi'\Phi, \\
c_1 &= -h\Phi'\Xi\Phi + 2h^2\sum_{i=1}^{N}\Phi'\Xi\gamma_i^1\gamma_i^{1\prime}\Xi'\Phi,
\end{aligned}$$

we obtain

$$\ln\left(E_t\left[\exp\left(-h\Lambda_{t+1}\right)\right]\right) = a_1 + b_1 X_t + X_t' c_1 X_t.$$

Log survival probabilities are therefore linear-quadratic functions of the states, and we can write them recursively. For horizon n, we have

$$\begin{aligned}
& E_t\left[\exp\left(-h\Lambda_{t+1}\right)\exp\left(-h\Lambda_{t+2}\right)...\exp\left(-h\Lambda_{t+n}\right)\right] \\
= \;& E_t\left[\exp\left(-h\lambda_0 - h\lambda X_{t+1} - hX_{t+1}'\Xi X_{t+1}\right)\exp\left(a_{n-1} + b_{n-1}X_{t+1} + X_{t+1}'c_{1-n}X_{t+1}\right)\right] \\
= \;& E_t\left[\exp\left(-h\lambda_0 - h\lambda\Phi X_t - hX_t'\Phi'\Xi\Phi X_t\right)\exp\left(-h\left(\lambda + 2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} - h\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right)\right. \\
& \left.\times\exp\left(a_{n-1} + b_{n-1}\left(\Phi X_t + \Sigma\varepsilon_{t+1}\right) + \left(X_t'\Phi' + \varepsilon_{t+1}'\Sigma'\right)c_{n-1}\left(\Phi X_t + \Sigma\varepsilon_{t+1}\right)\right)\right] \\
= \;& \exp\left(a_{n-1} - h\lambda_0 + b_{n-1}\Phi X_t - h\lambda\Phi X_t - hX_t'\Phi'\Xi\Phi X_t + X_t'\Phi'c_{n-1}\Phi X_t\right) \\
& \times E_t\left[\exp\left(b_{n-1}\Sigma\varepsilon_{t+1} - h\left(\lambda + 2X_t'\Phi'\Xi\right)\Sigma\varepsilon_{t+1} + 2X_t'\Phi'c_{n-1}\Sigma\varepsilon_{t+1} + \varepsilon_{t+1}'\Sigma'c_{n-1}\Sigma\varepsilon_{t+1} - h\varepsilon_{t+1}'\Sigma'\Xi\Sigma\varepsilon_{t+1}\right.\right. \\
= \;& \exp\left(a_{n-1} - h\lambda_0 + \left(b_{n-1} - h\lambda\right)\Phi X_t + X_t'\left(\Phi'c_{n-1}\Phi - h\Phi'\Xi\Phi\right)X_t\right) \\
& \times E_t\left[\exp\left(\left(b_{n-1} - h\lambda + 2X_t'\Phi'\left(c_{n-1} - h\Xi\right)\right)\Sigma\varepsilon_{t+1} + \varepsilon_{t+1}'\Sigma'\left(c_{n-1} - h\Xi\right)\Sigma\varepsilon_{t+1}\right)\right].
\end{aligned}$$

Defining

$$\begin{aligned}
\alpha &\equiv \left(b_{n-1} - h\lambda + 2X_t'\Phi'\left(c_{n-1} - h\Xi\right)\right), \\
w_{t+1} &\equiv \Sigma\varepsilon_{t+1},
\end{aligned}$$

we can write the expectation as

$$\begin{aligned} E_t[\cdot] &= E_t\left[\exp\left(\alpha w_{t+1} + w'_{t+1}(c_{n-1} - h\Xi) w_{t+1}\right)\right] \\ &= \frac{|\gamma|}{\text{abs}\,|\Sigma|} \prod_{i=1}^{N} \exp\left(\frac{(\alpha\gamma_i)^2}{2}\right), \end{aligned}$$

where

$$\gamma \equiv \left(\left(\Sigma\Sigma'\right)^{-1} - 2\left(c_{n-1} - h\Xi\right)\right)^{-1/2}.$$

We get

$$\begin{aligned} & E_t\left[\exp\left(\alpha w_{t+1} + w'_{t+1}(c_{n-1} - h\Xi) w_{t+1}\right)\right] \\ = \; & \frac{|\gamma|}{\text{abs}\,|\Sigma|} \prod_{i=1}^{N} \exp\left(\frac{\left(\left(b_{n-1} - h\lambda + 2X'_t\Phi'(c_{n-1} - h\Xi)\right)\gamma_i\right)^2}{2}\right), \end{aligned}$$

so that the log-probability of survival becomes

$$\begin{aligned} \ln(1 - PD(t, t+n)) &= a_{n-1} - h\lambda_0 + (b_{n-1} - h\lambda)\Phi X_t + X'_t\left(\Phi' c_{n-1}\Phi - h\Phi'\Xi\Phi\right)X_t \\ &\quad + \ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}\left(\left(b_{n-1} - h\lambda + 2X'_t\Phi'(c_{n-1} - h\Xi)\right)\gamma_i\right)^2. \end{aligned}$$

Evaluating the squared term:

$$\begin{aligned} & \left(\left(b_{n-1} - h\lambda + 2X'_t\Phi'(c_{n-1} - h\Xi)\right)\gamma_i\right)^2 \\ = \; & \left(b_{n-1} - h\lambda + 2X'_t\Phi'(c_{n-1} - h\Xi)\right)\gamma_i\gamma'_i\left(b'_{n-1} - h\lambda' + 2\left(c'_{n-1} - h\Xi'\right)\Phi X_t\right) \\ = \; & (b_{n-1} - h\lambda)\gamma_i\gamma'_i\left(b'_{n-1} - h\lambda'\right) + 4(b_{n-1} - h\lambda)\gamma_i\gamma'_i\left(c'_{n-1} - h\Xi'\right)\Phi X_t \\ & + 4X'_t\Phi'(c_{n-1} - h\Xi)\gamma_i\gamma'_i\left(c'_{n-1} - h\Xi'\right)\Phi X_t, \end{aligned}$$

we get

$$\begin{aligned} & \ln(1 - PD(t, t+n)) \\ = \; & a_{n-1} - h\lambda_0 + \ln\frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N}(b_{n-1} - h\lambda)\gamma_i\gamma'_i\left(b'_{n-1} - h\lambda'\right) \\ & + (b_{n-1} - h\lambda)\Phi X_t + 2\sum_{i=1}^{N}(b_{n-1} - h\lambda)\gamma_i\gamma'_i\left(c'_{n-1} - h\Xi'\right)\Phi X_t \\ & + X'_t\left(\Phi'(c_{n-1} - h\Xi)\Phi\right)X_t + 2\sum_{i=1}^{N}X'_t\Phi'(c_{n-1} - h\Xi)\gamma_i\gamma'_i\left(c'_{n-1} - h\Xi'\right)\Phi X_t. \end{aligned}$$

We can therefore write

$$\ln(1 - PD(t, t+n)) = a_n + b_n X_t + X'_t c_n X_t,$$

where

$$
\begin{aligned}
a_n &= a_{n-1} - h\lambda_0 + \ln \frac{|\gamma|}{\text{abs}\,|\Sigma|} + \frac{1}{2}\sum_{i=1}^{N} \left(b_{n-1} - h\lambda\right) \gamma_i \gamma_i' \left(b_{n-1}' - h\lambda'\right), \\
b_n &= \left(b_{n-1} - h\lambda\right) \Phi + 2\sum_{i=1}^{N} \left(b_{n-1} - h\lambda\right) \gamma_i \gamma_i' \left(c_{n-1}' - h\Xi'\right) \Phi, \\
c_n &= \Phi' \left(c_{n-1} - h\Xi\right) \Phi + 2\sum_{i=1}^{N} \Phi' \left(c_{n-1} - h\Xi\right) \gamma_i \gamma_i' \left(c_{n-1}' - h\Xi'\right) \Phi.
\end{aligned}
$$

Hence, the probability of default over the period t to $t+n$ is

$$
\begin{aligned}
PD\left(t, t+n\right) &= 1 - \exp\left(\ln\left(1 - PD\left(t, t+n\right)\right)\right) \\
&= 1 - \exp\left(a_n + b_n X_t + X_t' c_n X_t\right).
\end{aligned}
$$

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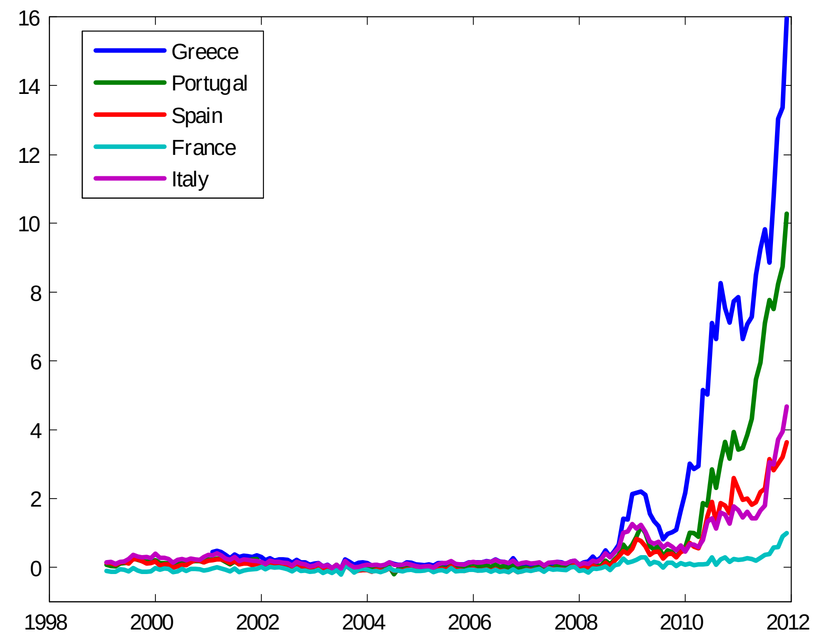
Table 1: Parameter estimates

State variable dynamics: $X_t^i = \Phi^i X_{t-1}^i + \Sigma^i \varepsilon_t^i,\ X_t^i = \left(C_t, g_t^i, d_t^i\right)$,
Default intensities: $\Lambda_t^i = \lambda_0^i + \lambda^i X_t^i + \left(X_t^i\right)' \Xi^i X_t^i$,
Market prices of risk: $\psi_t^i = \psi_0^i + \psi^i X_t^i$

Parameter	Greece	Portugal	Spain	France	Italy
$\lambda_0 \times 10^2$	0.036* (0.021)	0.021** (0.003)	0.039** (0.005)	0.003** (0.001)	0.026** (0.007)
$\lambda_C \times 10^2$	−0.220 (0.809)	−0.074 (0.145)	−0.017 (0.023)	−0.008 (0.009)	−0.024 (0.026)
$\lambda_g \times 10^2$	−0.031** (0.007)	−0.001 (0.001)	−0.015** (0.003)	−0.004** (0.001)	−0.003 (0.004)
$\lambda_d \times 10^2$	−0.200 (0.339)	−0.199** (0.055)	0.441** (0.021)	0.019** (0.008)	0.410** (0.056)
$\Xi_{d,d}$	0.116** (0.002)	0.041** (0.002)	0.021** (0.001)	0.003** (0.000)	0.094** (0.001)
$\phi_{C,C}$			0.837** (0.031)		
$\phi_{g,g}$	0.975** (0.002)	0.996** (0.040)	0.931** (0.040)	0.961** (0.002)	0.808** (0.007)
$\phi_{d,d}$	0.902** (0.006)	0.975** (0.011)	0.977** (0.001)	0.996** (0.001)	0.985** (0.002)
$\phi_{d,C} \times 10^2$	−0.322 (1.680)	−0.140 (0.658)	−0.032 (0.041)	−0.072 (0.145)	−0.002* (0.001)
$\phi_{d,g} \times 10^2$	−0.160** (0.026)	−0.016** (0.005)	−0.177 (0.203)	−0.018** (0.008)	−0.311** (0.026)
σ_g	0.146** (0.011)	0.182** (0.015)	0.155** (0.013)	0.158** (0.002)	0.167** (0.004)
σ_d	0.030* (0.016)	0.013 (0.018)	0.013** (0.004)	0.005** (0.000)	0.004** (0.001)
$\psi_{C,C}$	0.055** (0.009)	−0.014 (0.086)	−0.157** (0.038)	−0.043** (0.008)	−0.159** (0.064)
$\psi_{g,g}$	−0.198** (0.004)	−0.123* (0.071)	−0.011 (0.015)	−0.250** (0.002)	−0.318** (0.005)
$\psi_{d,d}$	−0.792** (0.067)	−1.743** (0.197)	−1.681** (0.111)	−1.979** (0.061)	−3.352** (0.076)
$\psi_{d,C}$	−0.002 (0.027)	−0.021 (0.191)	−0.022 (0.032)	0.134 (0.093)	−0.007 (0.017)
$\psi_{d,g}$	0.014 (0.161)	−0.013 (0.021)	0.033 (0.173)	−0.009** (0.004)	−0.106** (0.017)

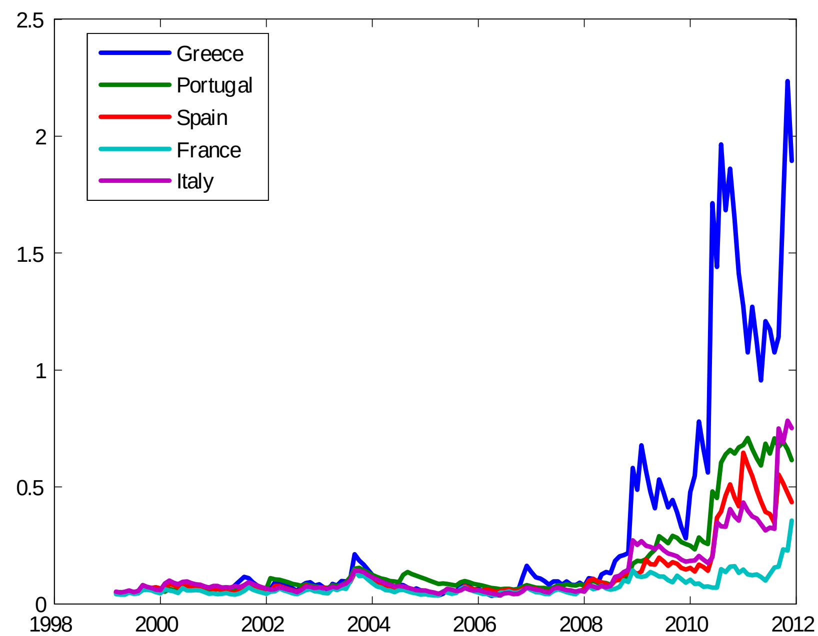
Parameter estimates are obtained using the maximum likelihood method. The estimates also include parameters for the variances of the measurement errors for the variables that are assumed to be imperfectly observed (all bond spreads and our expected GDP growth and debt-to-GDP variables), as well as the constant terms in the market prices of risk. To conserve space, we do not report these estimates here. The variance of the common factor C is normalised to one. During the estimation, the spreads are scaled as (percent per year spreads)/1200; GDP growth is scaled as (percent per year)/100, while debt-to GDP is in decimal form. Figures in parenthesis are asymptotic standard errors based on a numerical estimate of the Hessian matrix. ** denotes statistical significance at the 5% level; * at the 10% level.

Figure 1a: 10-year yield spreads relative to Germany



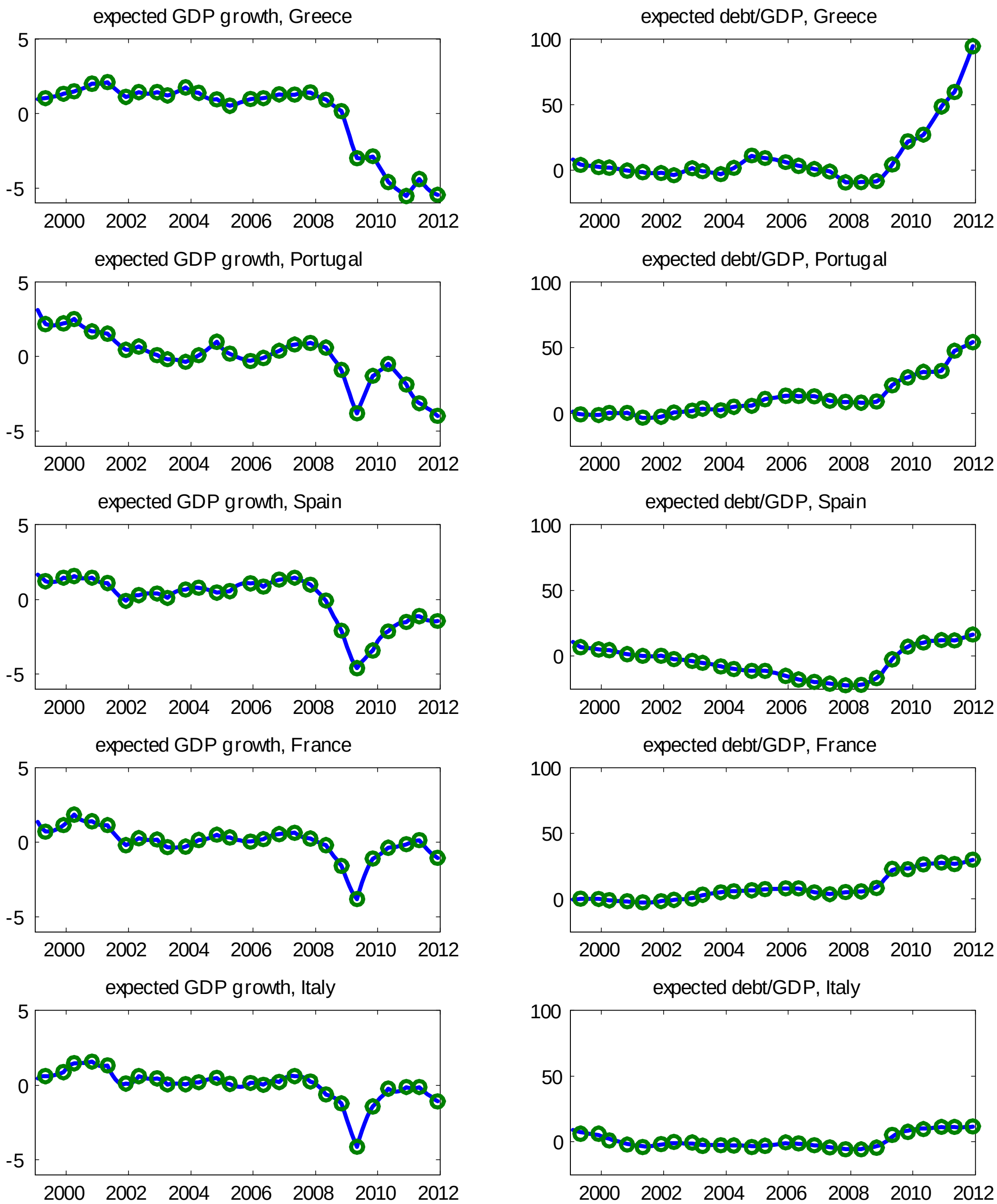
Continuously compounded zero-coupon rates, in annual percent.

Figure 1b: Conditional volatility of 10-year yield spreads



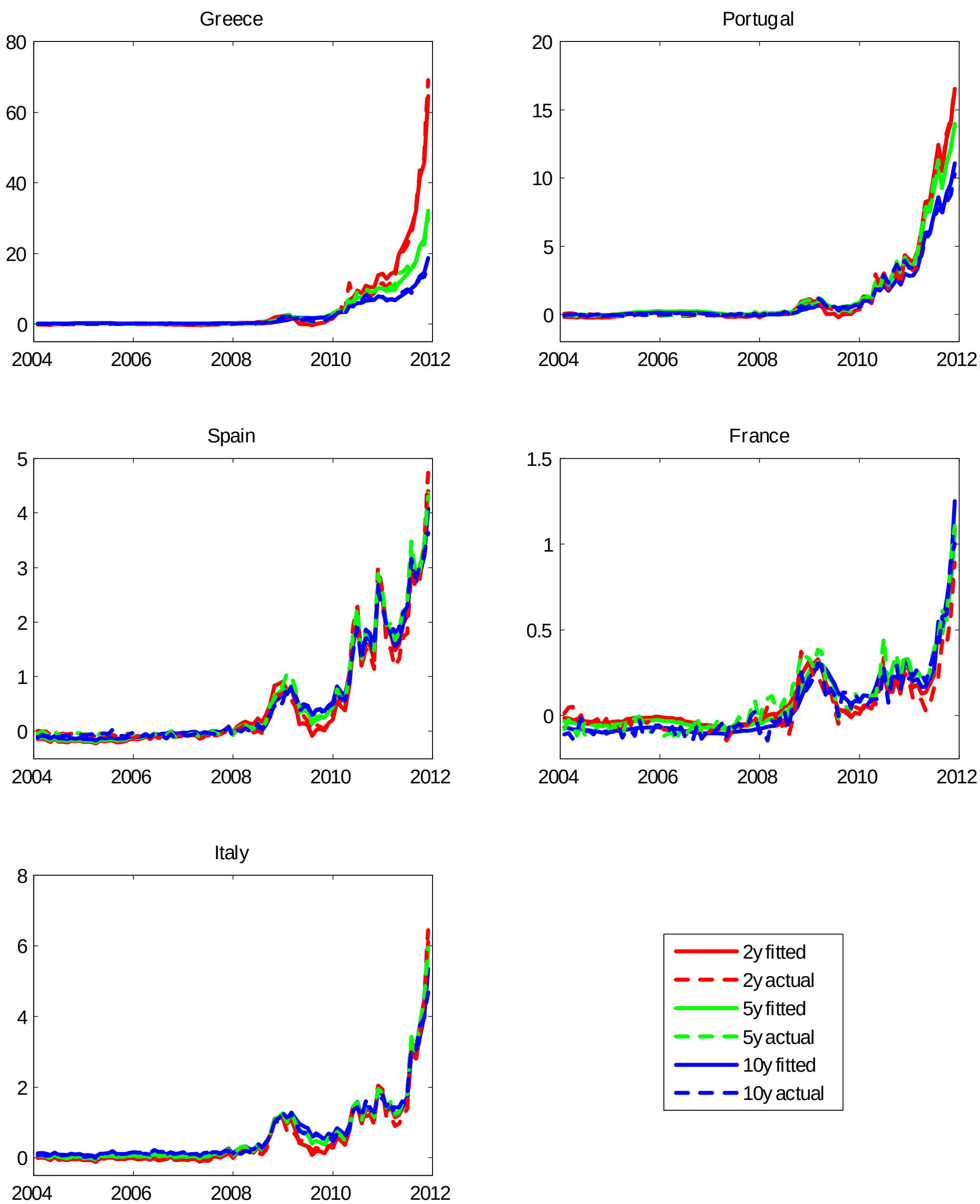
Conditional volatilities estimated using a GARCH(1,1) model on the continuously compounded zero-coupon rates in panel (a).

Figure 2: Macroeconomic data: expected growth and debt/GDP



The solid curves show filtered monthly values for 1-year ahead expected GDP growth and expcted debt/GDP (in percent) in deviations from their respective mean values. The circles show the corresponding semi-annual 'observed' values based on survey data published by the European Commission.

Figure 3: Fitted and actual yield spreads relative to Germany



The spreads are expressed in percent per year.

Figure 4: Estimated common factor

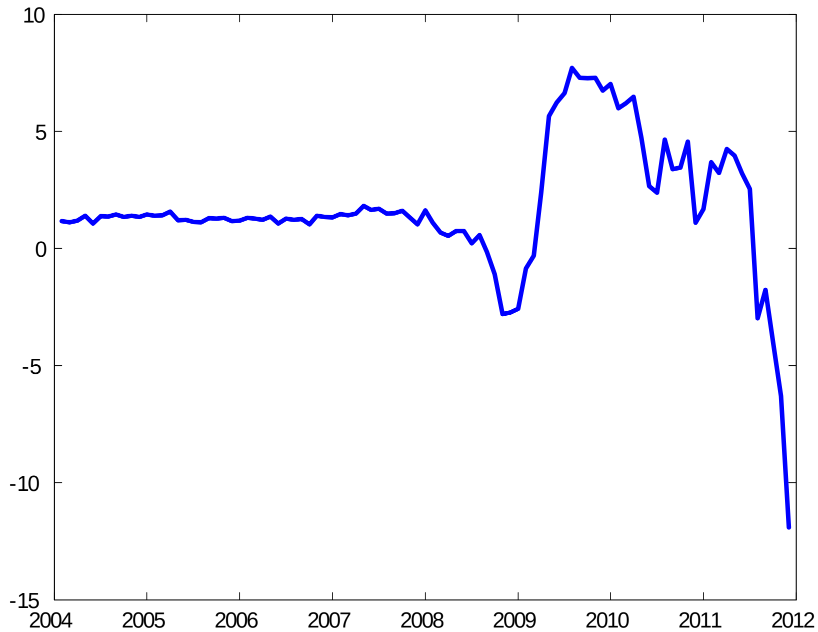
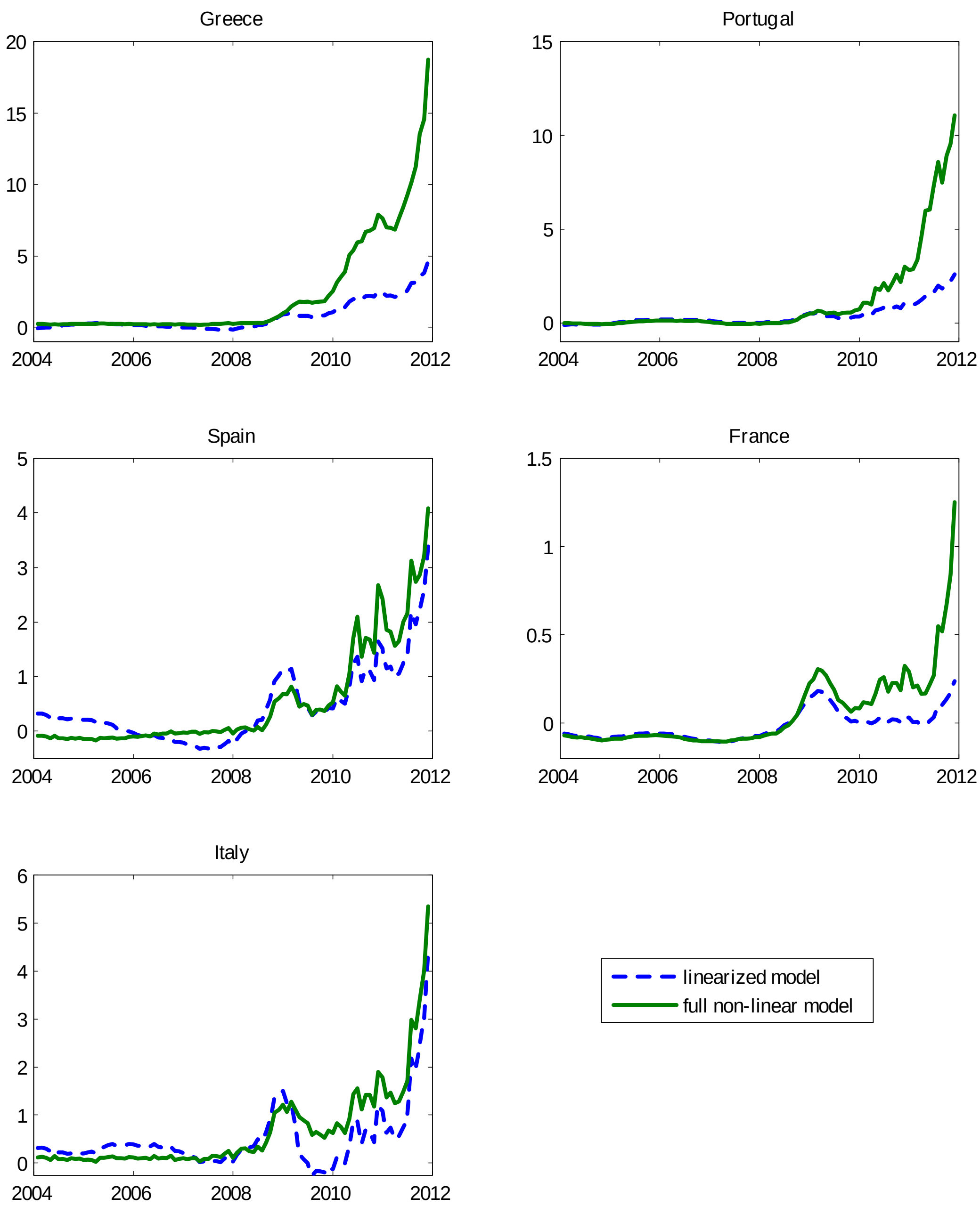
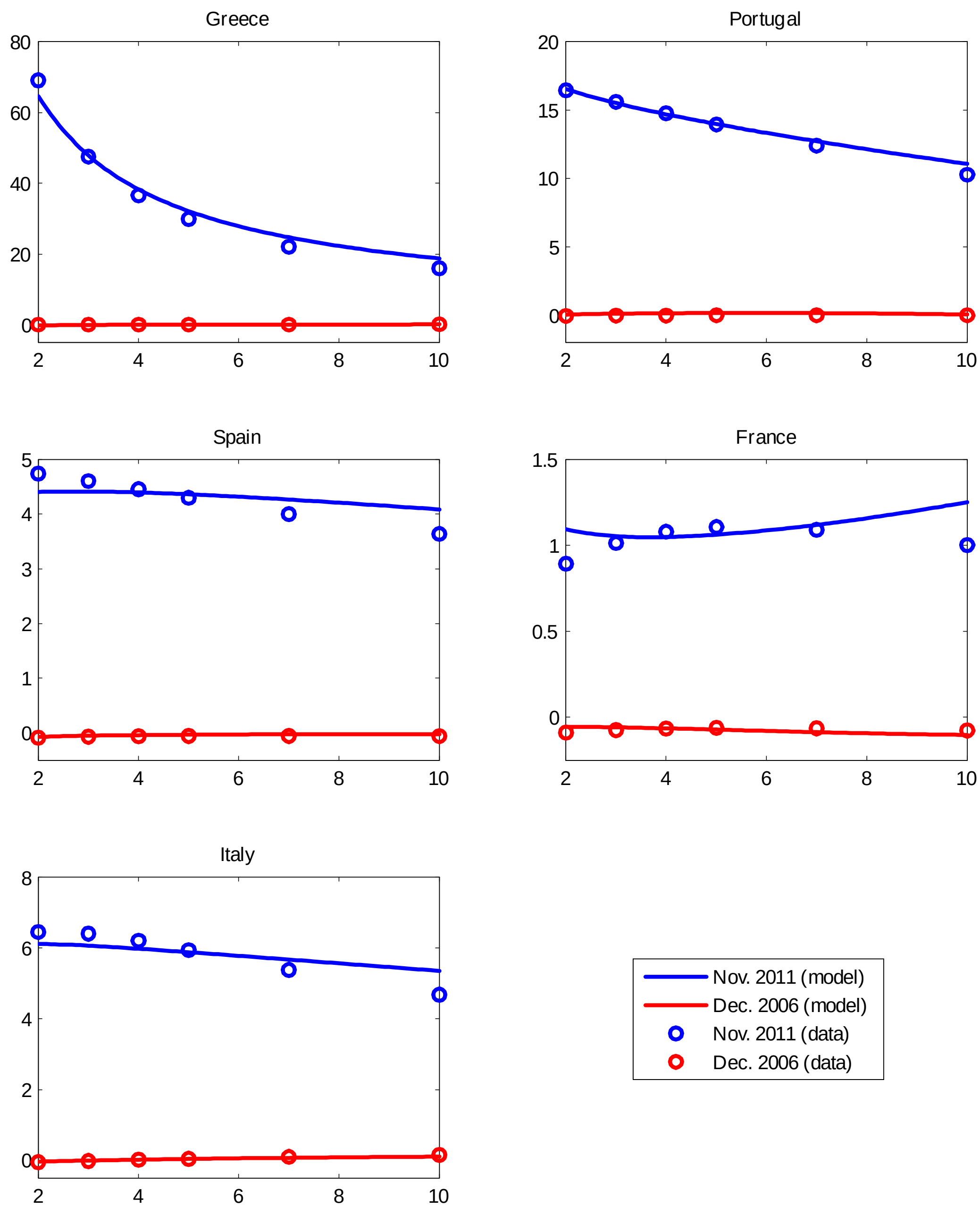


Figure 5: 10-year spread: linearized and non-linear versions of the model



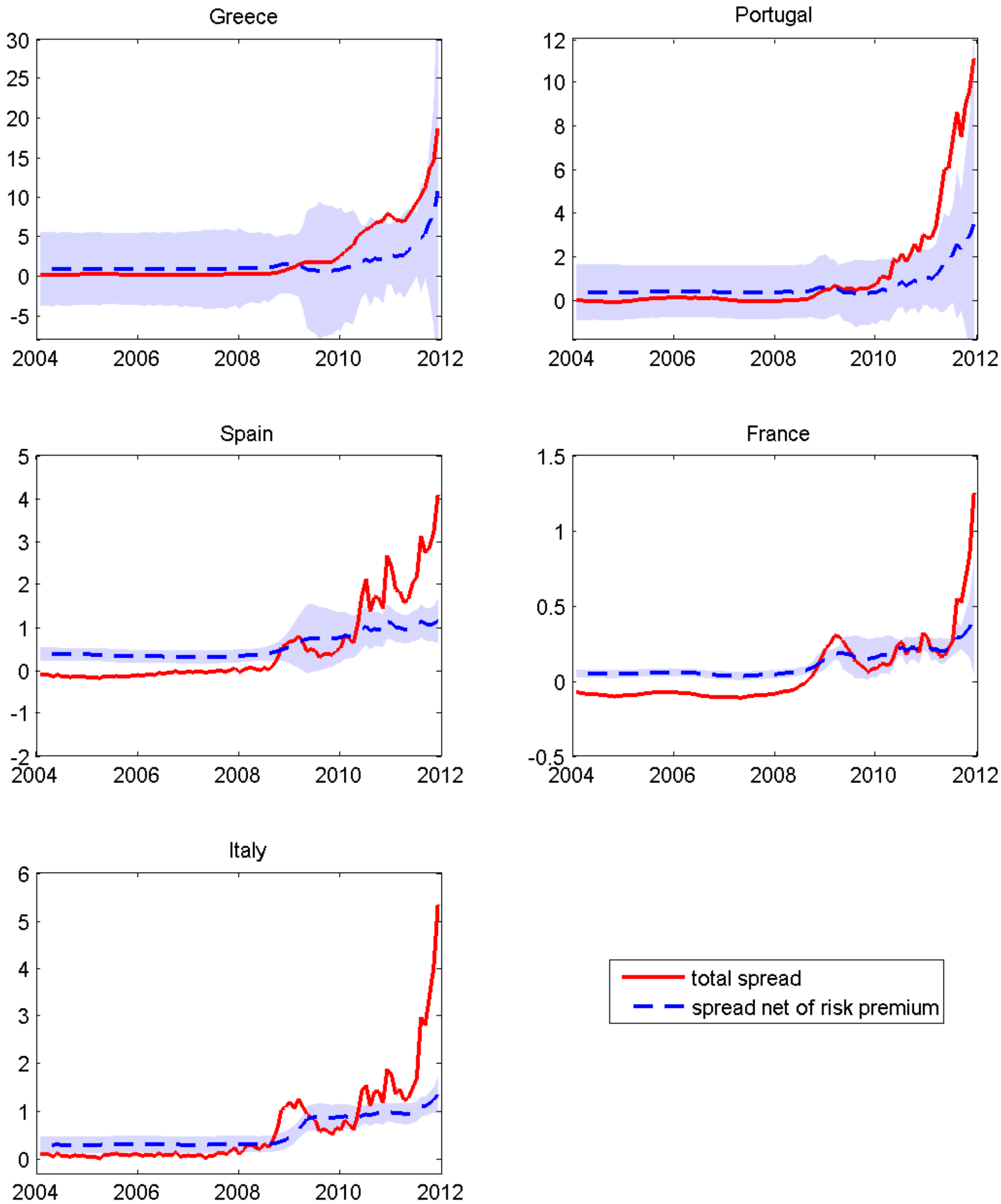
The solid lines show the 10-year sovereign bond spreads implied by the estimated full non-linear model; the dashed lines are corresponding spreads obtained by taking a linear approximation of the full model around the mean values of the state variables.

Figure 6: Fitted and actual term structure of yield spreads



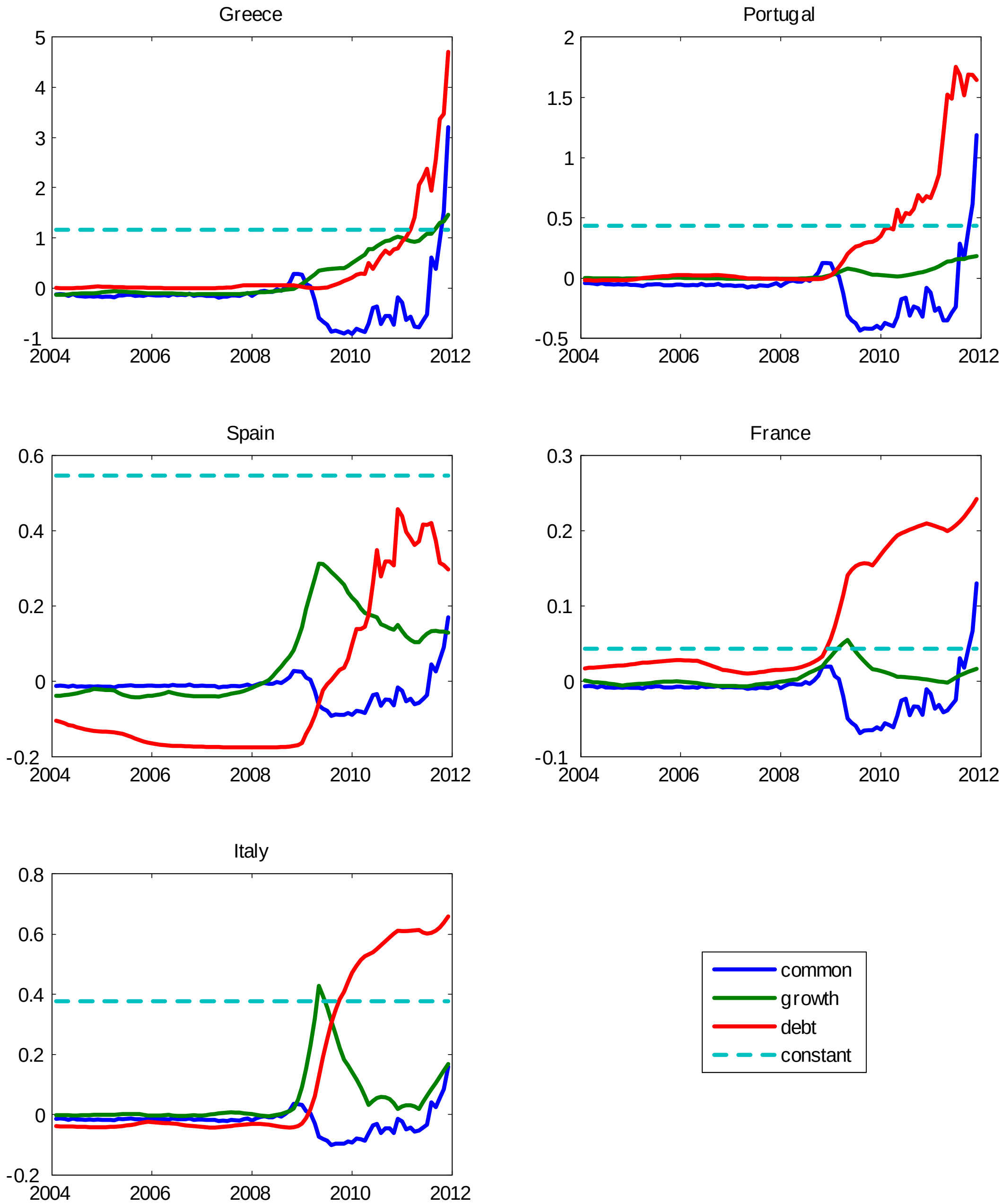
The spreads are expressed in percent per year.

Figure 7: 10-year spread - estimated total and net of distress risk premium



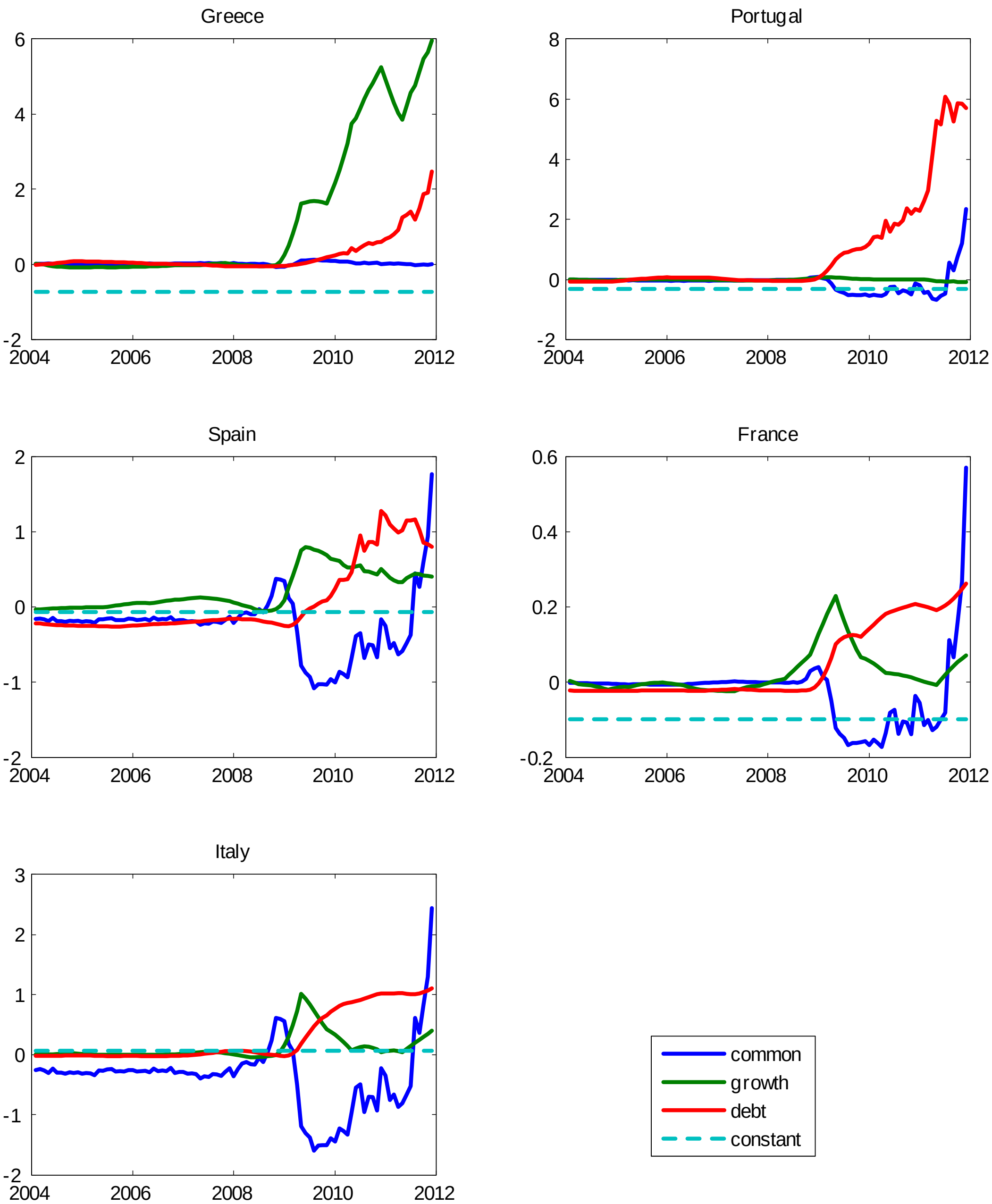
The solid lines show the total estimated 10-year sovereign bond spread relative to Germany, while the dashed lines show the corresponding spreads minus the estimated distress risk premium component, obtained by setting all market price of risk parameters to zero. The shaded areas show plus/minus two standard deviations, calculated using the delta method based on a numerical estimate of the Hessian matrix of the parametes.

Figure 8: Decomposition of the expected default component by state variable



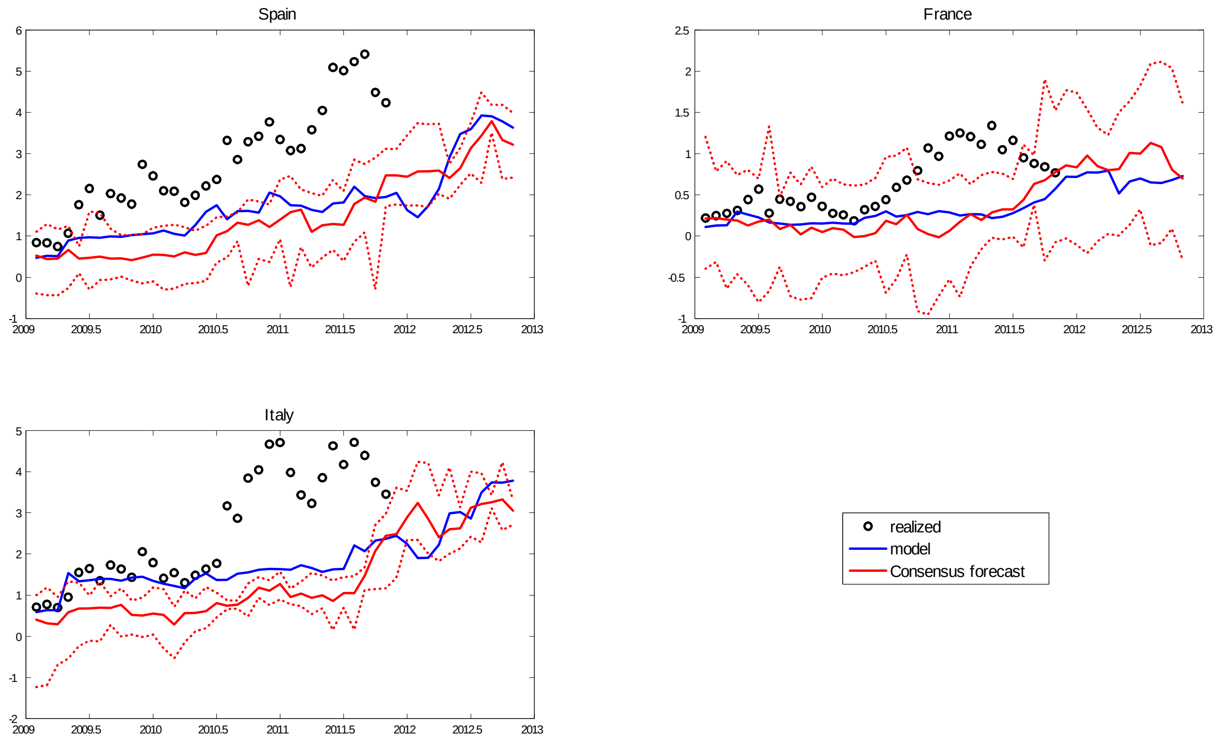
The expected default component is defined as the part of the overall spread that is not due to the distress risk premium. The contribution of each state variable is the sum of the linear and the quadratic parts. In addition, based on an analysis of how the interaction terms correlate with individual state variables, the interaction term between the common factor and debt-to-GDP is attributed to the common fator, while the cross-term between growth and debt is attributed to growth. The third interaction term is negligible and therefore ignored.

Figure 9: Decomposition of the distress risk premium by state variable



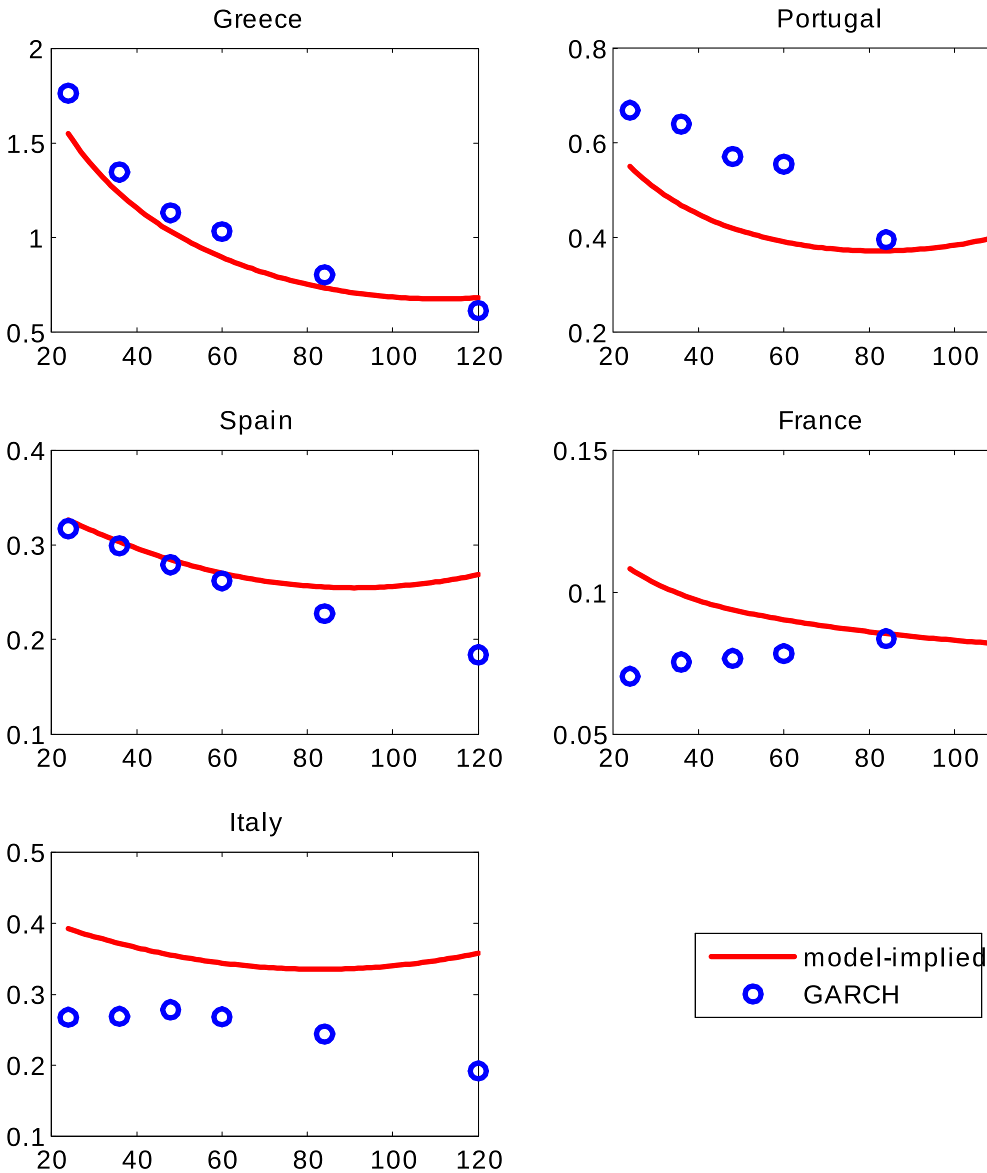
The distress risk premium component is defined as the overall spread minus the spread obtained if all market price of risk parameters are set to zero. The contribution of each state variable is the sum of the linear and the quadratic parts. In addition, based on an analysis of how the interaction terms correlate with individual state variables, the interaction term between the common factor and debt-to-GDP is attributed to the common fator, while the cross-term between growth and debt is attributed to growth. The third interaction term is negligible and therefore ignored.

Figure 10: One-year ahead forecasts of 10-year spreads



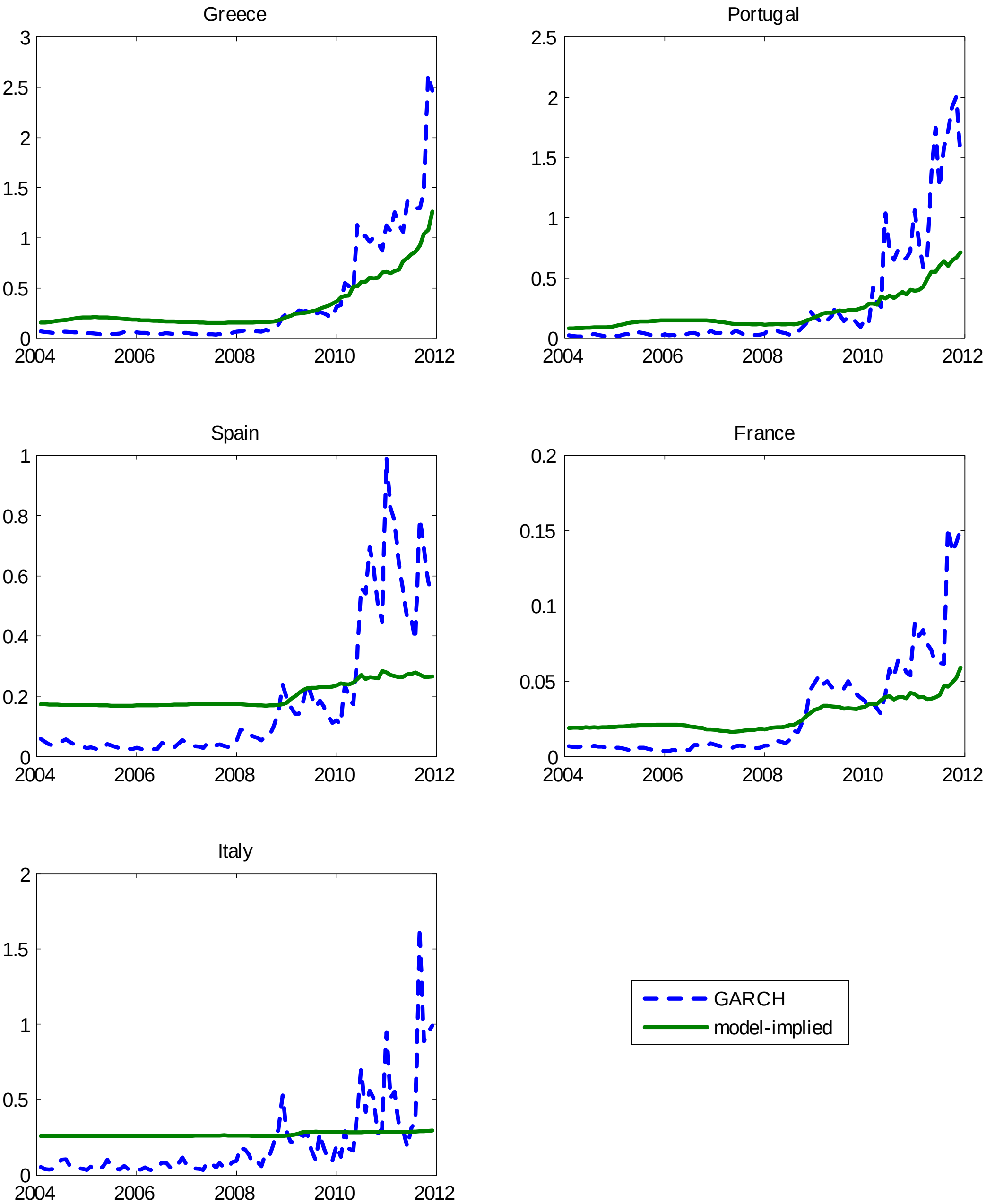
The blue solid lines show model forecasts of 10-year spreads one year ahead (at the time of the forecast); solid red lines are corresponding Consensus forecasts, and dashed red lines are the highes/lowest reported Consensus values, The circles are realized one-year ahead values (available only up to end-November 2011).

Figure 11: Term structure of average conditional spread volatilities



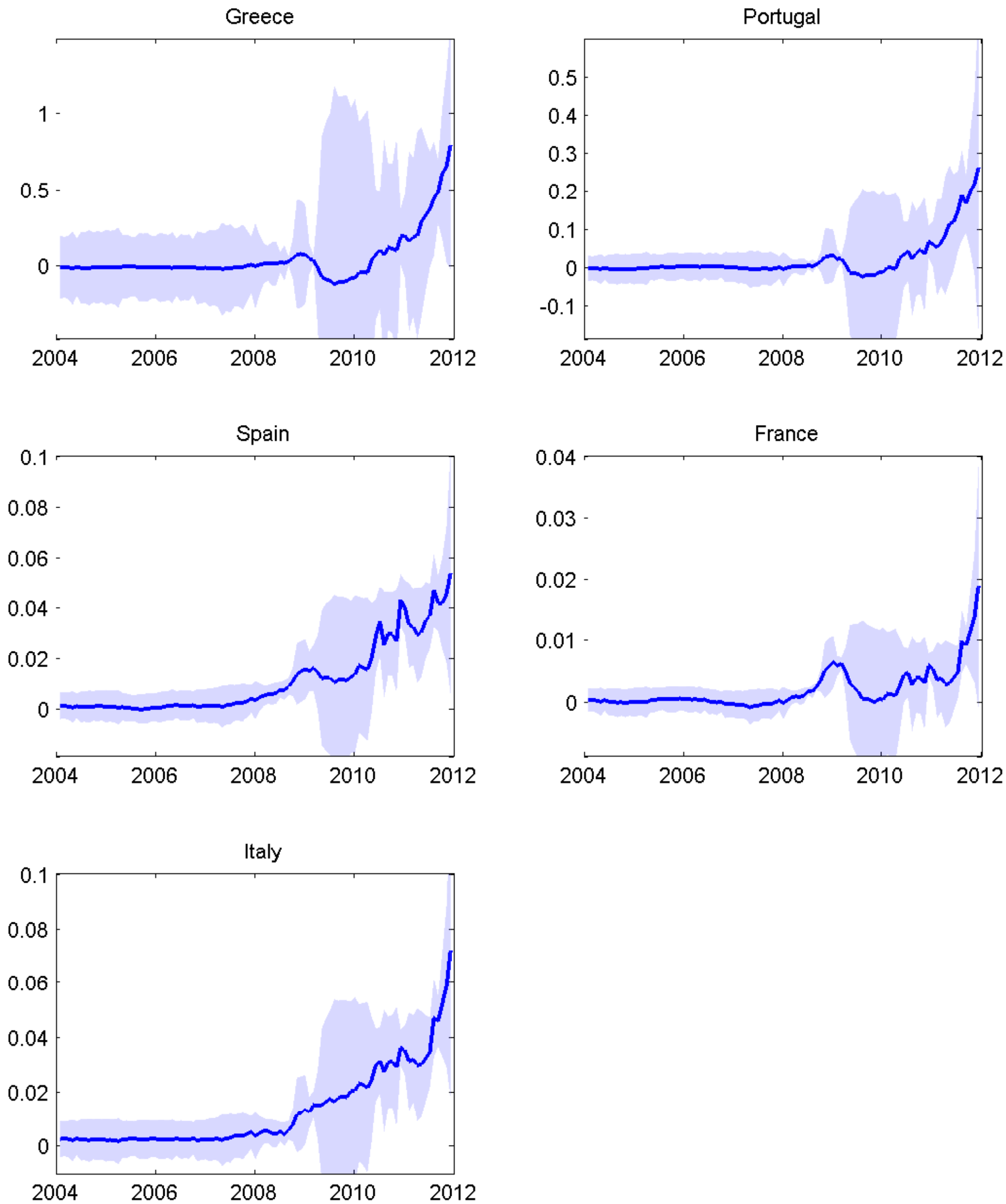
Term structure of average conditional volatilities implied by our affine-quadratic model and estimated using a GARCH(1,1) model on continuously compounded zero-coupon spreads.

Figure 12: Conditional volatilities of 5-year bond spreads



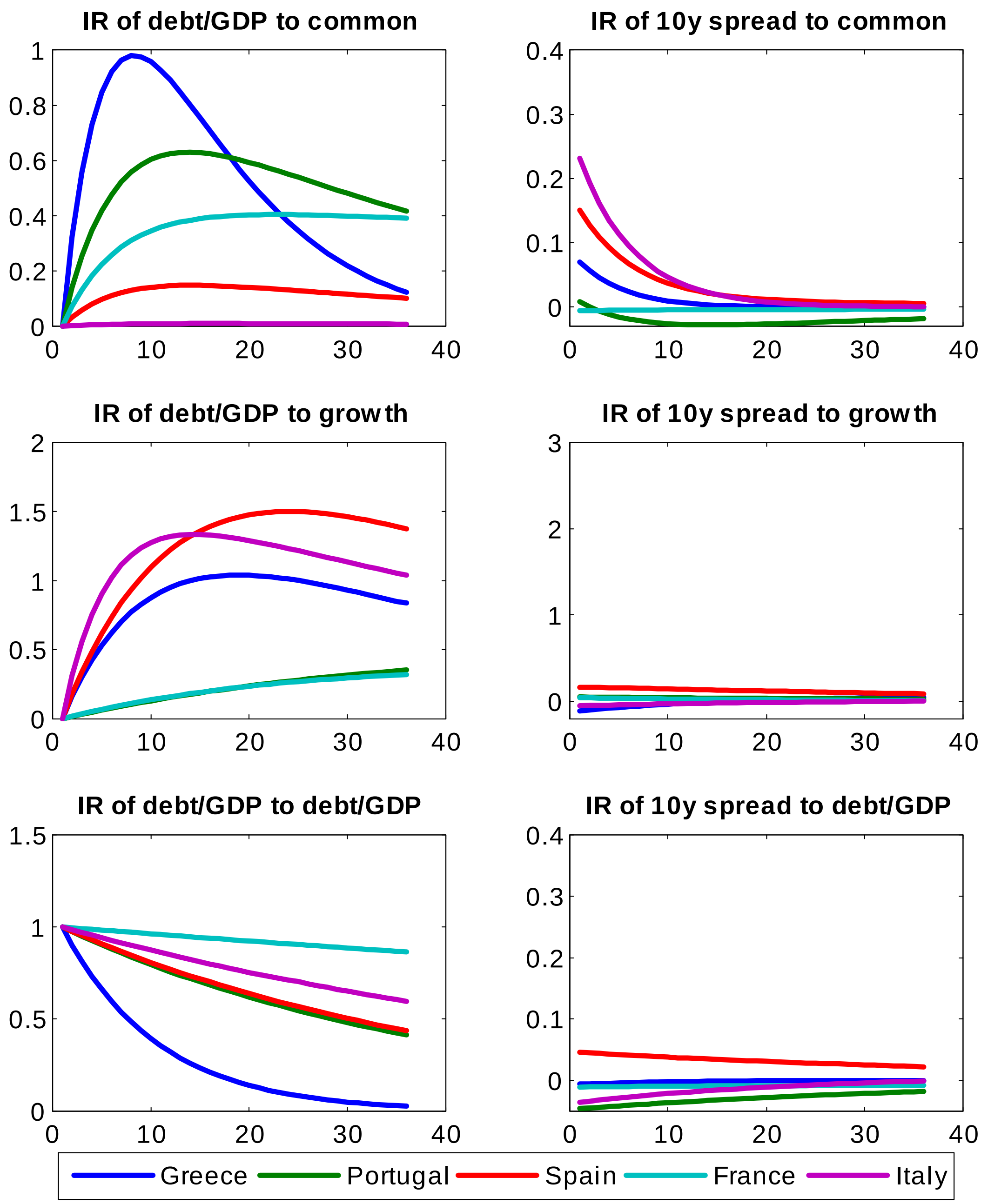
Conditional volatilities implied by our affine-quadratic model and estimated using a GARCH(1,1) model on continuously compounded zero-coupon spreads.

Figure 13: One-year default probabilities



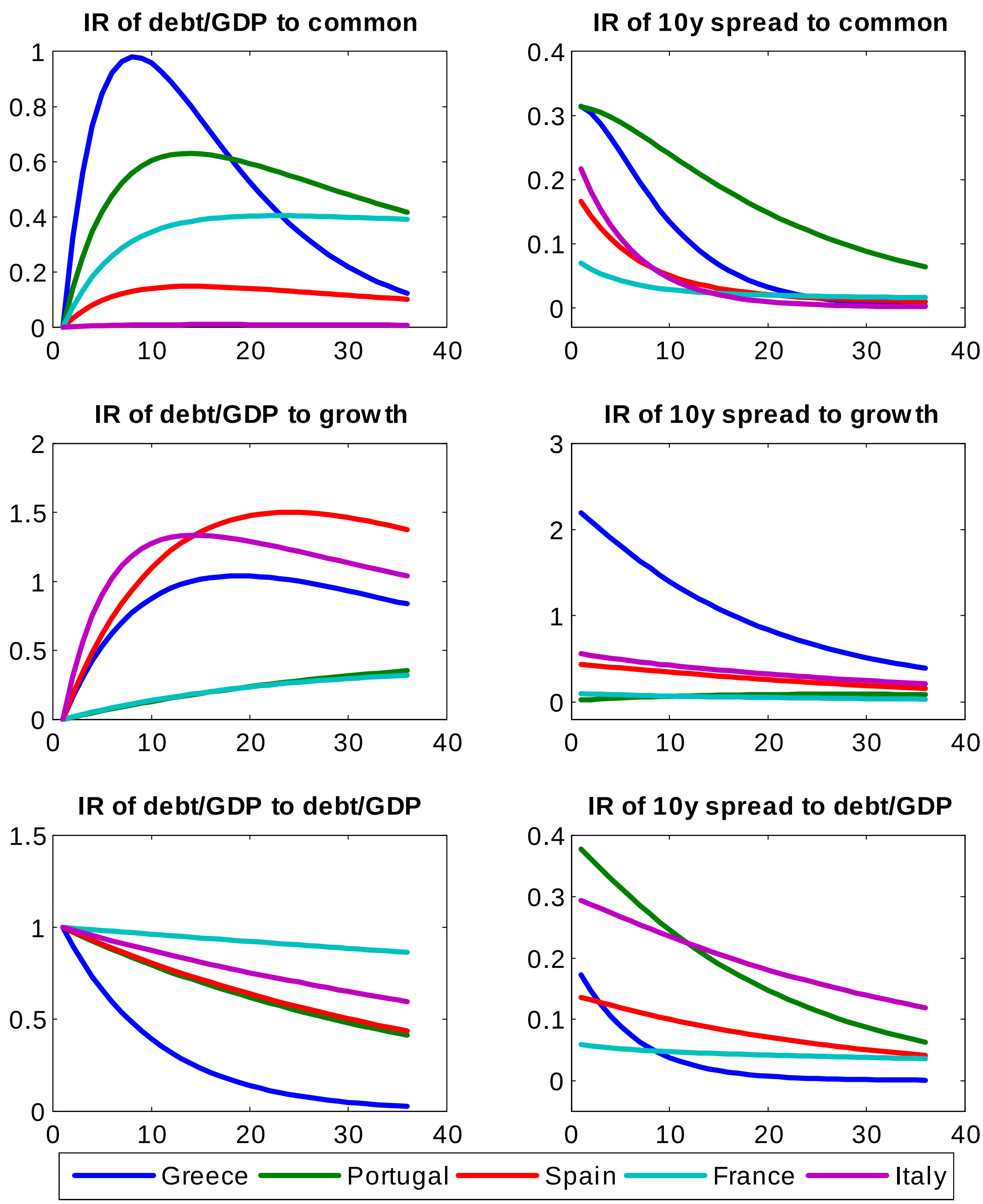
This assumes a loss given default of 50% of the market value. These probabilities are based on the expectation under the objective probability measure of the risk-neutral default intensities. The shaded areas show plus/minus two standard deviations, calculated using the delta method based on a numerical estimate of the Hessian matrix of the parametes.

Figure 14: Impulse responses to shocks in January 2001



The shocks are defined as: common factor = -1 (one standard deviation); growth = 1 percentage point fall in GDP growth; debt/GDP = 1 percentage point increase.

Figure 15: Impulse responses to shocks in November 2011



The shocks are defined as: common factor = -1 (one standard deviation); growth = 1 percentage point fall in GDP growth; debt/GDP = 1 percentage point increase.